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REPEATED FLOW ANALYSIS IN INFINITE-SERVER QUEUEING TANDEM WITH MMPP ARRIVALS AND FEEDBACK AT THE SECOND STAGE

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Abstract. Tandem of infinite-server queues with MMPP arrivals feedback at the second stage is considered. Service times at the stages of the tandem have non-exponential distribution. We use the method of Markovian summation to obtain the Kolmogorov differential equations for the probability distribution of the number of arrivals at the second stage (repeated arrivals). An expression for the characteristic probability distribution function of the number of repeated arrivals during given period is derived under the asymptotic condition of high intensity of the MMPP arrivals.

1. Introduction

Queueing systems with an unlimited number of servers and feedback [1, 2] can be used to describe various real processes in economics [3] as well in communication networks and other technical systems [4].

Queueing systems and tandems with an unlimited number of servers, Poisson arrival process and instant feedback were considered in [5, 6, 7]. The obtained equations for the probability distributions were solved analytically by the method of generating or characteristic functions. In these papers, the method of limit decomposition is used for study, but this method can be used only for the analysis of queueing models with Poisson arrivals.

Models with non-Poisson arrivals were considered in [8, 9]. In these papers, systems with exponential service were considered. To solve balance equations, the authors used asymptotic analysis methods under the asymptotic condition of increasing of service time.

In the paper, we consider tandem with MMPP arrivals, non-exponential service times and a feedback at the second stage. We consider non-stationary regime and we propose to use the Markov summation method [4, 10, 11, 12] in order to obtain the Kolmogorov (balance) differential equations for the probability distribution of the number of arrivals in a flow of repeated calls (arrivals at the second stage) over a certain time interval. To solve established balance equations, we use the asymptotic analysis methods [13, 14] under the asymptotic condition of high intensity of MMPP arrivals.

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Key words and phrases. Tandem of infinite-server queues, feedback, Markov summation method.

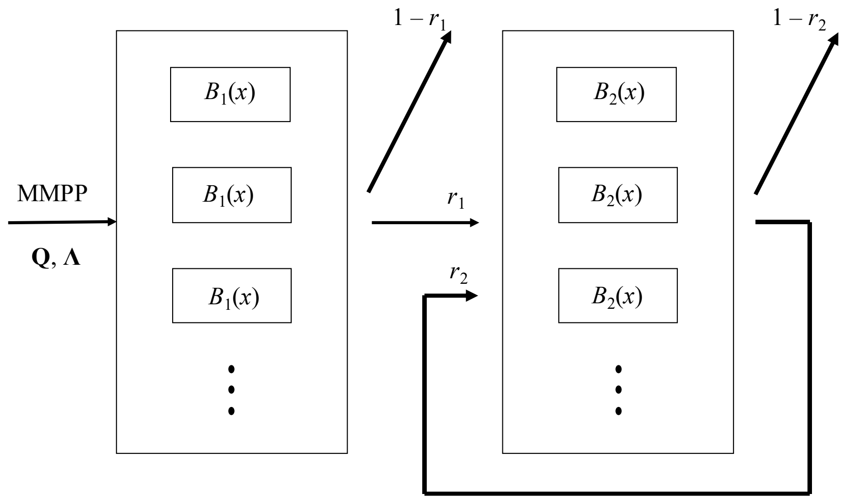

FIGURE 1. Tandem of queues with MMPP arrivals and feedback at the second stage.

2. Mathematical model

Consider a queue tandem (Fig. 1) with unlimited number of servers at each stage. Customers arrive according to Markovian modulated Poisson process (MMPP). The process is given by generator matrix $\mathbf{Q} = ||q_{ij}||$, $i, j = 1, 2, ..., K$ and conditional intensisites $\lambda_1, ..., \lambda_K$ which we compose into a diagonal matrix $\mathbf{\Lambda} = \text{diag}\{\lambda_n\}$, $n = 1, 2, ..., K$. Denote the underlying Markov chain of the MMPP as $k(t) \in \{1, 2, ..., K\}$. Arriving customer instantly occupies a server at the first stage of the system. Service time at this stage has distribution function $B_1(x)$. When the service is complete, the customer may leave the system with probability $(1 - r_1)$ or may go to the second stage for the further service with probability r_1. Service time at the second stage has distribution function $B_2(x)$. When the service is complete at the second stage, the customer may go to the second stage again with probability r_2 or may leave the system with probability $(1 - r_2)$.

Consider arrivals at the second stage of the tandem. They consist of customers that completed their service at the first stage and go to the second and customers that completed their service at the second stage and go for the repeated service at this stage. We will call this aggregate flow of customers as repeated arrivals, or as repeated flow, or r-flow.

We suppose that at the initial moment of time $t_0 = 0$, the system is empty and we analyze the number of customers arrived in the repeated flow during period $[0, T]$, where $T > 0$ is given time of observation. So, we consider non-stationary regime of the system evolution.

The goal of the study is to find probability distribution of the number of arrivals in the repeated flow during period $[0, T]$.

3. Markov summation method

Each customer arrived at the system in instant t generates arrivals in the r-flow that will occur after moment t. Let us denote by $m(t)$ the number of arrivals in r-flow generated by customers that arrived before time moment t, that is, the number of arrivals at the second stage of the tandem occurred during the period $[0, T]$ generated by customers arrived during period $[0, t]$, where $t \leq T$.

Considering two-dimensional stochastic process $\{m(t), k(t)\}$, we use the notation

$$P_k(m, t) = \mathbb{P}\{m(t) = m, k(t) = k\}$$

for its probability distribution. Also, let us use the following notations:

- $\xi(t)$ is the number of events in the r-flow generated during the period $[t, T]$ by a single customer arrived at the system at time moment t,
- $g(i, t) = \mathbb{P}\{\xi(t) = i\}$ is the probability that a customer arrived at the system at time moment t will generate i arrivals in the r-flow up to instant T.

Based on the total probability formula, we can write the following equations:

$$P_k(m, t + \Delta t) = P_k(m, t)(1 - \lambda\Delta t)(1 + q_{kk}\Delta t) +$$
$$\sum_{\nu \neq k} P_\nu(m, t)q_{\nu k}\Delta t + \lambda_k \Delta t \sum_{i=0}^{m} P_k(m - i, t)q(i, t) + o(\Delta t).$$

Then we obtain the Kolmogorov differential equations for probability distribution $P_k(m, t)$:

$$\frac{\partial P_k(m, t)}{\partial t} = -\lambda_k P_k(m, t) + \lambda_k \sum_{i=0}^{m} P_k(m - i, t)g(i, t) + \sum_{\nu} P_\nu(m, t)q_{\nu k}.$$

Consider characteristic functions

$$H_k(u, t) = \sum_{m=0}^{\infty} e^{jum} P_k(m, t),$$

$$G(u, t) = \sum_{i=0}^{\infty} e^{jui} g(i, t),$$

where $j = \sqrt{-1}$. Then we obtain the following differential equations for the characteristic function of the process $m(t)$:

$$\frac{\partial H_k(u, t)}{\partial t} = \lambda_k H_k(u, t)(G(u, t) - 1) + \sum_{\nu} H_\nu(u, t)q_{\nu k}.$$

The characteristic function $G(u, t)$ of process $\xi(t)$ was found in the paper [12] and has the form

$$G(u, t) = 1 + r_1(e^{ju} - 1)B_1(T - t) +$$
$$\frac{r_1 r_2}{2\pi}(e^{ju} - 1)e^{ju} \int_{-\infty}^{+\infty} \frac{b_1^*(\alpha)b_2^*(\alpha)\left(1 - e^{-j\alpha(T-t)}\right)}{(1 - r_2 e^{ju} b_2^*(\alpha))\, i\alpha} d\alpha,$$

where

$$b_1^*(\alpha) = \int_0^{\infty} e^{j\alpha\tau} dB_1(\tau),$$

$$b_2^*(\alpha) = \int_0^\infty e^{j\alpha\tau} dB_2(\tau).$$

We denote

$$\varphi(u,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{b_1^*(\alpha) b_2^*(\alpha) \left(1 - e^{-j\alpha(T-t)}\right)}{\left(1 - r_2 e^{ju} b_2^*(\alpha)\right) i\alpha} d\alpha,$$

then we can write the following equations

$$\frac{\partial H_k(u,t)}{\partial t} = \lambda_k H_k(u,t) r_1 (e^{ju} - 1) \left[B_1(T-t) + r_2 e^{ju} \varphi(u,t)\right] + \sum_\nu H_\nu(u,t) q_{\nu k}. \tag{3.1}$$

Denoting $\mathbf{H}(u,t) = \{H_1(u,t), H_2(u,t), ..., H_K(u,t)\}$, we derive the matrix equation

$$\frac{\partial \mathbf{H}(u,t)}{\partial t} = \mathbf{H}(u,t) \left(\mathbf{Q} + r_1 (e^{ju} - 1) \left[B_1(T-t) + r_2 e^{ju} \varphi(u,t)\right] \mathbf{\Lambda}\right) \tag{3.2}$$

with the initial condition

$$\mathbf{H}(u,0) = \mathbf{R}, \tag{3.3}$$

where $\mathbf{R}$ is a vector of the stationary distribution of the underlying Markov chain. Vector $\mathbf{R}$ satisfies the following linear system:

$$\begin{cases} \mathbf{RQ} = \mathbf{0}, \\ \mathbf{Re} = 1, \end{cases} \tag{3.4}$$

where $\mathbf{e}$ is a column vector with all entries equal to 1. The exact solution of equation (3.2) is not possible in general case, but it may be solved under an asymptotic condition. We will consider this equation under the condition of growing MMPP arrivals' rate.

4. Asymptotic analysis

We represent the intensity of the arrival process in the form $N\lambda$, where λ is a fixed value determined by the expression

$$\lambda = \mathbf{R\Lambda e},$$

and N is some parameter which characterizes high intensity of the arrival process ($N \to \infty$ in theoretical studies). Then the high-rate MMPP will be given by the matrices $N\mathbf{Q}$ and $N\mathbf{\Lambda}$.

In this case we can rewrite equation (3.2) in the form

$$\frac{1}{N} \frac{\partial \mathbf{H}(u,t)}{\partial t} = \mathbf{H}(u,t) \left(\mathbf{Q} + r_1 (e^{ju} - 1) \left[B_1(T-t) + r_2 e^{ju} \varphi(u,t)\right] \mathbf{\Lambda}\right) \tag{4.1}$$

with initial condition (3.3). We will obtain a solution of problem (4.1) in the form of approximations which we call first-order asymptotic and second-order asymptotic.

The main result is the following theorem.

Theorem 4.1. *The asymptotic characteristic function of the distribution of the number of repeated arrivals over the time interval* $[0, T]$ *under the condition of high rate of MMPP arrival process has the form*

$$h(u,T) = M\{e^{jum(T)}\} = \exp\left\{ juN\lambda r_1 \left[\int_0^T B_1(y)dy + \frac{r_2}{2\pi}\int_{-\infty}^{+\infty} \frac{b_1^*(\alpha)b_2^*(\alpha)}{j\alpha(1-r_2b_2^*(\alpha))}\left(T - \frac{1-e^{-j\alpha T}}{j\alpha}\right)d\alpha \right] + \frac{(ju)^2}{2} r_1 N \left[\lambda \int_0^T B_1(y)dy + \lambda\frac{r_2}{2\pi}\int_{-\infty}^{+\infty} \frac{b_1^*(\alpha)b_2^*(\alpha)\left(3-r_2b_2^*(\alpha)\right)}{j\alpha(1-r_2b_2^*(\alpha))^2}\left(T - \frac{1-e^{-j\alpha T}}{j\alpha}\right)d\alpha + \kappa\int_0^T \left(B_1(y) + \frac{r_2}{2\pi}\int_{-\infty}^{+\infty}\frac{b_1^*(\alpha)b_2^*(\alpha)}{j\alpha(1-r_2b_2^*(\alpha))}\left(\frac{1-e^{-j\alpha T}}{j\alpha}\right)d\alpha\right)^2 dy \right]\right\}, \tag{4.2}$$

where $\kappa = 2\mathbf{g}(\mathbf{\Lambda} - \lambda\mathbf{I})\mathbf{e}$ *and row vector* $\mathbf{g}$ *satisfies the linear matrix equation*

$$\mathbf{gQ} = r_1(\lambda\mathbf{R} - \mathbf{R\Lambda}).$$

Proof. We will carry out the proof of the theorem in two steps.

4.1. First-order Asymptotic Analysis. Let us make the following substitutions in (4.1) and (3.3):

$$\frac{1}{N} = \varepsilon, \quad u = \varepsilon w, \quad \mathbf{H}(u,t) = \mathbf{F}_1(w,t,\varepsilon). \tag{4.3}$$

Then we derive the problem

$$\varepsilon\frac{\partial \mathbf{F}_1(w,t,\varepsilon)}{\partial t} = \mathbf{F}_1(w,t,\varepsilon)\left(\mathbf{Q} + r_1(e^{ju}-1)\left[B_1(T-t) + r_2e^{ju}\varphi(u,t)\right]\mathbf{\Lambda}\right), \tag{4.4}$$

$$\mathbf{F}_1(w,0,\varepsilon) = \mathbf{R}. \tag{4.5}$$

1. Substituting $\varepsilon = 0$ into (4.4) and denoting $\mathbf{F}_1(w,t) = \lim_{\varepsilon\to 0}\mathbf{F}_1(w,t,\varepsilon)$, we obtain

$$\mathbf{F}_1(w,t)\mathbf{Q} = \mathbf{0}. \tag{4.6}$$

Then, taking into account the first expression in (3.4), we can conclude that we can express $\mathbf{F}_1(w,t)$ in the form

$$\mathbf{F}_1(w,t) = \mathbf{R}\Phi_1(w,t), \tag{4.7}$$

where $\Phi_1(w,t)$ is some scalar function that satisfies the equality

$$\Phi_1(w,0) = 1. \tag{4.8}$$

2. We multiply (4.4) by vector $\mathbf{e}$, substitute (4.7) and taking into account that $\mathbf{RQ} = \mathbf{0}$, $\mathbf{Re} = 1$, $\mathbf{R\Lambda e} = \lambda$, we obtain a differential equation for the function $\Phi_1(w,t)$

$$\varepsilon\frac{\partial \Phi_1(w,t)}{\partial t} = \Phi_1(w,t)r_1\lambda(e^{jw\varepsilon}-1)\left[B_1(T-t) + r_2e^{jw\varepsilon}\varphi(w,t,\varepsilon)\right]. \tag{4.9}$$

For $\varepsilon \to 0$, we can write the following expansions for the exponent $e^{jw\varepsilon}$ and for the function $\varphi(w,t,\varepsilon)$:

$$e^{jw\varepsilon} = 1 + jw\varepsilon + O(\varepsilon^2), \tag{4.10}$$

$$\varphi(w,t,\varepsilon) = \varphi(w,t,0) + \varphi'_\varepsilon(w,t,0)\varepsilon + O(\varepsilon^2) = a_0(t) + a_1(t)jw\varepsilon + O(\varepsilon^2), \tag{4.11}$$

where

$$a_0(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} b_1^*(\alpha)\frac{b_2^*(\alpha)}{1 - r_2 b_2^*(\alpha)}\left(\frac{1 - e^{-j\alpha(T-t)}}{j\alpha}\right) d\alpha, \tag{4.12}$$

$$a_1(t) = r_2 \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} b_1^*(\alpha)\left(\frac{b_2^*(\alpha)}{1 - r_2 b_2^*(\alpha)}\right)^2\left(\frac{1 - e^{-j\alpha(T-t)}}{j\alpha}\right) d\alpha. \tag{4.13}$$

Taking into account (4.9) and making a transition $\varepsilon \to 0$, we obtain the following differential equation for the function $\Phi_1(w,t)$:

$$\frac{\partial \Phi_1(w,t)}{\partial t} = r_1 \lambda jw \Phi_1(w,t)\left[B_1(T-t) + r_2 a_0(t)\right].$$

The solution of this equation with initial condition (4.8) is as follows:

$$\Phi_1(w,t) = exp\left\{jwr_1\lambda \int_0^t \psi(x)dx\right\},$$

where

$$\psi(x) = B_1(T-x) + \frac{r_2}{2\pi}\int_{-\infty}^{+\infty} b_1^*(\alpha)\frac{b_2^*(\alpha)}{1 - r_2 b_2^*(\alpha)}\left(\frac{1 - e^{-j\alpha(T-x)}}{j\alpha}\right) d\alpha. \tag{4.14}$$

Therefore, the first-order asymptotic solution of equation (4.4) has the form

$$\mathbf{F}_1(w,t) = \lim_{\varepsilon\to 0}\mathbf{F}_1(w,t,\varepsilon) = \mathbf{R}\exp\left\{jwr_1\lambda\int_0^t \psi(x)dx\right\}. \tag{4.15}$$

4.2. Second-order Asymptotic Analysis. Taking into account (4.15), we introduce vector function $\mathbf{H}_2(u,t)$ that satisfies the expression

$$\mathbf{H}(u,t) = \mathbf{H}_2(u,t)\exp\left\{juNr_1\lambda\int_0^t \psi(x)dx\right\}. \tag{4.16}$$

Substituting this expression into (4.1) and (3.3), we obtain the problem for function $\mathbf{H}_2(u,t)$:

$$\frac{1}{N}\frac{\partial \mathbf{H}_2(u,t)}{\partial t} + \mathbf{H}_2(u,t)ju\lambda r_1\psi(t) =$$

$$\mathbf{H}_2(u,t)\left(\mathbf{Q} + r_1(e^{ju}-1)\left[B_1(T-t) + r_2 e^{ju}\varphi(u,t)\right]\mathbf{\Lambda}\right), \tag{4.17}$$

$$\mathbf{H}_2(u,0) = \mathbf{R}. \tag{4.18}$$

1. Let us make here the following substitutions:

$$\frac{1}{N} = \varepsilon^2, \quad u = \varepsilon w, \quad \mathbf{H}_2(u,t) = \mathbf{F}_2(w,t,\varepsilon). \tag{4.19}$$

We obtain

$$\varepsilon^2\frac{\partial \mathbf{F}_2(w,t,\varepsilon)}{\partial t} + \mathbf{F}_2(w,t,\varepsilon)jw\varepsilon\lambda r_1\psi(t) =$$

$$\mathbf{F}_2(w,t,\varepsilon)\left(\mathbf{Q} + r_1(e^{jw\varepsilon}-1)\left[B_1(T-t) + r_2 e^{jw\varepsilon}\varphi(u,t)\right]\mathbf{\Lambda}\right), \tag{4.20}$$

$$\mathbf{F}_2(w,0,\varepsilon) = \mathbf{R}. \tag{4.21}$$

If we let $\varepsilon \to 0$ in (4.20) and denote $\mathbf{F}_2(w,t) = \lim_{\varepsilon \to 0} \mathbf{F}_2(w,t,\varepsilon)$, we obtain

$$\mathbf{F}_2(w,t)\mathbf{Q} = \mathbf{0}.$$

Therefore, we can assume that the function $\mathbf{F}_2(w,t)$ has the form

$$\mathbf{F}_2(w,t) = \mathbf{R}\Phi_2(w,t), \tag{4.22}$$

where $\Phi_2(w,t)$ is some scalar function that satisfies the equality

$$\Phi_2(w,0) = 1. \tag{4.23}$$

2. Using expression (4.22), we can represent function $\mathbf{F}_2(w,t,\varepsilon)$ in the expansion form

$$\mathbf{F}_2(w,t,\varepsilon) = \Phi_2(w,t)\left[\mathbf{R} + \mathbf{g} \cdot j e \varepsilon \psi(t)\right] + \mathbf{O}(\varepsilon^2), \tag{4.24}$$

where $\mathbf{g}$ is some row vector, $\mathbf{O}(\varepsilon^2)$ is a row vector of infinitesimals of the order ε^2.

Substituting (4.24), (4.10) and (4.11) into (4.20) and making transition $\varepsilon \to 0$, we obtain matrix equation for vector $\mathbf{g}$:

$$\mathbf{g} \cdot \mathbf{Q} = r_1(\lambda\mathbf{R} - \mathbf{R}\boldsymbol{\Lambda}).$$

3. We multiply (4.20) by vector $\mathbf{e}$, substitute (4.10), (4.11) and take into account that $\mathbf{RQ} = \mathbf{0}$, $\mathbf{Re} = 1$, $\mathbf{R}\boldsymbol{\Lambda}\mathbf{e} = \lambda$. Then we divide the left and right sides of the obtained equality by ε^2 and make transition $\varepsilon \to 0$. As a result, we obtain the following differential equation for the function $\Phi_2(w,t)$:

$$\frac{\partial \Phi_2(w,t)}{\partial t} = \Phi_2(w,t)\frac{(jw)^2}{2}\left[r_1\lambda\left(\psi(t) + 2r_2a_0(t) + 2r_2a_1(t)\right) + \right.$$
$$\left. 2r_1\mathbf{g}(\boldsymbol{\Lambda} - \lambda\mathbf{I})\mathbf{e}\psi^2(t)\right].$$

Solving this equation under initial condition (4.23) and using the notation

$$\kappa = 2\mathbf{g}(\boldsymbol{\Lambda} - \lambda\mathbf{I})\mathbf{e},$$

we obtain

$$\Phi_2(w,t) = \exp\left\{\frac{(jw)^2}{2}\left[r_1\lambda\int_0^t \psi(x)dx + 2r_1r_2\lambda\int_0^t \left(a_0(x) + a_1(x)\right)dx + \right.\right.$$
$$\left.\left. r_1\kappa\int_0^t \psi^2(x)dx\right]\right\}. \tag{4.25}$$

Substituting expression (4.25) into (4.22), we obtain the form of the asymptotic solution of the problem (4.20), (4.21):

$$\mathbf{F}_2(w,t) = \lim_{\varepsilon\to 0}\mathbf{F}_2(w,t,\varepsilon) =$$
$$= \mathbf{R}\exp\left\{\frac{(jw)^2}{2}\left[r_1\lambda\int_0^t \psi(x)dx + 2r_1r_2\lambda\int_0^t \left(a_0(x) + a_1(x)\right)dx + \right.\right. \tag{4.26}$$
$$\left.\left. r_1\kappa\int_0^t \psi^2(x)dx\right]\right\}.$$

Let us return to function $\mathbf{H}(u,t)$. Using results (4.15), (4.26) and performing substitutions that are inverse to (4.3), (4.16), (4.19), we obtain the following expression for this function:

$$\mathbf{H}(u,t) = \mathbf{R}\exp\left\{juN\lambda r_1\int_0^t \psi(x)dx+\right.$$
$$\left.\frac{(jw)^2}{2}\left[r_1\lambda\int_0^t \psi(x)dx + 2r_1r_2\lambda\int_0^t (a_0(x)+a_1(x))\,dx + r_1\kappa\int_0^t \psi^2(x)dx\right]\right\}.$$

Let us multiply both sides of this equality by $\mathbf{e}$, substitute $t=T$, then we derive the following expression for the characteristic function $h(u,T)$ of the number of repeated arrivals under the asymptotic condition of high intensity of the incoming flow:

$$h(u,T) = \exp\left\{juN\lambda r_1\int_0^T \psi(x)dx+\right.$$
$$\left.\frac{(jw)^2}{2}\left[r_1\lambda\int_0^T \psi(x)dx + 2r_1r_2\lambda\int_0^T (a_0(x)+a_1(x))\,dx + r_1\kappa\int_0^T \psi^2(x)dx\right]\right\}. \tag{4.27}$$

Substituting here expressions (4.12)–(4.14), we obtain (4.2).

The theorem is proved. □

So, we establish that the probability distribution of the number of repeated calls in the system with high-rate MMPP arrivals can be approximated by Gaussian distribution with the following parameters:

$$\text{Mean} = N\lambda r_1\int_0^T \psi(x)dx, \tag{4.28}$$

$$\text{Variance} = r_1\lambda\int_0^T \psi(x)dx + 2r_1r_2\lambda\int_0^T (a_0(x)+a_1(x))\,dx + r_1\kappa\int_0^T \psi^2(x)dx. \tag{4.29}$$

5. Numerical Analysis

Let us analyze an accuracy and establish an applicability area of the obtained Gaussian approximation (4.27)–(4.29). To do this we perform simulation experiments for various values of parameter N and compare their results with obtained approximation. For accuracy estimation we use Kolmogorov distance

$$d = \max_{i=0,1,\dots} |F(i) - G(i)|,$$

where $F(i)$ is an empiric distribution function of the number of repeated arrivals (arrivals at the second stage of the considered tandem) obtained on the base of results of simulation experiment and $G(i)$ is discretized Gaussian distribution function with parameters (4.28) and (4.29).

We performed a big number of the experiments and we obtained very similar results for all of them. Let us introduce one of the results here. Consider queueing

TABLE 1. Kolmogorov distances d between Gaussian approximation and empiric distribution based on the simulation results for the case $T = 1$ for various values of parameter N.

N	10	25	50	100	250	500	1000
$N \cdot T$	10	25	50	100	250	500	1000
d	0.1499	0.0958	0.0674	**0.0490**	**0.0318**	**0.0240**	**0.0201**

tandem with feedback at the second stage. The arrival process is MMPP with high intensity given by the following matrices of generator and conditional intensities:

$$\mathbf{Q} = N \cdot \begin{pmatrix} -1 & 0.5 & 0.5 \\ 1 & -1.2 & 0.2 \\ 0.5 & 0.8 & -1.3 \end{pmatrix}, \qquad \mathbf{\Lambda} = N \cdot \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2.5 \end{pmatrix}.$$

We choose various values for intensity parameter N to find boundaries of the applicability area of the approximation (4.27). Service times at both stages of the tandem are gamma distributed with shape and rate parameters $\alpha_1 = \beta_1 = 0.5$ for the first stage and $\alpha_2 = \beta_2 = 1.5$ for the second stage. Probabilities r_1 and r_2 (see Sec. 2) are chosen as follows: $r_1 = 0.75$, $r_2 = 0.5$.

Observation time T we choose equal to 1 at the first step of the analysis. In Table 1, you may find values of Kolmogorov distances between empiric distribution and Gaussian approximation for various values of parameter N. As we see, the approximation becomes more accurate while N is increasing, i.e., while increasing the intensity of MMPP arrivals. If we choose $d \leq 0.05$ to the approximation may be named as enough accurate, then we may conclude that an applicability area of the approximation is $N \geq 100$ for the case $T = 1$ (this area is highlighted by boldface font in the table). In addition, visual representation of probability mass functions for empiric distribution of the number of repeated arrivals in comparison with discretized Gaussian approximation for various values of parameter N are presented in Fig. 2.

It is obvious, that an accuracy of Gaussian approximation (4.27) significantly depends not only on parameter N but on length of chosen interval of observation T. Due to this, we consider other values of T. Notice that actually Gaussian approximation becomes more accurate when its mean is increasing. So, we will consider dependence of the accuracy on $N \cdot T$. In Tables 2 and 3, such results are presented for $T = 10$ and $T = 100$. In Fig. 3, you may find visual representation of Kolmogorov distance behavior while $N \cdot T$ is increasing. As we see, in all cases ($T = 1, 10, 100$) the Gaussian approximation becomes more accurate while $N \cdot T$ is increasing and it becomes enough accurate ($d \leq 0.05$) when $N \cdot T \geq 100$. This value determines an applicability area of the approximation. For great values of length of the observation period (for $T = 100$ and greater), the low boundary of the applicability area becomes less (e.g., for $T = 100$ we have $d = 0.0468 < 0.5$ while $N \cdot T = 50$).

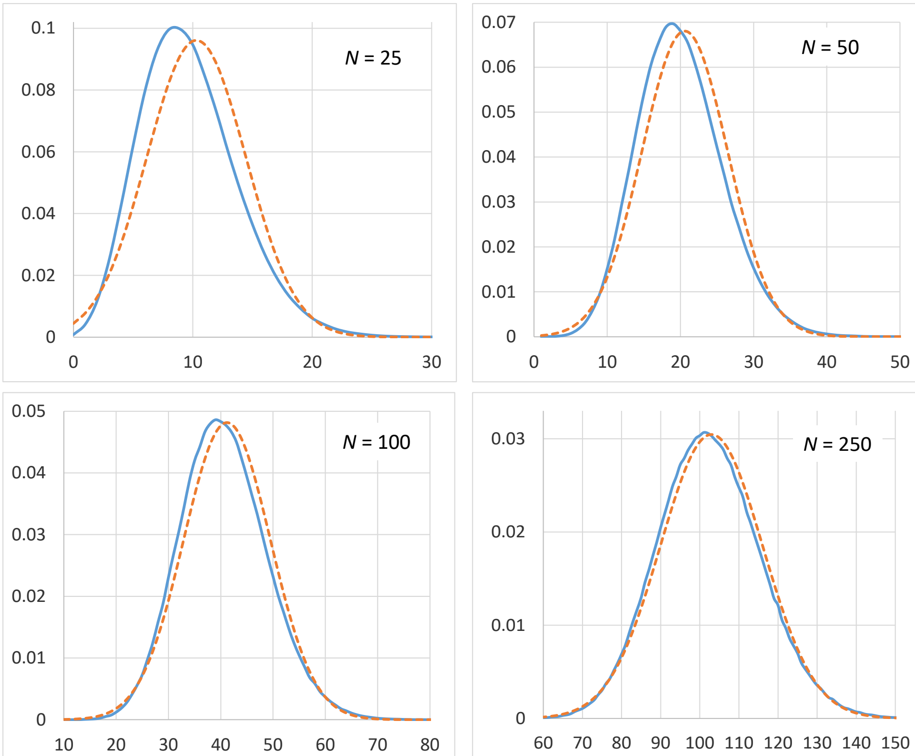

FIGURE 2. Comparisons of the Guassian approximation (dashed line) and empiric distribution (solid line) for various values of the parameter N.

TABLE 2. Kolmogorov distances d between Gaussian approximation and empiric distribution based on the simulation results for the case $T = 10$ for various values of parameter N.

N	1	2.5	5	10	25	50	100
$N \cdot T$	10	25	50	100	250	500	1000
d	0.1172	0.0742	0.0533	**0.0385**	**0.0272**	**0.0220**	**0.0198**

TABLE 3. Kolmogorov distances d between Gaussian approximation and empiric distribution based on the simulation results for the case $T = 100$ for various values of parameter N.

N	0.1	0.25	0.5	1	2.5	5	10
$N \cdot T$	10	25	50	100	250	500	1000
d	0.1046	0.0654	**0.0468**	**0.0344**	**0.0246**	**0.0201**	**0.0196**

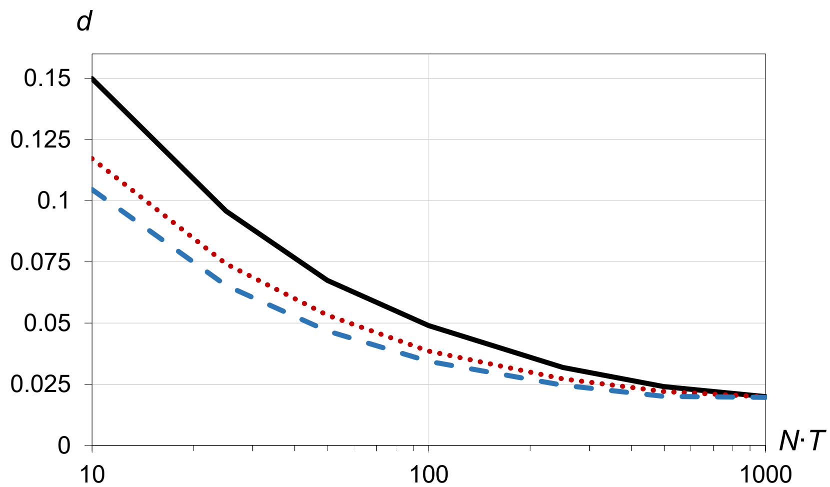

FIGURE 3. Changes in Kolmogorov distance d while $N \cdot T$ is increasing for the cases $T = 1$ (solid line), $T = 10$ (dotted line), and $T = 100$ (dashed line). Logarithmic scale is used for horizontal axis.

6. Conclusions

In the paper, we consider queueing tandem with MMPP arrivals, non-exponential service times and feedback at the second stage. The goal of the study is to find probability distribution of the number of arrivals at the second stage (repeated arrivals). Using Markov summation method, we establish equations to be solved for the problem solution. The equations are solved under asymptotic condition of high intensity of MMPP arrivals. As a result, we obtain Gaussian distribution which can be used as an approximation for the probability distribution of the number of repeated arrivals. Using simulation experiments, we establish accuracy and applicability area of the obtained approximation for various values of intensity parameter N and observation time T. It is shown that the approximation has enough small error for $N \cdot T \geq 100$.

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