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A NOTE ON DEDEKIND η-FUNCTION IDENTITIES OF LEVEL 14 GIVEN BY SOMOS

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Abstract. Recently Vasuki and Veeresha were able to prove Dedekind η-function identities of level 14 conjectured by Somos. In this paper, we show how these Somos's identities are the consequence of Ramanujan's modular equations of degree 7. Also, in their paper authors are using a known multiplier to prove Somos's identities and we explained the method of obtaining that multiplier in this paper.

1. Introduction

We always assume $|q| < 1$ throughout this paper and recall the q-series notation as follows:

$$(x)_n := (x;q)_n := \prod_{k=0}^{\infty}(1 - xq^{k-1}) \qquad n \geq 1.$$

The theta function $\mathfrak{f}(x,y)$ in Ramanujan's notation is

$$\mathfrak{f}(x,y) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2}, \qquad |xy| < 1.$$

The Jacobi's triple-product identity [6, p. 35] is given by

$$\mathfrak{f}(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty.$$

Two primary results on $\mathfrak{f}(x,y)$ [6, p. 36] are:

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2} = (q;q)_\infty$$

$$\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty.$$

Observe that, for $q = e^{2\pi i\tau}$ then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$ and $\eta(\tau)$ is the Dedekind eta-function defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) \qquad Im(\tau) > 0,$$

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and τ is a complex number. Together with, next to Ramanujan define

$$\chi(q) = (-q; q^2)_\infty.$$

For comfort we address $f(-q^n) = f_n$. Also one can deduce the following easily:

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{1.1}$$

A theta-function identity, which relates f_1 to f_n is known as n^{th} level theta function identity. Ramanujan [10, 11] documented many identities which associates quotients of the function f_1 at various arguments. For instance [7], let

$$P := \frac{f_1^2}{q^{1/2} f_7^2} \qquad \text{and} \qquad Q := \frac{f_2^2}{q f_{14}^2}$$

then

$$PQ + \frac{49}{PQ} = \left(\frac{Q}{P}\right)^3 - 8\frac{Q}{P} - 8\frac{P}{Q} + \left(\frac{P}{Q}\right)^3. \tag{1.2}$$

Thereafter the publication of [7], many authors along with C. Adiga et al. [1, 2, 3], N. D. Baruah [4, 5], M. S. M. Naika [9], K. R. Vasuki et al. [17, 18] and numerous mathematicians constructed supplementary modular equations of the type (1.2) of various levels to evaluate Weber-class invariants, several continued fractions, two-parameter evaluation of theta functions and many more. Inspired by the above, M. Somos [12] employed a computer to explore allover 6200 new classic Dedekind η-function identities of assorted levels and left out contributing the proof. He runs GP/PARI writing to check every identity to verify whether it is correlative to an identity in P-Q forms. He executes GP/PARI manuscripts and it functions as a sophisticated calculator but could not offer any proof for them. Furthermore, Yuttanan [20] demonstrated some of Somos's identities from the well-known Ramanujan modular equations and obtained some fascinating results on colored partitions. Furthermore, Somos discovered nearly 26 identities of level 14 and these identities have been proved by Vasuki and Veeresha [17]. Srivatsa Kumar and Veeresha [13] attained partition identities for the same. Further, Srivatsa Kumar et al. [15] achieved the arguments for the level 6 by employing 3^{rd} degree modular equations of Ramanujan. In the proofs [19] authors multiplied a polynomial in a and b into a known modular equation. In other words, authors succeeded in illustrating the fact that all the theorems "modulo some polynomials in a and b" are essentially equivalent. In [19], authors have not discussed how the various polynomials appeared in their proofs. This is important because it will help the readers to fully understand their method.

Before pursuing to prove the above Somos's identities, we have chosen initially to review modular equation. A modular equation [6] of n^{th} degree is an equation describing α and β which is stated by

$$n\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)},$$

where

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \qquad |z|<1, $$

denotes an ordinary hypergeometric series with

$$ (a)_n := a(a+1)(a+2)...(a+n-1). $$

Then, we say that β is of degree n over α and call the ratio

$$ m := \frac{z_1}{z_n}, $$

the multiplier, where $z_1 := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

2. Proof of Dedekind η-function identities of level 14

Theorem 2.1. *We have*

$$ f_1 f_2^8 f_7^4 f_{14}^3 + q f_1^4 f_2 f_7^7 f_{14}^4 + 8q^2 f_1 f_2^4 f_7^4 f_{14}^7 - q^3 f_1^5 f_{14}^{11} - f_2^5 f_7^{11} = 0. $$

Proof. From [17, Theorem 3.2], we have if

$$ A := \frac{f_1}{q^{1/24} f_2} \qquad \text{and} \qquad B := \frac{f_7}{q^{7/24} f_{14}} $$

then

$$ (AB)^3 + \frac{8}{(AB)^3} + 7 = \left(\frac{A}{B}\right)^4 + \left(\frac{B}{A}\right)^4. \tag{2.1} $$

On dividing (2.1) throughout by $f_1 f_2^8 f_7^4 f_{14}^3$, we find that

$$ 1 + q\frac{f_1^3 f_7^3 f_{14}}{f_2^7} + 8q^2 \frac{f_{14}^4}{f_2^4} - q^3 \frac{f_1^4 f_{14}^8}{f_2^8 f_7^4} - \frac{f_7^7}{f_2^3 f_1 f_{14}^3} = 0. $$

Employing A, B, P and Q as defined as in (2.1) and (1.2), the above equation reduces to

$$ 1 + \frac{(AB)^3}{Q^2} + \frac{8}{Q^2} - \frac{(AB)^3}{P^2} - \frac{P^2}{Q^4} = 0 $$

which is equivalent to

$$ (AB)^3 = \frac{P^2(P^2 - 8Q^2 - Q^4)}{Q^2(P^2 - Q^2)}. \tag{2.2} $$

Using (2.2) in (2.1), fact that $(A/B)^2 = P/Q$, and then factorizing the resulting identity, we obtain

$$ R(P,Q)S(P,Q) = 0, $$

allowing that

$$ R(P,Q) = Q^6 - 49P^2Q^2 - 8P^2Q^4 - 8P^4Q^2 - P^4Q^4 + P^6 $$

and

$$\begin{aligned}S(P,Q)=&1561P^4Q^{12}+75P^4Q^{16}+837P^6Q^{10}+826P^6Q^{12}-837P^8Q^8\\&-113P^8Q^{10}-339P^{10}Q^6+84P^8Q^{12}-327P^{10}Q^8+294P^{12}Q^4\\&+19P^8Q^{14}-76P^{10}Q^{10}+111P^{12}Q^6-54P^{14}Q^2+P^8Q^{16}\\&-5P^{10}Q^{12}+10P^{12}Q^8-9P^{14}Q^4+195P^6Q^{14}+13P^6Q^{16}\\&+Q^{18}P^4+Q^{20}+744P^2Q^{14}+694P^4Q^{14}+5P^2Q^{18}\\&+166P^2Q^{16}+192Q^{16}+3P^{16}+24Q^{18}.\end{aligned}$$

However $R(P,Q)$ is same as (1.2) in another form, it confirms (2.1). $\square$

Theorem 2.2. *We have*

$$f_1^3f_2^7f_7^4+qf_1^7f_2^3f_{14}^4+49q^2f_1^3f_2^3f_7^4f_{14}^4-f_1^{10}f_7^3f_{14}-8qf_2^{10}f_7f_{14}^3=0, \tag{2.3}$$

Proof. On dividing throughout by $f_1^3f_2^7f_7^4$, we have

$$1+q\frac{f_1^4f_{14}^4}{f_2^4f_7^4}+49q^2\frac{f_{14}^4}{f_2^4}-\frac{f_1^7f_{14}}{f_2^7f_7}-8q\frac{f_2^3f_{14}^3}{f_1^3f_7^3}=0.$$

Employing A, B, P and Q as defined in (2.1) and (1.2), the above equation reduces to

$$1+\frac{P^2}{Q^2}+\frac{49}{Q^2}-\frac{A^3B^3P^2}{Q^2}-\frac{8}{A^3B^3}=0,$$

which is equivalent to

$$P^6(AB)^6-(P^2+Q^2+49)(AB)^3+8Q^2=0.$$

On solving the above quadratic equation for $(AB)^3$, we obtain

$$(AB)^3=\frac{P^2+Q^2+49\pm\sqrt{P^4-30P^2Q^2+98P^2+98Q^2+Q^4+2401}}{2P^2}. \tag{2.4}$$

Using (1.1) in (2.1) and using the fact that $(A/B)^2=P/Q$ and then factorising the resulting identity, we obtain

$$P^2Q^2(P^6-49P^2Q^2-8P^4Q^2-8P^2Q^4-P^4Q^4+Q^6)=0.$$

However the second factor is same as (1.2) in another form, it confirms (2.3). $\square$

2.1. Remark. Since the proof of the other identities are same, we neglect the proof. Now we write down some of η-function identities of level 14 which are due to Somos [12] and proved by Vasuki and Veeresha [19]:

$$\begin{gathered}f_1^7f_2^4f_{14}^3+49qf_1^3f_2^4f_7^4f_{14}^3+7f_1^6f_2f_7^7-7q^2f_1^7f_{14}^7-8f_2^{11}f_7f_{14}^2=0,\\f_1^{11}f_7^2f_{14}+7qf_2^7f_7^7-f_1^4f_2^7f_7^3-49q^2f_1^4f_2^3f_7^3f_{14}^4-56q^3f_1f_2^6f_{14}^7=0,\\f_1^7f_7^6f_{14}+7qf_1^4f_2^3f_7^3f_{14}^4+7q^2f_2^3f_7^7f_{14}^4-f_2^7f_7^7-56q^4f_1f_2^2f_{14}^{11}=0,\\f_1^4f_2^4f_7^3f_{14}^3+qf_2^4f_7^7f_{14}^3+q^2f_1^4f_7^3f_{14}^7-f_1^3f_2f_7^{10}-8q^3f_1f_2^3f_{14}^{10}=0,\\7f_1^2f_2f_7^{11}+qf_1^7f_{14}^7-7q^2f_1^3f_7^4f_{14}^7-7f_1^3f_2^4f_7^4f_{14}^3-8qf_2^7f_7f_{14}^6=0,\\f_1^{14}f_{14}^2+8f_2^{14}f_7^2-49qf_1^3f_2^7f_7^5f_{14}-49q^2f_1^7f_2^3f_7f_{14}^5-9f_1^7f_2^7f_7f_{14}=0,\end{gathered}$$

and many more identities of the similar type.

3. Method of getting the multiplier

Now we explain the concept of obtaining the multiplier used in the proofs of [19]. On dividing (2.3) throughout by $f_1^3 f_2^7 f_7^4$ and using (1.1), we obtain

$$1 + q\frac{\chi^4(-q)}{\chi^4(-q^7)} + 49q^2\left(\frac{f_{14}}{f_2}\right)^4 - \frac{\chi^7(-q)}{\chi(-q^7)} - \frac{8q}{\chi^3(-q)\chi^3(-q^7)} = 0. \tag{3.1}$$

Also from (1.1), we observe that

$$\frac{f_2}{f_{14}} = \frac{\chi^2(q^7)}{\chi^2(q)}\frac{\varphi(q)}{\varphi(q^7)}. \tag{3.2}$$

For convenience, set $a(q) := q^{-1/24}\chi(q) = a$ and $b(q) := q^{-7/24}\chi(q^7) = b$, letting q to $-q$ in (3.1) and making use of (3.2), we obtain

$$1 - \frac{a^4}{b^4} + 49\frac{a^8}{b^8}\frac{\varphi^4(q^7)}{\varphi^4(q)} - \frac{a^7}{b} + \frac{8}{a^3b^3} = 0. \tag{3.3}$$

For $y = \pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}$ and $z = {}_2F_1(\frac{1}{2},\frac{1}{2};1;x)$, then we have from Entries 10(i) and 12(v) [6, pp.122-124], we observe that

$$\varphi(e^{-y}) := \sqrt{z} \tag{3.4}$$

and

$$\chi(e^{-y}) := 2^{1/6}\left\{x(1-x)e^{-y}\right\}^{-1/24}. \tag{3.5}$$

From [6, Entry 19(ix), p.314-324] we have, if β is of 7^{th} degree in α and m is the multiplier then

$$m = \frac{1 - 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{\{(1-\alpha)(1-\beta)^{1/8} - (\alpha\beta)^{1/8}\}} \quad\text{and}\quad \frac{7}{m} = \frac{1 - 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}{\{(1-\alpha)(1-\beta)\}^{1/8} - (\alpha\beta)^{1/8}} \tag{3.6}$$

which readily gives

$$\frac{m^2}{7} = \frac{1 - 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{1 - 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}. \tag{3.7}$$

Now, on transcribing (3.7) into theta functions by employing (3.4) and (3.5), we obtain

$$\frac{\varphi^4(q)}{7\varphi^4(q^7)} = \frac{1 - \frac{8a}{b^7}}{\frac{8b}{a^7} - 1}. \tag{3.8}$$

On employing (3.8) in (3.3), we obtain

$$\frac{7a^8}{b^8}\left(\frac{8b}{a^7} - 1\right) = \left(\frac{a^7}{b} - \frac{8}{a^3b^3} + \frac{a^4}{b^4} - 1\right)\left(1 - \frac{8a}{b^7}\right). \tag{3.9}$$

Also, from [6, Entry 19(ix), p.314-324] we have, if

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad\text{and}\qquad Q := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}$$

then

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right). \tag{3.10}$$

On transcribing (3.10) into theta functions by employing (3.4) and (3.5), we obtain

$$\frac{a^4}{b^4} + \frac{b^4}{a^4} + 7 = \frac{8}{a^3b^3} + a^3b^3 \tag{3.11}$$

On factorizing (3.9), we obtain a known modular equation (3.11) and the multiplier $a^{-3}b^{-11}(a^3b^3+8)$ and this multiplier is not unique. If we change the dividing term in the Somos's identity, we obtain a new multiplier. In [19], authors are using this technique to obtain a multiplier. Since the proofs of Somos's identities are monotonous, we proved only one of his identity and the remaining identities can be proved by the same technique. Also, the similar technique can be applied to get the multipliers in [14, 15] with modular equations of degree 3 and 5.

Conclusion

From the above observation, we conclude that all the Somos's 6200 η-function identities of various levels are the alternative forms of Ramanujan's modular equations and one can prove these identities from the known modular equations of various degrees given by Ramanujan.

References

[1] Adiga, C., Naika, M.S.M., Shivashankara, K.: On some P-Q eta-function identities of Ramanujan, Indian Journal of Mathematics, **44(3)**, 253-267 (2002)

[2] Adiga, C., Kim, T., Naika, M.S.M., Madhusudhan, H.S.: On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions, Indian Journal of Pure and Applied Mathematics, **35(9)**, 1047-1062 (2004)

[3] Adiga, C., Bulkhali, N.A.S., Ranganatha, D., Srivastava, H.M.: Some new modular relations for the Rogers-Ramanujan type functions of order eleven with applications to partitions, Journal of Number Theory, **158**, 281-297 (2016)

[4] Baruah, N.D.: Modular equations for Ramanujan's cubic continued fraction, Journal of Mathematical Analysis and Applications, **268(1)**, 244-255 (2002)

[5] Baruah, N.D.: On some of Ramanujan's Schläfli-type "mixed" modular equations, Journal of Number Theory, **100(2)**, 270-294 (2003)

[6] Berndt, B.C.: Ramanujan's Notebooks, Part III, Springer, New York (1991)

[7] Berndt, B.C.: Ramanujan's Notebooks, Part IV, Springer, New York (1996)

[8] Cao, Z.: On Somos's dissection identities, Journal of Mathematical Analysis and Applications, **365**, 659-667 (2010)

[9] Naika, M.S.M.: A note on cubic modular equations of degree two, Tamsui Oxford Journal of Mathematical Sciences, **22(1)**, 1-8 (2006)

[10] Ramanujan, S.: *Notebooks (2 Volumes)*, Tata Institute of Fundamental Research, Bombay 1957.

[11] Ramanujan, S.: The Lost Notebook and other unpublished papers, Narosa, New Delhi (1988)

[12] M. Somos, Personal communication.

[13] Srivatsa Kumar, B.R., Veeresha, R. G.: Partition identities arising from Somos's theta-function identities, *Annali Dell 'Universita' Di Ferrara*, vol. 63, pp. 303-313, 2017.

[14] Srivatsa Kumar, B.R., Anu Radha, D.: Somos's theta-function identities of level 10, Turkish Journal of Mathematics, **42**, 763-773 (2018)

[15] Srivatsa Kumar, B.R., Rajanna, K.R., Narendra, R.: Theta-function identities of level 8 and its application to partition, Afrika Matematika, **20**, 257-267 (2019)

[16] Srivatsa Kumar, B.R., Rajanna, K.R., Narendra,R.: Theta function identities of level 6 and their application to partitions, Filomat, **33(11)**, 3323-3335 (2019)
[17] Vasuki, K.R., Sreeramamurthy, T.G.: A note on P-Q modular equations, Tamsui Oxford Journal Mathematical Science, **21(2)**, 109-120 (2005)
[18] Vasuki, K.R.: On some of Ramanujan's P-Q modular equations, Journal of Indian Mathematical Society, **73(3-4)**, 131-143 (2006)
[19] Vasuki, K.R., Veeresha, R.G.: On Somos's theta-function identities of level 14, Ramanujan Journal, **42**, 131-144 (2017)
[20] Yuttanan, B.: New modular equations in the spirit of Ramanujan, Ramanujan Journal, **29**, 257-272 (2012)

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