Effect of time-delay on a plankton-fish interaction model with food limited growth rate and harvesting

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Abstract

In this paper, we investigate the effect of time-delay on a harvested fish-plankton model with a food-limited growth rate of plankton. The corresponding ODE version of the proposed model has already been studied in detail by Gupta et al. [7] where, the authors established the conditions for the stability and bifurcation of various steady states. It is observed that the inclusion of time-delay causes instability through Hopf bifurcation to the co-existing steady states after a certain threshold for delayed parameter. It is also observed that an increase in delayed parameter beyond the Hopf bifurcation threshold results in chaotic behaviour of the proposed system. The positivity and boundedness of solutions for the delayed model are derived with the help of differential inequalities. In addition, the conditions for Hopf bifurcation and the stability of periodic solution through it are also presented. Numerical results are also provided to support our analytic findings.

Key words : Time-delay; Stability; Hopf bifurcation; Chaos.

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1   INTRODUCTION

Time-delay plays an important role in the dynamics of prey-predator interaction, which has been perceived to contribute critically to the stable or unstable outcomes of prey populations due to predation. The original motivation for studying delayed models mainly comes from their applications [9]. Time-delay is also an omnipresent phenomenon in ecological systems and has a prime role in affecting population dynamics. In most natural systems, the population of one species does not respond immediately to the changes in the environment or the interaction with the other species. A large number of ecological models involving one or more time-delays have been evolved and examined by several researchers [2, 4, 9, 13, 12, 14]. In particular, Chen et al. [2] have derived the sufficient conditions for the existence of periodic solutions of a food-limited population model with toxicants and the presence of state-dependent delays. It is believed that the time delays have a destabilizing effect on the system of population dynamics. These are responsible for population oscillations within a deterministic environment. Smith [11] has also observed that the species needs more food for growth and maintenance when they are growing, and when they have reached their saturation, they require less food for maintenance only.

Gopalsamy et al. [6] studied a population model for a single species with a food-limited growth

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rate and time lag, which they extended to understand the effect of environmental periodicity. An explicit method to obtain the stability of periodic solutions through Hopf-bifurcation in a delayed food limited population model for two species is given by Wan and Wei [16]. Davidson et al. [3] investigated the existence, uniqueness, and asymptotic stability of the positive equilibrium states in which the population is supposed to diffuse and the average growth rate is a function of a specific delayed argument. Li and He [10] have proposed a delayed food-limited model to demonstrate the existence of Hopf-bifurcation and to understand the behaviour of bifurcating periodic solutions in the presence of time-delay and feedback control. The existence and uniqueness of the solution of a similar model have been analyzed by Dou and Li [4]. Tang and Chen [17] have derived the conditions for global stability of the positive steady-state for a single species food-limited population model with time-delay and impulsive effects. Bairagi and Jana [1] explored a delay-induced predator-prey model in the presence of habitat complexity and introduced gestation delay in the predator's response function. Wang and Li [15] introduced the existence of monotone travelling fronts in a diffusive food-limited model with nonlocal delay with the help of different kernel functions. So and Yu [12] considered a delayed food-limited population model and derived the conditions for the uniform and asymptotic stability of the positive equilibrium state.

Motivated by the preceding work, we present a model with the goal of investigating the role of time-delay on a predator-prey model with a food-limited growth rate. The remaining part of the manuscript is arranged in the following order. In section [**2**], we present a brief description of a plankton-fish interaction model without and with delay in the growth term of plankton population. The feasibility of solutions of the delayed system is also guaranteed in this section. In section [**3**], we analyze the sufficient conditions that ensures the existence of periodic solution through Hopf bifurcation. Taking time-delay as a bifurcation parameter, we explore the direction of Hopf bifurcation which provides the stability of bifurcated periodic solution in section [**4**] with the help of center manifold theorem. In in section [**5**] numerical simulations are provided in support of theoretical findings which is followed by a brief conclusion.

2 Mathematical model

In this section, we briefly present the model for plankton-fish interaction with nonlinear harvesting [7] in absence of time-delay and introduce the delayed version of the model with an ecological justification.

2.1 Non-delayed model

The following model for interaction of plankton and fish with nonlinear harvesting, where plankton follows a food limited growth rate with linear predation rate is proposed by Gupta et al. [7]

$$
\begin{cases}
\frac{dN(t)}{dt} = \frac{rN(t)(k-N(t))}{k+aN(t)} - mN(t)P(t), \\
\frac{dP(t)}{dt} = nN(t)P(t) - dP(t) - \frac{hP(t)}{c+P(t)}
\end{cases}
\tag{2.1}
$$

with the initial condition

$$
N(0) > 0, \quad P(0) > 0.
\tag{2.2}
$$

Here, $N(t)$ denotes the population density plankton population and $P(t)$ denotes fish population at time t. The parameters m, n and d represent the predation rate, growth rate of fish due to predation and natural death rate of fish respectively. The remaining parameters have their usual ecological representations as above. The parameters r, k, m, n, a, d, h and c are all positive due to the ecological and economic restrictions. This model is studied in detail for stability and bifurcation

of various equilibrium states. To decode the system from various aspects of ecology, Gupta et al. [7] performed a detailed analysis for stability and bifurcation of various steady-states. It is shown that the system (2.1) exhibits saddle-node bifurcation, Hopf bifurcation, and Bogdanov-Takens bifurcation around interior steady states. In particular, the system (2.1) has two interior steady-states, one of which is always unstable and the other one is stable under certain parametric conditions. The stability results of various steady states can be summerized as in the following table:

Table 1: Stability of the equilibrium states in the non-delayed model

Equilibrium states	Stability
$L_0(0,0)$	Always unstable
$L_1(k,0)$	LAS if $h > c(nk-d)$
$L_{1*}(N_{1*}, P_{1*})$	Always unstable
$L_{2*}(N_{2*}, P_{2*})$	LAS if $hP_{2*}(k+aN_{2*})^2 > rkN_{2*}(1+a)(c+P_{2*})^2$

Here, P_{1*} and P_{2*} are the positive roots of the quadratic equation $\tau_0 P^2 + \tau_1 P + \tau_2 = 0$ and the first component of interior equilibrium states are

$$N_{1*} = \frac{dc + dP_{1*} + h}{n(c + P_{1*})} \quad \text{and} \quad N_{2*} = \frac{dc + dP_{2*} + h}{n(c + P_{2*})}.$$

Where, $\tau_0 = mkn + mad$, $\tau_1 = -rkn + rd + mknc + madc + mah$, $\tau_2 = -rknc + rdc + rh$.

2.2 Delayed model

In the literature, time delays into mathematical models for population dynamics have been incorporated due to maturation time, capturing time, selective harvesting and many other reasons. We assume that a population requires more food for growth and maintenance than a saturated one which is for maintenance only. An additional assumption we consider here is that the average growth rate of plankton is a function of an specified delayed argument $t - \tau$. This motivates us to study the effect of time-delay $\tau(> 0)$ in food-limited growth term of the system (2.1) which takes the form

$$\begin{cases} \frac{dN(t)}{dt} = \frac{rN(t)(k-N(t-\tau))}{k+aN(t-\tau)} - mN(t)P(t), \\ \frac{dP(t)}{dt} = nN(t)P(t) - dP(t) - \frac{hP(t)}{c+P(t)} \end{cases} \tag{2.3}$$

with initial conditions

$$N_0(\phi) = \theta_1(\phi), \quad P_0(\phi) = \theta_2(\phi), \ \phi \in [-\tau, 0]. \tag{2.4}$$

Since, we deal with population in system (2.3), therefore all initial condition will be positive i.e.

$$\theta_i(\phi) \geq 0, \quad \theta_i(0) > 0, \quad i = 1, 2 \tag{2.5}$$

where, $(\theta_1(\phi), \theta_2(\phi)) \in C([-\tau, 0], \mathbb{R}_+^2)$ and $C([-\tau, 0], \mathbb{R}_+^2)$ be the Banach space of continuous mapping from $[-\tau, 0]$ to $\mathbb{R}_+^2 = \{(N(t), P(t)) \in \mathbb{R}^2 \mid N(t) \geq 0, P(t) \geq 0\}$.

2.3 Well-posedness of the delayed system

In this section, we will show that the solutions of system (2.3) defined on $[-\tau, A)$ where, $A \in (0, \infty)$ are positive and ultimately bounded.

Lemma 1. *The positive quadrant $Int(\mathbb{R}_+^2)$ is invariant for system (2.3) w.r.t. initial conditions (2.4) and (2.5).*

Proof. To show that for all $t \in (0, A)$, $N(t) > 0$ and $P(t) > 0$. On the contrary, we assume that it is not true i.e. there exists a $0 < T < A$ such that for all $t \in [0, T)$, $P(t) > 0$ and $N(t) > 0$ and either $N(T) = 0$ or $P(T) = 0$. From system (2.3), we have

$$N(t) = N(0) \exp \left(\int_0^t \left(\frac{r(k - N(s - \tau))}{k + aN(s - \tau)} - mP(s) \right) ds \right)$$

or

$$N(t) = N(0) \exp \left(\int_{-\tau}^{t-\tau} \frac{r(k - N(u))}{k + aN(u)} du - \int_0^t mP(s) ds \right)$$

and

$$P(t) = P(0) \exp \left(\int_0^t \left(nN(s) - d - \frac{h}{c + P(s)} \right) ds \right),$$

for all $t \in [-\tau, T)$ and $s - \tau = u$. As $N(t)$ and $P(t)$ are defined and continuous on $(-\tau, T)$ there exists an $M > 0$ such that

$$N(t) = N(0) \exp \left(\int_{-\tau}^{t-\tau} \frac{r(k - N(u))}{k + aN(u)} du - \int_0^t mP(s) ds \right) \geq N(0) \exp(-MT)$$

and

$$P(t) = P(0) \exp \left(\int_0^t \left(nN(s) - d - \frac{h}{c + P(s)} \right) ds \right) \geq P(0) \exp(-MT),$$

for all $t \in [-\tau, T)$. Taking the limit as $t \to T$ and using initial conditions (2.2), we get

$$N(T) \geq N(0) \exp(-MT) > 0 \quad \text{and} \quad P(T) \geq P(0) \exp(-MT) > 0$$

which contradicts our assumption that either $N(T) = 0$ or $P(T) = 0$. Therefore, $N(t) > 0$ and $P(t) > 0$ for all $t \in (0, A)$. $\qquad \square$

Lemma 2. *The solutions of the system (2.3) w.r.t. initial conditions (2.4) and (2.5) are bounded.*

Proof. In the first equation of system (2.3), we make use of the above result to get

$$\frac{dN(t)}{dt} \leq rN(t) \left(1 - \frac{N(t - \tau)}{k} \right).$$

From the above equation it is clear that $\frac{dN(t)}{dt} \leq rN(t)$. Integrating the above inequality from $t - \tau$ to t, for $t > \tau$, we have $N(t) \leq N(t - \tau)e^{r\tau}$ or $N(t - \tau) \geq N(t)e^{-r\tau}$. So, we obtain

$$\frac{dN(t)}{dt} \leq rN(t) \left(1 - \frac{e^{-r\tau}}{k} N(t) \right).$$

Hence, $\limsup_{r \to \infty} N(t) = ke^{r\tau}$. Similarly, from the second equation of system (2.3), for sufficiently large t, we have, $\frac{dP(t)}{dt} \leq nN(t)P(t) \leq nke^{r\tau}P(t)$. This gives $P(t) \leq P(0)e^{nke^{r\tau}}$, which proves the lemma. $\qquad \square$

3 Local stability and Hopf bifurcation

In this section, we discuss the change in stability behavior of the interior equilibrium state $L_{2*}(N_{2*}, P_{2*})$ and existence of Hopf bifurcation about it whenever the time delay τ crosses a threshold value τ_0

of Hopf bifurcation. We take the transformations $x(t) = N(t) - N_{2*}$ and $y(t) = P(t) - P_{2*}$. Using Taylor's series expansion the system (2.3) becomes

$$
\begin{cases}
\frac{dx}{dt} = a_{11}x(t) + a_{12}y(t) + b_{11}x(t-\tau) + \sum_{i+j+l\geq 2} \frac{1}{i!j!l!}\phi^{(1)}_{ijl}x^i(t)x^j(t-\tau)y^l(t), \\[2mm]
\frac{dy}{dt} = a_{21}x(t) + a_{22}y(t) + \sum_{i+j\geq 2} \frac{1}{i!j!}\phi^{(2)}_{ij}x^i(t)y^j(t),
\end{cases}
\tag{3.1}
$$

where, $a_{11} = 0$, $a_{12} = -mN_{2*} < 0$, $a_{21} = nP_{2*} > 0$, $a_{22} = \frac{hP_{2*}}{(c+P_{2*})^2} > 0$, $b_{11} = -\frac{rkN_{2*}(1+a)}{(k+aN_{2*})^2} < 0$,

$f^{(1)}_{ijl} = \frac{\partial^{i+j+l}f^{(1)}}{\partial x^i \partial x(t-\tau)^j \partial y^l}(N_{2*}, P_{2*})$ and $f^{(2)}_{ij} = \frac{\partial^{i+j}f^{(2)}}{\partial x^i \partial y^l}(N_{2*}, P_{2*})$.

The Linearized system corresponding to (3.1) is given by:

$$
\begin{cases}
\frac{dx}{dt} = a_{11}x(t) + a_{12}y(t) + b_{11}x(t-\tau), \\[2mm]
\frac{dy}{dt} = a_{21}x(t) + a_{22}y(t).
\end{cases}
\tag{3.2}
$$

The characteristic equation corresponding to the system (3.2) is

$$
\lambda^2 + \Lambda_0\lambda + \Lambda_1 + (\Lambda_2 + \Lambda_3\lambda)e^{-\lambda\tau} = 0,
\tag{3.3}
$$

where, $\Lambda_0 = -a_{22}$, $\Lambda_1 = -a_{12}a_{21}$, $\Lambda_2 = a_{22}b_{11}$, $\Lambda_3 = -b_{11}$.

The stability of the interior equilibrium state $L_{2*}(N_{2*}, P_{2*})$ is given as follows:
Case I: When $(\tau = 0)$, then equation (3.3) can be written as

$$
\lambda^2 + (\Lambda_0 + \Lambda_3)\lambda + (\Lambda_1 + \Lambda_2) = 0.
\tag{3.4}
$$

By the Routh-Hurwitz criterion, all roots of the equation (3.4) have negative real parts if the conditions $\Lambda_0 + \Lambda_3 > 0$ and $\Lambda_1 + \Lambda_2 > 0$ are satisfied together. Thus, in absence of delay, the interior equilibrium L_{2*} of system (2.3) is locally asymptotically stable if $\Lambda_0 + \Lambda_3 > 0$ and $\Lambda_1 + \Lambda_2 > 0$.

Case II: When $(\tau \neq 0)$, we investigate the existence of purely imaginary roots and the instability caused by time-delay. Now, by putting $\lambda = \iota\omega$, where, $\omega > 0$ in the equation (3.3), we have

$$
(\iota\omega)^2 + \Lambda_0(\iota\omega) + \Lambda_1 + (\Lambda_2 + \Lambda_3\iota\omega)e^{-\iota\omega\tau} = 0
$$
$$
-\omega^2 + \iota\Lambda_0\omega + \Lambda_1 + (\Lambda_2 + \iota\Lambda_3\omega)(\cos\omega\tau - \iota\sin\omega\tau) = 0.
$$

After separating real and imaginary parts, we get

$$
-\omega^2 + \Lambda_1 + \Lambda_2\cos\omega\tau + \Lambda_3\omega\sin\omega\tau = 0,
$$
$$
\Lambda_0\omega - \Lambda_2\sin\omega\tau + \Lambda_3\omega\cos\omega\tau = 0.
$$

It follows that ω satisfies the following quartic equation,

$$
\omega^4 + \left(-2\Lambda_1 + \Lambda_0^2 - \Lambda_3^2\right)\omega^2 + \Lambda_1^2 - \Lambda_2^2 = 0.
\tag{3.5}
$$

Thus, we obtain,

$$
\omega^2 = \frac{-2\Lambda_1 + \Lambda_0^2 - \Lambda_3^2 \pm \sqrt{\left(-2\Lambda_1 + \Lambda_0^2 - \Lambda_3^2\right)^2 - 4(\Lambda_1^2 - \Lambda_2^2)}}{2}.
$$

We take two basic assumptions $-2\Lambda_1 + \Lambda_0{}^2 - \Lambda_3{}^2 > 0$ and $\Lambda_1^2 - \Lambda_2^2 < 0$ for ω to be real positive. The expression for τ can be given as follows:

$$\tau_n = \frac{1}{\omega} \arctan\left(\frac{\omega\left(\omega^2 \Lambda_3 - \Lambda_1 \Lambda_3 + \Lambda_0 \Lambda_2\right)}{\omega^2\left(\Lambda_2 - \Lambda_0 \Lambda_3\right) - \Lambda_1 \Lambda_2} \right) + \frac{n\pi}{\omega}$$

where, $n = 0, 1, 2, 3 \ldots$

3.1 Transversality condition for Hopf bifurcation

We need to verify the transversality condition $\frac{d}{d\tau}\left[Re\lambda(\tau)\right] > 0$, to ensure the occurrence of Hopf bifurcation. As λ depends upon τ so λ can be taken as $\lambda(\tau)$. Therefore, differentiating equation (3.3) with respect to τ and manipulating the resulting equation with the help of equation (3.3), we get,

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = \frac{2\lambda + \Lambda_0}{e^{-\lambda\tau}\lambda\left(\Lambda_2 + \Lambda_3\lambda\right)} + \frac{\Lambda_3}{\lambda\left(\Lambda_2 + \Lambda_3\lambda\right)} - \frac{\tau}{\lambda}.$$

Putting $\lambda = \iota\omega$, the real part is given by

$$Re\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = \frac{2\omega^2 - 2\Lambda_1 + \Lambda_0{}^2 - \Lambda_3{}^2}{\Lambda_2{}^2 + \omega^2 \Lambda_3{}^2}. \tag{3.6}$$

From equation (3.5), a unique positive root of ω exists, if

$$\begin{aligned} -2\Lambda_1 + \Lambda_0{}^2 - \Lambda_3{}^2 &> 0 \\ \Lambda_1{}^2 - \Lambda_2{}^2 &< 0. \end{aligned} \tag{3.7}$$

Thus, we conclude that transversality condition is satisfied and hence Hopf bifurcation occurs when τ passes through the critical value τ_0. We summarized this result in the following theorem.

Theorem 1. *Suppose that the interior equilibrium state L_{2*} satisfies the conditions given in (3.7). Then (i) The equilibrium state L_{2*} is locally asymptotically stable if $\tau \in [0, \tau_0)$ and unstable if $\tau > \tau_0$.*
(ii) The system (3.2) undergoes Hopf-bifurcation with respect to the bifurcation parameter $\tau = \tau_0$.

4 Direction of Hopf bifurcation

In the previous section, we have observed that the system (3.2) undergoes a Hopf-bifurcation as τ approaches towards the critical value τ_0. Now, we are interested to find the direction of Hopf bifurcation with the help of normal form theory and center manifold theorem introduced by Hassard et al. [8]. For this purpose, we consider $x(t) = N(t) - N_{2*}$, $y(t) = P(t) - P_{2*}$ and $\tau = \tau_0 + \epsilon$, where, $\epsilon \in \mathbb{R}$. Therefore, $\epsilon = 0$ is the bifurcation point of the system. Normalizing the delay τ by re-scaling $t \to \frac{t}{\tau}$ and considering,

$$\dot{X}(t) = L_\epsilon(X_t) + f(X_t, \epsilon), \tag{4.1}$$

where, $X(t) = (x(t), y(t)) \in \mathbb{R}^2$, $L_\epsilon : C \to \mathbb{R}^2$ and $f : \mathbb{R} \times C \to \mathbb{R}^2$, for $\phi = (\phi_1, \phi_2)^T \in C([-1, 0], \mathbb{R}_+^2)$, i.e.

$$L_\epsilon(\phi) = (\tau_0 + \epsilon)\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}\right],$$

where, $a_{11} = 0$, $a_{12} = -mN$ $a_{21} = nP$, $a_{22} = \frac{hP}{(c+P)^2}$, $b_{11} = -\frac{rkN(1+a)}{(k+aN)^2}$.

$$f(\phi, \epsilon) = (\tau_0 + \epsilon) \begin{pmatrix} f^{(1)}_{110}\phi_1(0)\phi_1(-1) + f^{(1)}_{101}\phi_1(0)\phi_2(0) + \frac{1}{2}f^{(1)}_{020}\phi_1^2(-1) \\ f^{(2)}_{11}\phi_1(0)\phi_2(0) + \frac{1}{2}f^{(2)}_{02}\phi_2^2(0) \end{pmatrix},$$

where, $f^{(1)} = \frac{rN(t)(k-N_\tau(t))}{k+aN_\tau(t)} - mN(t)P(t)$, $f^{(2)} = nN(t)P(t) - dP(t) - \frac{hP(t)}{c+P(t)}$, $f^{(1)}_{ijl} = \frac{\partial^{i+j+l}f^{(1)}}{\partial N^i \partial N_\tau^j \partial P^l}(N_0, P_0)$ and $f^{(2)}_{ij} = \frac{\partial^{i+j}f^{(2)}}{\partial N^i \partial P^l}(N_0, P_0)$ which gives $f^{(1)}_{110} = -\frac{rk(1+a)}{(k+aN)^2}$, $f^{(1)}_{101} = -m$, $f^{(1)}_{020} = -\frac{2arkN(1+a)}{(k+aN)^3}$, $f^{(2)}_{11} = n$, $f^{(2)}_{02} = \frac{2hc}{(c+P)^3}$.

Now, by Riesz representation theorem, there exist a 2×2 matrix $\eta(\theta, \epsilon)$ whose each components have bounded variation such that for all $\phi \in C([-1, 0], \mathbb{R}^2)$

$$L_\epsilon(\phi) = \int_{-1}^0 \phi(\theta)d\eta(\theta, \epsilon). \tag{4.2}$$

We choose,

$$\eta(\theta, \epsilon) = (\tau_0 + \epsilon)\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \delta(\theta) + \begin{pmatrix} b_{11} & 0 \\ 0 & 0 \end{pmatrix} \delta(\theta + 1) \right], \tag{4.3}$$

where, δ is a Dirac delta function for $\phi \in C([-1, 0], \mathbb{R}^2)$. Now, we define

$$A(\epsilon)\phi = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-1, 0) \\ \int_{-1}^0 \phi(\theta)d\eta(\theta, \epsilon), & \theta = 0 \end{cases} \tag{4.4}$$

and

$$B(\epsilon)\phi = \begin{cases} 0, & \theta \in [-1, 0) \\ f(\phi, \epsilon), & \theta = 0. \end{cases} \tag{4.5}$$

Thus, system (4.1) can be re-written as

$$\frac{dx_t}{dt} = A(\epsilon)x_t + B(\epsilon)x_t, \tag{4.6}$$

with $x_t(\theta) = x_t(t + \theta)$ for $\theta \in [-1, 0]$. Now, for $\varphi \in C^1([0, 1], \mathbb{R}^2)$, we define

$$A^*(\epsilon)(\varphi) = \begin{cases} -\frac{d\varphi}{ds}, & s \in (0, 1] \\ \int_{-1}^0 \varphi(-t)d\eta(t, 0), & s = 0. \end{cases} \tag{4.7}$$

For $\phi \in [-1, 0]$ and $\varphi \in [0, 1]$, we consider a bilinear product:

$$< \varphi(s), \phi(\theta) > = \bar{\varphi}^T(0)\phi(0) - \int_{-1}^0 \int_0^\theta \bar{\varphi}^T(\zeta - \theta)d\eta(\theta)\phi(\zeta)d\zeta, \tag{4.8}$$

where, $\eta = \eta(\theta, 0)$. Here, $A = A(0)$ and $A^* = A^*(0)$ are adjoint operators such that $\pm \iota\tau_0\omega$ are the eigenvalues of $A(0)$ and $\mp \iota\tau_0\omega$ are eigenvalues $A^*(0)$.

Now, we assume that $q(\theta) = (q_1, q_2)^T e^{\iota\theta\tau_0\omega}$ is the eigenvector of A with respect to the eigenvalue $\iota\tau_0\omega$. So, from the definition of $A(0)$, we have $(A(0) - \iota\omega\tau_0 I)q(\theta) = 0$ and

$$\tau_0 \begin{pmatrix} a_{11} - \iota\omega + b_{11}e^{-\iota\omega\tau_0} & a_{12} \\ a_{21} & a_{22} - \iota\omega \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

After solving the above matrix equation, we get

$$q(\theta) = \left(1, \frac{\iota\omega - b_{11}e^{-\iota\omega\tau_0} - a_{11}}{a_{12}}\right)^T e^{\iota\omega\tau_0\theta}. \tag{4.9}$$

Similarly, if $q^*(s) = D(q_1^*, q_2^*)e^{\iota\omega\tau_0 s}$ be the eigenvector of A^* associated with the eigenvalue $-\iota\omega\tau_0$. Then, $A^*q^*(s) = -\iota\tau_0\omega q^*(s)$ and from the definition of A^*, we get

$$\tau_0 \begin{pmatrix} a_{11} + \iota\omega + b_{11}e^{-\iota\omega\tau_0} & a_{21} \\ a_{12} & a_{22} + \iota\omega \end{pmatrix} \begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, from the above matrix equation, we obtain

$$q^*(s) = D\left(1, \frac{-(b_{11}e^{-\iota\omega\tau_0} + a_{11} + \iota\omega)}{a_{21}}\right)^T e^{\iota\omega\tau_0 s} \tag{4.10}$$

Now, for showing $< q^*(s), q(\theta) >= 1$, we have to calculate D. We know that

$$\begin{aligned}
< q^*(s), q(\theta) > &= \bar{q}^{*T}(0)q(0) - \int_{-1}^0 \int_0^\theta \bar{q}^{*T}(\zeta - \theta)d\eta(\theta)q(\zeta)d\zeta \\
&= \bar{D}\{(\bar{q}_1^*, \bar{q}_2^*)(q_1, q_2)^T - \int_{-1}^0 \int_0^\theta (\bar{q}_1^*, \bar{q}_2^*)e^{-\iota(\zeta-\theta)\omega\tau_0}d\eta(\theta)(q_1, q_2)^T e^{\iota\omega\tau_0\zeta}d\zeta\} \\
&= \bar{D}\{1 + \bar{q}_2^* q_2 - \int_{-1}^0 \int_0^\theta (\bar{q}_1^*, \bar{q}_2^*)e^{\iota\theta\omega\tau_0}d\eta(\theta)(q_1, q_2)^T d\zeta\} \\
&= \bar{D}(1 + q_2\bar{q}_2^* + \tau_0 b_{11}e^{-\iota\omega\tau_0}).
\end{aligned}$$

Thus, if we take,

$$\bar{D} = (1 + q_2\bar{q}_2^* + \tau_0 b_{11}e^{-\iota\omega\tau_0})^{-1} \text{ i.e. } D = \frac{1}{(1 + \bar{q}_2 q_2^* + \tau_0 b_{11}e^{\iota\omega\tau_0})},$$

we obtain $< q^*(s), q(\theta) >= 1$. Since, we know that $< \varphi, A\phi >=< A^*\varphi, \phi >$, so $-\iota\omega\tau_0 < q^*, \bar{q} >=< q^*, A\bar{q} >=< A^*q^*, \bar{q} >=< -\iota\omega\tau_0 q^*, \bar{q} >= \iota\omega\tau_0 < q^*, q > .$ This shows that $< q^*(s), \bar{q}(\theta) >= 0$.

Now, we compute the coordinates to describe the center manifold C_0 at $\epsilon = 0$, for which, we define

$$z(t) =< q^*, x_t > \text{ and } V(t, \theta) = x_t(\theta) - 2\Re\{z(t)q(\theta)\}. \tag{4.11}$$

On the center manifold C_0,

$$V(t, \theta) = V(z(t), \bar{z}(t), \theta),$$

which can be written as

$$V(z(t), \bar{z}(t), \theta) = V_{20}(\theta)\frac{z^2}{2} + V_{11}(\theta)z\bar{z} + V_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{4.12}$$

Here, z and $\bar{z}$ are the local coordinates for the center manifold C_0 in the direction of q and $\bar{q}$, respectively. For real value of x_t, V becomes real. Hence, only the real solutions are considered. Now, for solutions $x_t \in C_0$

$$\dot{z}(t) =< q^*, \dot{x}_t >=< q^*, A(0)x_t + B(0)x_t >=< q^*, A(0)x_t > +\bar{q}^*(0)f(0, x_t) = \iota\omega\tau_0 z(t) + \bar{q}^*(0)^T f_0(z, \bar{z}).$$

Thus

$$\dot{z}(t) = \iota\omega\tau_0 z(t) + \xi(z, \bar{z}), \tag{4.13}$$

where,

$$\xi(z, \bar{z}) = \bar{q}^*(0)^T f_0(z, \bar{z}) = \xi_{20}\frac{z^2}{2} + \xi_{11}z\bar{z} + \xi_{02}\frac{\bar{z}^2}{2} + \xi_{21}\frac{z^2\bar{z}}{2} + \dots \tag{4.14}$$

Now,

$$x_t(\theta) = V(z, \bar{z}, \theta) + 2\Re\{z(t)q(\theta)\}$$

with

$$\begin{pmatrix} x_{1t}(\theta) \\ x_{2t}(\theta) \end{pmatrix} = \begin{pmatrix} V^{(1)}(\theta) \\ V^{(2)}(\theta) \end{pmatrix} + z \begin{pmatrix} 1 \\ q_2 \end{pmatrix} e^{\iota\omega\tau_0\theta} + \bar{z} \begin{pmatrix} 1 \\ \bar{q}_2 \end{pmatrix} e^{-\iota\omega\tau_0\theta}$$

which gives

$$x_{1t}(\theta) = V^{(1)}(\theta) + ze^{\iota\omega\tau_0\theta} + \bar{z}e^{-\iota\omega\tau_0\theta} \text{ and } x_{2t}(\theta) = V^{(2)}(\theta) + zq_2e^{\iota\omega\tau_0\theta} + \bar{z}\bar{q}_2e^{-\iota\omega\tau_0\theta}.$$

We also have

$$\phi_1(0) = V_{20}^{(1)}(0)\frac{z^2}{2} + V_{11}^{(1)}(0)z\bar{z} + V_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + z + \bar{z} + \dots$$
$$\phi_2(0) = V_{20}^{(2)}(0)\frac{z^2}{2} + V_{11}^{(2)}(0)z\bar{z} + V_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + zq_2 + \bar{z}\bar{q}_2 + \dots$$
$$\phi_1(-1) = V_{20}^{(1)}(-1)\frac{z^2}{2} + V_{11}^{(1)}(-1)z\bar{z} + V_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + ze^{-\iota\omega\tau_0} + \bar{z}e^{\iota\omega\tau_0} + \dots$$
$$\phi_2(-1) = V_{20}^{(2)}(-1)\frac{z^2}{2} + V_{11}^{(2)}(-1)z\bar{z} + V_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + zq_2e^{-\iota\omega\tau_0} + \bar{z}\bar{q}_2e^{\iota\omega\tau_0} + \dots$$

which gives

$$f_0(z, \bar{z}) = \tau_0 \begin{pmatrix} \Gamma_{11}z^2 + \Gamma_{12}z\bar{z} + \Gamma_{13}\bar{z}^2 + \Gamma_{14}z^2\bar{z} \\ \Gamma_{21}z^2 + \Gamma_{22}z\bar{z} + \Gamma_{23}\bar{z}^2 + \Gamma_{24}z^2\bar{z} \end{pmatrix} + \dots$$

$$\xi(z, \bar{z}) = D(1, \bar{q}_4)\tau_0 \begin{pmatrix} \Gamma_{11}z^2 + \Gamma_{12}z\bar{z} + \Gamma_{13}\bar{z}^2 + \Gamma_{14}z^2\bar{z} \\ \Gamma_{21}z^2 + \Gamma_{22}z\bar{z} + \Gamma_{23}\bar{z}^2 + \Gamma_{24}z^2\bar{z} \end{pmatrix} + \dots$$

Now, comparing the coefficients of z^2, $z\bar{z}$, $\bar{z}^2$ and $z^2\bar{z}$, from both the sides, we get

$$\xi_{20} = 2\bar{D}\tau_0(\Gamma_{11}+\bar{q}_4\Gamma_{21}), \ \xi_{02} = 2\bar{D}\tau_0(\Gamma_{13}+\bar{q}_4\Gamma_{23}), \ \xi_{11} = \bar{D}\tau_0(\Gamma_{12}+\bar{q}_4\Gamma_{22}) \text{ and } \xi_{21} = 2\bar{D}\tau_0(\Gamma_{14}+\bar{q}_4\Gamma_{24}),$$

where, $\Gamma_{11} = f_{110}^{(1)}e^{-i\omega\tau_0} + \frac{1}{2}f_{20}^{(2)}e^{-2i\omega\tau_0} + f_{101}^{(1)}q_2,$

$\Gamma_{12} = f_{20}^{(2)} + f_{110}^{(1)}e^{-\iota\omega\tau_0} + f_{101}^{(1)}\bar{q}_2 + f_{110}^{(1)}e^{\iota\omega\tau_0} + f_{101}^{(1)}q_2,$

$\Gamma_{13} = \frac{1}{2}f_{20}e^{2\iota\omega\tau_0} + f_{110}e^{\iota\omega\tau_0} + f_{101}\bar{q}_2,$

$\Gamma_{14} = \frac{1}{2}f_{20}^{(2)}V_{20}^{(1)}(-1)e^{\iota\omega\tau_0} + f_{101}^{(1)}V_{11}^{(1)}(0)q_2 + \frac{1}{2}f_{101}^{(1)}V_{20}^{(2)}(0) + \frac{1}{2}f_{101}V_{20}^{(1)}(0)\bar{q}_2 + \frac{1}{2}f_{110}^{(1)}V_{20}^{(1)}(-1) + f_{101}^{(1)}V_{11}^{(2)}(0) + f_{20}^{(2)}V_{11}^{(1)}(-1)e^{-\iota\omega\tau_0} + f_{110}^{(1)}V_{11}(0)e^{-\iota\omega\tau_0} + f_{110}V_{11}^{(1)}(-1) + \frac{1}{2}f_{110}^{(1)}V_{20}^{(1)}(0)e^{\iota\omega\tau_0},$

$\Gamma_{21} = f_{11}^{(1)}q_2 + \frac{1}{2}f_{02}^{(2)}q_2^2,$

$\Gamma_{22} = f_{02}^{(2)}q_2\bar{q}_2 + f_{11}^{(2)}\bar{q}_2 + f_{11}^{(2)}q_2,$

$\Gamma_{23} = f_{11}^{(2)}\bar{q}_2 + \frac{1}{2}f_{02}^{(2)}\bar{q}_2^2,$ and

$\Gamma_{24} = \frac{1}{2}f_{02}^{(2)}\bar{q}_2V_{20}^{(2)}(0) + f_{02}^{(2)}q_2V_{11}^{(2)}(0) + \frac{1}{2}f_{11}^{(2)}V_{20}^{(2)}(0) + f_{11}^{(2)}V_{11}^{(1)}(0) + f_{11}^{(2)}V_{11}^{(1)}(0)q_2 + \frac{1}{2}f_{11}^{(2)}V_{20}^{(1)}(0)\bar{q}_2.$

Since, ξ_{21} contains the terms of V_{11} and V_{20}, so, for finding the value of ξ_{21} for $\theta \in [-1,0]$, we have

$$\dot{V} = \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q}$$
$$= \begin{cases} A(0)V - 2Re(\bar{q}^*(0)f_0(z, \bar{z})q(\theta)), & \theta \in [-1, 0) \\ A(0)V - 2Re(\bar{q}^*(0)f_0(z, \bar{z})q(\theta)) + f_0(z, \bar{z}), & \theta = 0. \end{cases}$$

The above equation can be written as

$$\dot{V} = AV + H(z, \bar{z}, \theta), \tag{4.15}$$

where,

$$H(z(t), \bar{z}(t), \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{4.16}$$

We know that, $\dot{V} = V_z \dot{z} + V_{\bar{z}} \dot{\bar{z}}$ and using (4.12), we get

$$\dot{V} = (V_{20}(\theta)z + V_{11}\bar{z} + \dots)(\iota\omega\tau_0 z + \xi(z, \bar{z})) + (V_{11}(\theta)z + V_{02}\bar{z} + \dots)(-\iota\omega\tau_0\bar{z} + \bar{\xi}(z, \bar{z})). \tag{4.17}$$

Comparing the coefficients of z^2, $z\bar{z}$ and $\bar{z}^2$ from equations (4.17) and (4.16), we have

$$AV_{11}(\theta) = -H_{11}(\theta)$$
$$(A - 2\iota\omega\tau_0 I)V_{20}(\theta) = -H_{20}(\theta). \tag{4.18}$$

For $\theta \in [-1, 0)$, $H(z(t), \bar{z}(t), \theta) = -2Re\{\bar{q}^{*T} f_0(z, \bar{z})q(\theta)\} = -\xi(z, \bar{z})q(\theta) - \bar{\xi}(z, \bar{z})\bar{q}(\theta)$

$$= -\left(\xi_{20}\frac{z^2}{2} + \xi_{11}z\bar{z} + \xi_{02}\frac{\bar{z}^2}{2} + \xi_{21}\frac{z^2\bar{z}}{2} + \dots\right)q(\theta) - \left(\xi_{20}\frac{z^2}{2} + \xi_{11}z\bar{z} + \xi_{02}\frac{\bar{z}^2}{2} + \xi_{21}\frac{z^2\bar{z}}{2} + \dots\right)\bar{q}(\theta). \tag{4.19}$$

Again, comparing the equations (4.19) and (4.16), we get

$$H_{20}(\theta) = -\xi_{20}q(\theta) - \bar{\xi}_{02}\bar{q}(\theta) \text{ and } H_{11}(\theta) = -\xi_{11}q(\theta) - \bar{\xi}_{11}\bar{q}(\theta). \tag{4.20}$$

Now, from the definition of $A(\theta)$ and from the equations (4.20), for $\theta \in [-1, 0)$

$$\dot{V}_{20}(\theta) = A(0)V_{20}(\theta) = 2\iota\omega\theta\tau_0 V_{20}(\theta) - H_{20}(\theta)$$

$$\dot{V}_{20}(\theta) = 2\iota\omega\theta\tau_0 V_{20}(\theta) + \xi_{20}q(0)e^{\iota\omega\theta\tau_0} + \bar{\xi}_{02}\bar{q}(0)e^{-\iota\omega\theta\tau_0}.$$

Solving the differential equation for $V_{20}(\theta)$, we get

$$V_{20}(\theta) = \frac{\iota\xi_{20}q(0)e^{\iota\omega\theta\tau_0}}{\omega\tau_0} + \frac{\iota\bar{\xi}_{02}\bar{q}(0)e^{-\iota\omega\theta\tau_0}}{3\omega\tau_0} + E_1 e^{2\iota\omega\theta\tau_0}. \tag{4.21}$$

Similarly, by using the definition of $A(\theta)$ and equations (4.20), we solve for $V_{11}(\theta)$ to get

$$V_{11}(\theta) = \frac{-\iota\xi_{11}q(0)e^{\iota\omega\theta\tau_0}}{\omega\tau_0} + \frac{\iota\bar{\xi}_{11}\bar{q}(0)e^{-\iota\omega\theta\tau_0}}{\omega\tau_0} + E_2. \tag{4.22}$$

Here, E_1 and E_2 are two dimensional constant vectors and we can find these values from the definition of $A(0)$ and the equation (4.18). Also

$$\int_{-1}^{0} d\eta(\theta)V_{20}(\theta) = 2\iota\omega\tau_0 V_{20}(\theta) - H_{20}(0) \text{ and } \int_{-1}^{0} d\eta(\theta)V_{11}(\theta) = -H_{11}(0)$$

with $\eta(\theta, 0) = \eta(\theta)$ and taking $\theta = 0$, we get

$$H(z(t), \bar{z}(t), 0) = -2Re\{\bar{q}^{*T} f_0(z, \bar{z})q(0)\} = -\xi(z, \bar{z})q(0) - \bar{\xi}(z, \bar{z})\bar{q}(0) + f_0(z, \bar{z})$$

i.e.

$$H(z(t), \bar{z}(t), 0) = -\left(\xi_{20}\frac{z^2}{2} + \xi_{11}z\bar{z} + \xi_{02}\frac{\bar{z}^2}{2} + \dots\right)q(0) - \left(\bar{\xi}_{20}\frac{\bar{z}^2}{2} + \bar{\xi}_{11}z\bar{z} + \bar{\xi}_{02}\frac{z^2}{2} + \dots\right)\bar{q}(0) \tag{4.23}$$

$$+ f_0(z, \bar{z}).$$

Now, comparing the coefficients of z^2 and $z\bar{z}$ from both the sides, we get

$$H_{20}(0) = -\xi_{20}q(0) - \bar{\xi}_{02}\bar{q}(0) + 2\tau_0 \begin{pmatrix} \Gamma_{11} \\ \Gamma_{21} \end{pmatrix} \text{ and } H_{11}(0) = -\xi_{11}q(0) - \bar{\xi}_{11}\bar{q}(0) + \tau_0 \begin{pmatrix} \Gamma_{12} \\ \Gamma_{22} \end{pmatrix}$$

(4.24)

According, to the definition of $A(0)$ having eigenvector $q(0)$ with respect to $\iota\omega\tau$, we have

$$\left(\iota\omega\tau_0 - \int_{-1}^0 d\eta(\theta)e^{\iota\omega\theta\tau_0}\right)q(0) = 0 \text{ and } \left(-\iota\omega\tau_0 - \int_{-1}^0 d\eta(\theta)e^{-\iota\omega\theta\tau_0}\right)\bar{q}(0) = 0 \qquad (4.25)$$

By using the values of $V_{20}(0)$ and $V_{11}(0)$ together with the equation (4.20), we get the following matrix equation

$$\tau_0 \begin{pmatrix} 2\iota\omega - a_{11} - b_{11}e^{-2\iota\omega\tau_0} & -a_{12} \\ -a_{21} & 2\iota\omega - a_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \tau_0 \begin{pmatrix} \Gamma_{11} \\ \Gamma_{21} \end{pmatrix}. \qquad (4.26)$$

By using Cramer's rule, we get

$$e_1 = \frac{1}{D_1} \begin{vmatrix} \Gamma_{11} & -a_{12} \\ \Gamma_{21} & 2\iota\omega - a_{22} \end{vmatrix} \text{ and } e_2 = \frac{1}{D_1} \begin{vmatrix} 2\iota\omega - a_{11} - b_{11}e^{\iota\omega\tau_0} & \Gamma_{11} \\ -a_{21} & \Gamma_{21} \end{vmatrix},$$

where,

$$D_1 = \begin{vmatrix} 2\iota\omega - a_{11} - b_{11}e^{-\iota\omega\tau_0} & -a_{12} \\ -a_{21} & 2\iota\omega - a_{22} \end{vmatrix}.$$

Similarly, from,

$$\int_{-1}^0 d\eta(\theta)V_{11}(\theta) = -H_{11}(0),$$

we get

$$\tau_0 \begin{pmatrix} -a_{11} - b_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} \begin{pmatrix} e_3 \\ e_4 \end{pmatrix} = \tau_0 \begin{pmatrix} \Gamma_{12} \\ \Gamma_{22} \end{pmatrix}. \qquad (4.27)$$

After solving the above equations, we get

$$e_3 = \frac{1}{D_2} \begin{vmatrix} \Gamma_{12} & -a_{12} \\ \Gamma_{22} & -a_{22} \end{vmatrix} \text{ and } e_4 = \frac{1}{D_2} \begin{vmatrix} -a_{11} - b_{11} & \Gamma_{12} \\ -a_{21} & \Gamma_{22} \end{vmatrix},$$

where,

$$D_2 = \begin{vmatrix} -a_{11} - b_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix}.$$

From these calculations, we can find $V_{20}(\theta)$ and $V_{11}(\theta)$ to compute the value of ξ_{21}. Consequently, we can obtain the following quantities:

$$C_1(0) = \frac{\iota}{2\tau_0\omega}\left(\xi_{20}\xi_{11} - 2|\xi_{11}^2| - \frac{|\xi_{02}^2|}{3}\right) + \frac{\xi_{21}}{2},$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_0)\}},$$

$$\beta_2 = 2Re\{C_1(0)\},$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(\tau_0)\}}{\omega\tau_0}.$$

Here, the notations μ_2, β_2 and T_2 respectively determines the direction, stability and period of bifurcating periodic solutions. If, $\mu_2 > 0(\mu_2 < 0)$ then the Hopf bifurcation is supercritical (subcritical). Further, β_2, decides the stability of the bifurcating periodic solutions, so when $\beta_2 < 0$ $(\beta_2 > 0)$ the periodic solutions are stable (unstable). Periods of the bifurcating solutions increases (decreases) if $T_2 > 0$ ($T_2 < 0$) respectively. The following theorem gives the outline of the results.

Theorem 2. *The delayed system (2.3) undergoes a supercritical Hopf bifurcation as τ approaches towards τ_0 if $Re\{C_1(0)\} < 0$ and otherwise, if $Re\{C_1(0)\} > 0$ it possess subcritical Hopf bifurcation.*

5   Numerical Simulations

In this section, we apply numerical simulations to verify our analytical results obtained in the earlier sections. Since, Gupta et al. [7] have already discussed the corresponding ODE version of this model for two sets of parameters corresponding to the number of interior equilibrium states. Therefore, we consider these two sets of numerical values for which interior equilibrium are stable.

Example 1. *Gupta et al. [7] have observed that the interior equilibrium L_{2*} in the absence of delay (i.e. $\tau = 0$) is stable for the set of parameters $r = 0.25$, $k = 10$, $a = 0.2$, $m = 0.5$, $d = 0.9$, $h = 0.2$, $c = 0.2$, $n = 0.194$. The plots for the time-series and phase portrait for $\tau = 0$ around the interior equilibrium state $L_{2*} = (8.6256, 0.0586)$ are given in Figures 1(a), 1(b) and 1(c). So, here we check the impact of delayed parameter τ of the model (2.3) on the stability of the interior equilibrium. For this set of parameters, $\omega_0 = 0.2963$ (3.5) and the threshold value of time-delay τ is $\tau_0 = 3.9665$. The corresponding periodic solutions (time-series and phase portraits) of the system (2.3) through Hopf-bifurcation are given in Figures 2(a), 2(b), 2(c).*

To observe the dynamics of the system, we have constructed bifurcation diagrams in which we plotted the succssesive local maxima and minima of both the populations with respect to the bifurcation parameter τ. It can be seen from these bifurcation diagrams that the interior equilibrium L_{2*} is locally asymptotically stable for $\tau < \tau_0$ and it is unstable for $\tau > \tau_0$ which is clear from bifurcation diagrams 4(a) and 4(b). By using the expressions given in the last section we computed the quantities as $C_1(0) = -0.5816 - 0.1699I$, $\beta_2 = 0.0568$, $\mu_2 = -1.9692$ and $T_2 = 0.4693$, which indicates that the system exhibits stable supercritical Hopf-bifurcation. As we increase the value of τ beyond τ_0 the solutions become chaotic in nature. This is ensured in 3(a), 3(b), 3(c). The sensitivity of solutions are also verified to ensure the chaotic nature of the system in Figure 4(c) by taking two nearby initial conditions $(8.6256, 0.0586)$ (green color curve) and $(8.6356, 0.0686)$ (red color curve) with $\tau = 6.84$.

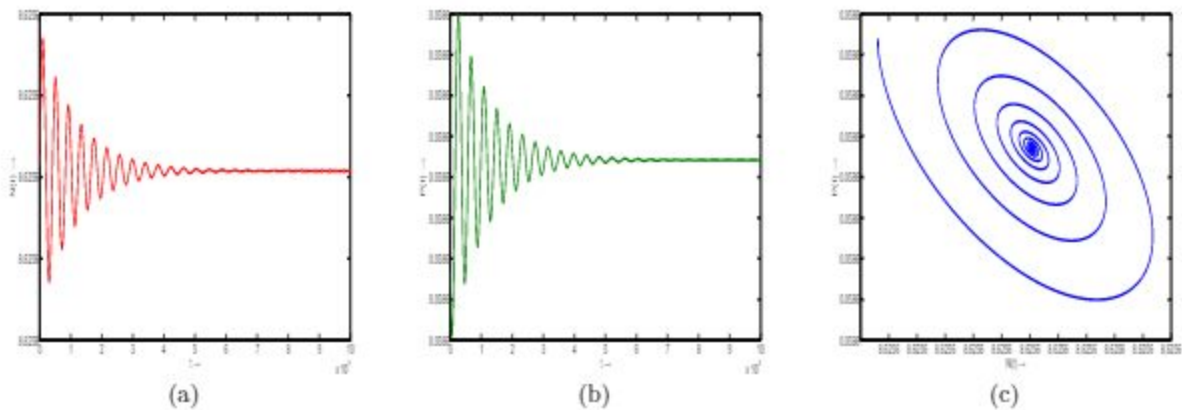(a)   (b)   (c)

Figure 1: Here, (a) and (b) represent time series for $N(t)$ and $P(t)$ respectively and (c) shows phase portrait between both the species corresponding to stable solutions in absence of delay.
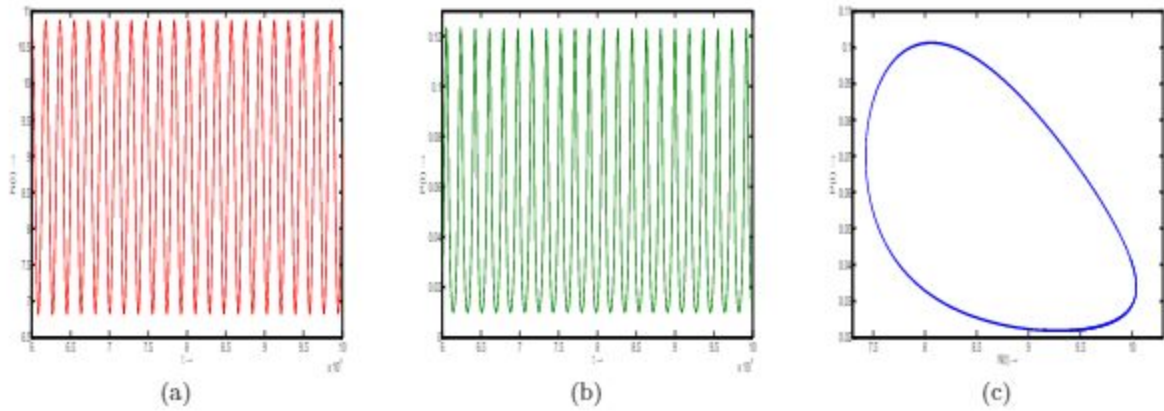
(a)   (b)   (c)

Figure 2: Here, (a) and (b) represent time series for $N(t)$ and $P(t)$ respectively and (c) shows phase portrait between both the species corresponding to periodic solutions at the threshold value of delay parameter $\tau = \tau_0 = 3.9665$.
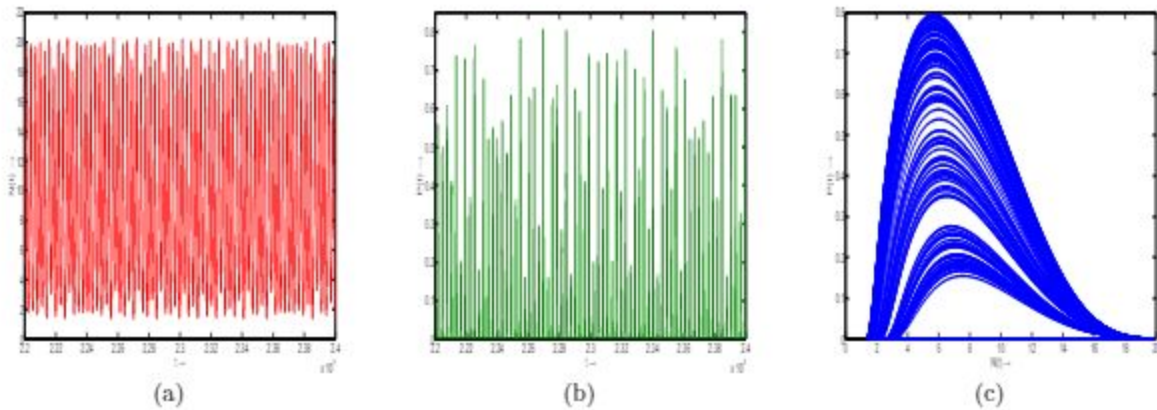

(a)   (b)   (c)

Figure 3: Here, (a) and (b) represent time series for $N(t)$ and $P(t)$ respectively and (c) reflects chaotic trajectory between both the species when $\tau_0 < \tau = 6.84$.
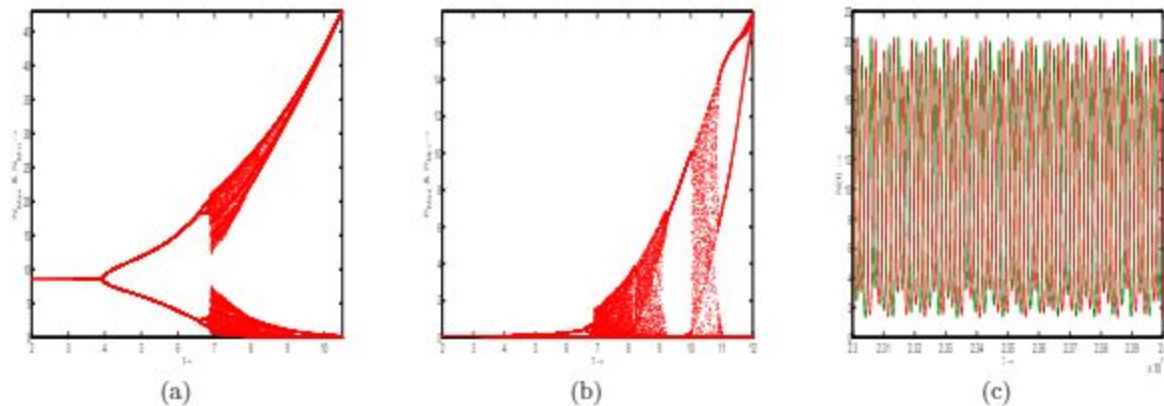

(a)   (b)   (c)

Figure 4: Here, (a) and (b) display bifurcation diagram for $N(t)$ and $P(t)$ with respect to τ respectively and (c) represents the sensitivity of plankton species.

Example 2. *For the second set of parameters $r = 0.1$, $k = 1$, $a = 0.3$, $m = 0.03$, $d = 0.07$, $h = 0.11$, $c = 0.4$, $n = 0.25$ there are two interior equilibrium states from which $L_{2*} = (0.5322, 1.3446)$ Figure 5(a) is stable in absence of delay τ [7]. It can be seen that system(2.3) exhibits periodic solution at the threshold value $\tau_0 = 11.8020$ Figure 5(b) and shows quasi-periodic behavior at $\tau = 13.92$ in Figure 5(c). For this set of parameter, we obtain $\omega_0 = 0.9112$ and $\tau_0 = 11.8020$, $C_1(0) = -242.1958 + 122.9812I$, $\beta_2 = 1.2237$, $\mu_2 = -484.3915$ and $T_2 = 203.9729$. Therefore, we conclude that the Hopf-bifurcation is supercritical in this case as well that is the amplitude of periodic solution bifurcating through Hopf-bifurcation increases with increasing value of τ. The further increase in value of τ beyond threshold value τ_0 provides quasi-periodic solutions which are shown in Figure 6(a), and it is also observed that the chaotic solution can be obtained for a very small range of $\tau \in (14.040, 14.064)$ and the corresponding chaotic solution is depicted in Figure 6(b). The bifurcation diagram of $N(t)$ with respect to τ is represented in Figure 6(c).*
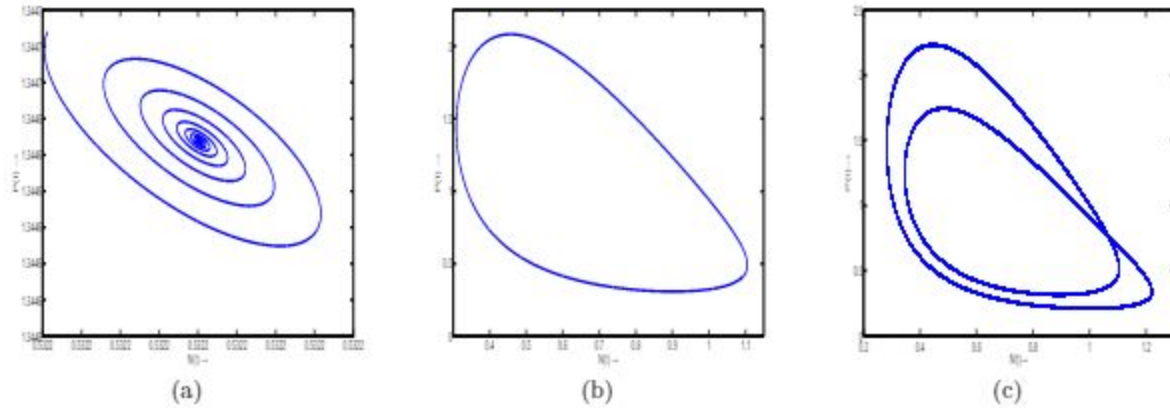

(a)　　　　　　　(b)　　　　　　　(c)

Figure 5: These pictures display the phase portrait of the system between $N(t)$ and $P(t)$, where, (a) represents stable solution in absence of delay, (b) displays periodic solution for $\tau = 11.80$ and (c) shows the quasi-periodic solution for $\tau = 13.92$.
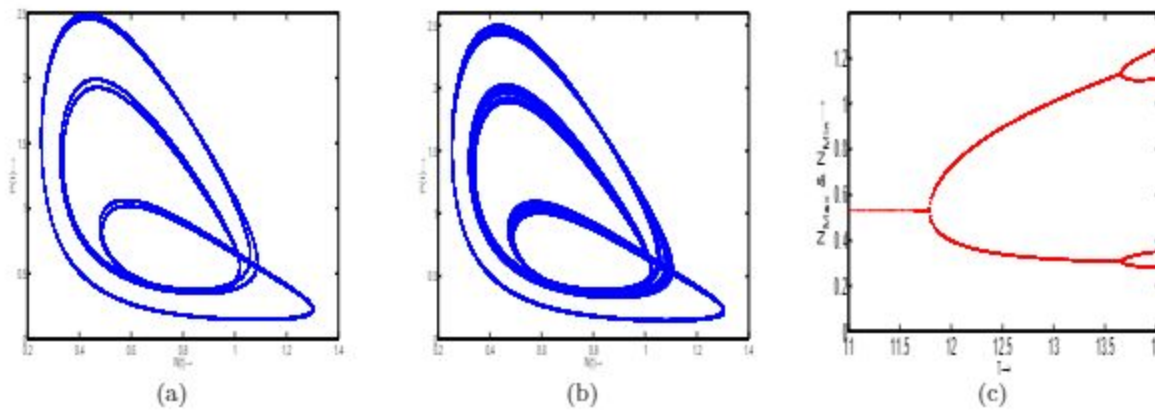

(a)　　　　　　　(b)　　　　　　　(c)

Figure 6: Here, (a) and (b) represents quasi-periodic phase portraits for and $\tau = 14.04$. (c) It represents bifurcation diagram for $N(t)$ with respect to τ.

These numerical examples show that to control chaotic behavior of solution of the system (2.3), the time-delay should not exceed a certain threshold. That is system (2.3) will present stable co-existence of species if the time-delay does not cross a critical threshold.

6 Conclusion

Phytoplankton are the main food source for the marine food chain. They consume energy from the Sun and utilize it to convert carbon dioxide into carbohydrates that sustain ocean life. Almost all small marine predators consume plankton as their major food intake. So if the plankton's growth rate is delayed, the growth rate of tiny predators, such as juvenile fish, will also be delayed. If the population of young fishes or tiny predators declines, the growth rate of large sea creatures suffers. As a result, a terrible food-chain would swiftly spread beyond the ocean's boundaries, because fish is essential to a healthy diet in many parts of the globe. Fish accounts for more than 20 percent of the average per capita animal protein consumption for approximately 3.3 billion individuals. Not only would those inhabitants be deprived of a key food source, but local trade would suffer, and their economies would begin to deteriorate. Thus, we conclude that plankton development and decay have a profound impact on the marine food-web and, as a consequence, on ecology.

In this work, we introduced a delay in the food-limited growth rate of prey (plankton) of a plankton-fish interaction model with nonlinear predator (fish) harvesting. The objective was to find out how delay affects the stability of the model (2.3). According to stability analysis, the presence of a time delay has no influence on the stability of the equilibrium states L_0 and L_1 lying on the boundaries. However, it acts as a destabilizing factor in the system near the interior equilibrium state, driving it to approach it asymptotically for values less than the critical value τ_0. We demonstrated that the solutions of the governing equations are limited for all future times. The feasibility of solutions has been guaranteed for the delayed model. Since, Gupta et al. [7] have already performed the stability analysis of all equilibrium states. We summerize these results in the table [1]. We have observed that the delayed model exhibits more complex and fascinating dynamical behaviour. A local periodic orbit appears about L_{2*} through Hopf bifurcation for $\tau = \tau_0$ and the solutions approach to this orbit asymptotically. A further increase in the value of τ causes the periodic orbit to lose its stability, and a strange attractor is formed, resulting in the chaotic instability of the system. This is evident by the fact that a temporal delay may cause a stable equilibrium state to become an unstable one, as well as being responsible for population oscillations.

We have considered two sets of examples in this paper with different parametric values. The first example is taken with such parameters that the system (2.3) has one interior equilibrium state, and after adding time-delay, the system shows a drastic change and becomes chaotic after the threshold value. The system in the second case displays chaotic behavior over a limited range and reflects a quasi-periodic nature after the threshold value $\tau_0 = 11.80$. The dynamics of the system change from a stable equilibrium to a periodic oscillation, and further bifurcations generate new excessive quasi-periodic fluctuations. We develop the explicit formulas that determine the stability and direction of the bifurcating periodic solutions using normal form theory and the center manifold theorem. From Theorem (2), we can determine the direction and stability of the Hopf bifurcation, and it is found to be supercritical and stable for the considered parameter values. Since we have no control over the duration of delays in the natural environment, the stability of the model system can be maintained by adjusting the strategies of harvesting. Thus, the coexistence of prey and predator populations may be maintained by limited harvesting, which also precludes the possibility of large-amplitude oscillations in population densities near their equilibrium levels. Otherwise, the high amplitude oscillation leads to the extinction of both species.

Compliance with Ethical Standards

Conflict of Interest:

The authors confirm that there are no known conflicts of interest associated with this manuscript.

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