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ON SOME PROPERTIES OF THE AREA OF A CONVEX HULL GENERATED BY A POISSON POINT PROCESS INSIDE A PARABOLA

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Abstract. This work is devoted to the study of the properties of functionals of the vertex process of convex hulls generated by independent observations of a two-dimensional random vector with a Poisson distribution inside a parabola, the intensity of which is is related to regularly varying functions. In the cases under consideration, the finiteness of the exponential moment of the area of a convex hull is proved using the modified P. Groenebeum [6] method. The results obtained can be further applied in the proofs of limit theorems for the area of a convex hull.

1. Introduction and formulation of the main results

Let $\Pi_n(\cdot)$ be an inhomogeneous Poisson point process (i.p.p.p.) with intensity $\Lambda_n(\cdot)$, and let $(X_1,Y_1);(X_2,Y_2);...(X_k,Y_k);...$ be the realization of $\Pi_n(\cdot)$ i.p.p.p. in R_n. We denote the convex hulls generated by these random points by C_n, where

$$R_n = \left\{(x,y) : \frac{x^2}{2b_n} \le y\right\}$$

and

$$\Lambda_n(A) = \begin{cases} \frac{1}{2\pi\sqrt{b_n}L(b_n)} \iint_A \frac{\partial}{\partial y}\left[\left(y-\frac{x^2}{2b_n}\right)^{\beta} L\left(b_n\Big/\left(y-\frac{x^2}{2b_n}\right)\right)\right]dxdy, & A\subset R_n;\\ 0, & at\ A\not\subset R_n, \end{cases}$$

is the intensive measure.

Here, b_n is the least root of equation

$$nb_n^{-\left(\beta+\frac{1}{2}\right)}L\left(b_n\right)=1, \tag{1.1}$$

where $L(x)$ is the slowly varying function in the sense of Karamata represented as:

$$L(u)=\exp\left\{\int_1^u \frac{\varepsilon(t)}{t}dt\right\},\quad \varepsilon(t)\to 0,\quad t\to\infty.$$

Convex hulls belong to the area of stochastic geometry. Due to the complexity of the objects, the research was limited only to the study of the average value of the main functionals, such as the number of vertices, the perimeter, and the area

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of the convex hull (see, for example, [4], [5] and [11]). The problems of asymptotic expressions for variance remained unsolved until the appearance of the articles by C. Buchta [1, 2] and J. Pardon [9, 10]. For the first time in this area of science, P.Groeneboom in [6] (approximating a Binomial point process with the Poisson process) proved the central limit theorem for the number of vertices of a convex hull in the case when the support of the initial uniform distribution is either a convex polygon or a unit circle. In that article, a technique based on the martingality of a stationary process with the strong mixing property was successfully applied. In [3], the method developed by P.Groeneboom [6] was applied to prove the central limit theorem for the area and perimeter of a convex hull when the initial distribution is uniform in the convex polygon. This study is conceptually close to the research conducted by [3], [6], [7] and [8]. Here we investigate some properties of the perimeter of a convex hull that can be applied to prove central limit theorems for the area of a convex hull.

According to P. Groeneboom's remark, we investigate these functionals for the case when $L(x) = 1$. Then from (1.1) we obtain

$$b_n = n^{\frac{2}{2\beta+1}} \tag{1.2}$$

Following in [6] and [7], all $a \in (a_-, a_+)$ we define the vertices of process $W_n(a) = (X_n(a), Y_n(a))$ as a point (X_i, Y_i) for which $Y_i - aX_i$ is minimal, where $a_- = -\pi \log n \big/ \sqrt{b_n}$, $a_+ = \pi \log n \big/ \sqrt{b_n}$.

It can be easily shown using the definition that $W_n(a)$ constitutes an unsteady Markov jump process.

If $0 = a_0 < a_1 < ... < a_k \le a$ are the time points of the process jumps $\{W_n(c),\ 0 \le c \le a\}$, then the area of the domain bounded by the lines $y = Y_n(0)$, $y = \frac{x^2}{2b_n}$ is denoted by A_0. For all $1 \le i \le k$, the area of the domain bounded by the lines $y = a_{i-1}\left(x - X_n(a_{i-1})\right) + Y_n(a_{i-1})$, $y = a_i\left(x - X_n(a_{i-1})\right) + Y_n(a_{i-1})$ is denoted by A_i. Finally, the area of the domain bounded by the lines $y = a_k\left(x - X_n(a_k)\right) + Y_n(a_k)$, $y = a\left(x - X_n(a_k)\right) + Y_n(a_k)$ is denoted by $A[a]$. Then, $A_n(0, a) = A_0 + A_1 + ... + A_k + A[a]$ is the area of the domain bounded by the trajectories of the vertex process $\{W_n(c),\ 0 \le c \le a\}$ and the lines $y = \frac{x^2}{2b_n}$.

The following theorem is true.

Theorem 1.1. *If* $|a| \le a_0 = \frac{c_0}{\sqrt{b_n}}$, *then there is such* $\tau_0 > 0$ *that*

$$E \exp\left(\tau \frac{A_n(0, a)}{\sqrt{b_n}}\right) < c_1,$$

is true for any $0 < \tau < \tau_0$.

2. Preliminary lemmas

Before proceeding with the proof of the theorem, we consider two lemmas proved in [7] (see, for example, Lemmas 3 and 4 in [7]).

Lemma 2.1. *Let* $s = y - ax + a^2 b_n/2$. *Then*

1) $P\left(W_n(0)\in(dx,dy)\right)$

$$=\frac{\beta}{2\pi\sqrt{b_n}}\exp\left\{-\frac{y^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}B\left(\beta+1;\frac{1}{2}\right)\right\}\left(y-\frac{x^2}{2b_n}\right)^{\beta-1}dxdy;$$

2) $P\left(W_n(a)\in(dx,dy)\right)$

$$=\frac{\beta}{2\pi\sqrt{b_n}}\exp\left\{-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}B\left(\beta+1;\frac{1}{2}\right)\right\}\left(y-\frac{x^2}{2b_n}\right)^{\beta-1}dxdy;$$

3) $P\left(W_n(a)=W_n(0)/W_n(0)=(x,y)\right)$

$$=\exp\left\{-\frac{1}{2\pi\sqrt{b_n}}\int_{x-ab_n}^{\sqrt{2b_ns}}\left(s-\frac{u^2}{2b_n}\right)^{\beta}du\ -\int_{x}^{\sqrt{2b_ny}}\left(y-\frac{u^2}{2b_n}\right)^{\beta}dy\right]\right\}.$$

Assume that

$$R_n\left(a\right)=X_n\left(a\right)-ab_n,\quad S_n\left(a\right)=Y_n\left(a\right)-\frac{X_n^2\left(a\right)}{2b_n}+\frac{R_n^2\left(a\right)}{2b_n},$$

$$T_n\left(a\right)=\left(R_n\left(a\right),S_n\left(a\right)\right).$$

Obviously that

$$T_n(0)=W_n(0)\ a.s.$$

and therefore

$$P\left(T_n(0)\in(dr,ds)\right)=P\left(W_n(0)\in(dr,ds)\right).$$

Lemma 2.2. *$T_n(a)$ is a stationary Markov jump process and*

1) $P\left(T_n(a)\in(dr,ds)\right)$

$$=\frac{\beta}{2\pi\sqrt{b_n}}\exp\left\{-\frac{s^{\beta+\frac{1}{2}}}{\sqrt{2\pi}}B\left(\beta+1;\frac{1}{2}\right)\right\}\left(s-\frac{r^2}{2b_n}\right)^{\beta-1}drds;$$

2) $P\left(T_n(a)=(r_1,s_1)/T_n(0)=(r_0,s_0)\right)$

$$=\exp\left\{-\frac{1}{\sqrt{2\pi}L\left(b_n\right)}\left[s_1^{\beta+\frac{1}{2}}\int_{\frac{r_1}{\sqrt{2b_ns_1}}}^{1}\left(1-t^2\right)^{\beta}dt-\right.\right.$$

$$\left.\left.-s_0^{\beta+\frac{1}{2}}\int_{\frac{r_0}{\sqrt{2b_ns_0}}}^{1}\left(1-t^2\right)^{\beta}dt\right]\right\},$$

where $r_1=r_0-ab_n,\ \ s_1=s_0-ar_0+\frac{a^2b_n}{2}$.

3) $P\left(T_n(a)\in\left(dr_1,ds_1\right)/T_n(0)=\left(r_0,s_0\right)\right)=P\left(T_n(a)\in(dr_1,ds_1)\right),$
if $ab_n-\sqrt{2b_ns_1}>\sqrt{2b_ns_0}$.

4) $P\left(T_n(a)\in\left(dr_2,ds_2\right)/T_n(0)=\left(r_1,s_1\right)\right)$

$$=\frac{1}{2\pi\sqrt{b_n}}\exp\left\{-\frac{1}{\sqrt{2\pi}}\left[s_2^{\beta+\frac{1}{2}}\int_{\frac{s_1-s_2}{a\sqrt{2b_ns_2}}+\frac{ab_n}{\sqrt{2b_ns_2}}}^{1}\left(1-t^2\right)^{\beta}dt\ -\right.\right.$$

$$\left.\left.-s_1^{\beta+\frac{1}{2}}\int_{\frac{s_1-s_2}{a\sqrt{2b_ns_1}}+\frac{ab_n}{\sqrt{2b_ns_1}}}^{1}\left(1-t^2\right)^{\beta}dt\right]\right\}\left(s_2-\frac{r_2^2}{2b_n}\right)^{\beta-1}dr_2ds_2.$$

Here assume that

$$(r_i, s_i) \in D = \left\{(r,s): \ s \geq \left(r^2\right)/\left(2b_n\right)\right\}, \ ab_n - \sqrt{2b_n s_2} \leq \sqrt{2b_n s_1},$$
$$s_2 + \left(a^2 b_n\right)/2 + ar_2 \geq s_1 \geq s_2 - \left(a^2 b_n\right)/2 + ar_1.$$

3. The proof of the Theorem 1.1

Let $W_n(0) = (x_0, y_0)$, then it is easy to make sure that

$$A(0,a) \leq A_0 + A(0,a,x_0,y_0), \tag{3.1}$$

where $A(0,a,x_0,y_0)$ is the area of the domain bounded by lines $y = a(x-x_0)+y_0$ and $y = \frac{x^2}{2b_n}$.

On the other hand, it can be easily shown using the definition that

$$A_0 = \int_{-\sqrt{2b_n y_0}}^{\sqrt{2b_n y_0}} \left(y_0 - \frac{x^2}{2b_n}\right) dx = y_0^{\frac{3}{2}} \sqrt{2b_n} \int_{-1}^{1} \left(1-t^2\right) dt = \frac{4\sqrt{2b_n}}{3} y_0^{\frac{3}{2}} \tag{3.2}$$

Similarly, if we assume that $s_0 = y_0 - ax_0 + \frac{a^2 b_n}{2}$ and $t = x - ab_n$, then it is easy to obtain

$$A(0,a,x_0,y_0) = \int_{ab_n - \sqrt{2b_n s_0}}^{ab_n + \sqrt{2b_n s_0}} \left(a(x-x_0) + y_0 - \frac{x^2}{2b_n}\right) dx$$
$$= \int_{-\sqrt{2b_n s_0}}^{\sqrt{2b_n s_0}} \left(s_0 - \frac{t^2}{2b_n}\right) dt = \frac{4\sqrt{2b_n}}{3} s_0^{\frac{3}{2}}. \tag{3.3}$$

On the other hand, if $y_0 > c_0$, then $s_0 \leq c_1 y_0 + c_2$. Hence, and from (3.1) – (3.3) we obtain

$$A(0,a) \leq \left(c_3 y_0^{\frac{3}{2}} + c_4\right)\sqrt{2b_n}$$

for all $y_0 > c_0$. Hence, and from Lemma 2.1 we have

$$E\exp\left(\tau \frac{A_n(0,a)}{\sqrt{b_n}}\right) = E\left\{E\left(\exp\left(\tau\frac{A_n(0,a)}{\sqrt{b_n}}\right) \Big/ W_n(0) = (x_0,y_0)\right)\right\}$$
$$\leq c_5 EE\left\{\exp\left(c_6\tau y_0^{\beta+\frac{1}{2}}\right) \Big/ W_n(0) = (x_0,y_0)\right\} = \iint \exp\left(c_6\tau y_0^{\beta+\frac{1}{2}}\right)$$
$$\cdot P\left(W_n(0) \in (dx_0, dy_0)\right) = \frac{\beta c_5}{2\pi\sqrt{b_n}} \int_0^\infty \exp\left(c_6 \tau y_0^{\beta+\frac{1}{2}}\right)$$
$$\cdot \exp\left(-\frac{y_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1; \frac{1}{2}\right)\right) \int_{-\sqrt{2b_n y_0}}^{\sqrt{2b_n y_0}} \left(y_0 - \frac{x_0^2}{2b_n}\right)^{\beta-1} dx_0 dy_0$$
$$\leq c_5 \int_0^\infty \exp\left(c_6\tau y_0^{\beta+\frac{1}{2}} - \frac{y_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1;\frac{1}{2}\right)\right) d\left(\frac{y_0^{\beta+\frac{1}{2}}}{\sqrt{2\pi}} B\left(\beta+1;\frac{1}{2}\right)\right) < \infty$$

for any $0 < \tau < \tau_0$.

The proof of the theorem is complete.

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