Journal of Mathematical Control Science & Applications, Vol. 8 No. 1 (January-June, 2022)
ISSN : 0974-0570

BLOW UP OF SOLUTIONS FOR A COUPLED VISCOELASTIC KIRCHHOFF SYSTEM WITH DISTRIBUTED DELAY TERMS

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ABSTRACT. Our goal in this paper is to study a coupled nonlinear viscoelastic Kirchhoff system with distributed delay terms and source terms. For a non negative initial energy , we investigated that the solutions blows up under suitable conditions.

1. Introduction

The study of the asymptotic behavior of wave equation is an old and large areas because it importance in the applications, for this reasons wave equation has been taking different formes and names according to the described phenomena and also the material using in the experiment. In this paper we will study one of themes named Kirchhoff equation in one dimension space, it describe the transversal small amplitude vibration of elastic strings. Also in some equation we add the viscoelastic term which impose a natural damping due to the special property of the observe vibration's material without forget its affect into dissipation of the energy. To give a problem describe a phenomena enough perfect, we must take a consideration de delay which appear in practical phenomena like physical, biological , economic and some of themes. Furthermore, delay term influenced on the stability and instability of the studied system and many papers has been discussed the both situations for example [4].

In a bounded domain Ω in R^n with a smooth boundary, we consider the following viscoelastic Kirchhoff system with distributed delay in the internal frictional damping terms :

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t h_1(t-s)\,\Delta u(s)\,ds + \mu_1 u_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, u_t(x, t-\varrho)\,d\varrho = f_1(u,v), & (x,t) \in \Omega \times R_+, \\ v_{tt} - M\left(\|\nabla v\|^2\right)\Delta v + \int_0^t h_2(t-s)\,\Delta v(s)\,ds + \mu_3 v_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|\, v_t(x, t-\varrho)\,d\varrho = f_2(u,v), & (x,t) \in \Omega \times R_+, \\ u(x,t) = 0,\ v(x,t) = 0, & x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t),\ v_t(x,-t) = k_0(x,t), & (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\mu_1,\ \mu_3 > 0$, represented the weights of non delayed damping terms. The $\tau_1,\ \tau_2$ are the margin of the delay's distribution with $0 \leq \tau_1 < \tau_2$, $\mu_2,\ \mu_4$ are

2000 *Mathematics Subject Classification.* 35L05; 35L70; 93D20.
Key words and phrases. Blow up, Distributed delay, Nonlinear source.

L^{∞} functions and the memory kernel functions h_1, h_2 are differentiable functions. $M(s) = m_0 + \alpha s^{\gamma}$ is a C^1 nonnegative function on R^+ with m_0, α and $\gamma \geq 0$ are nonnegative constants, to simplify our calculations we take $M(s) = 1 + s^{\gamma}$ in the problem (1.1). Finally, the source terms f_1 and f_2 are given as :

$$\begin{cases} f_1(u,v) = a_1 |u+v|^{2(p+1)} (u+v) + b_1 |u|^p \ u \ |v|^{p+2}, \\ f_2(u,v) = a_1 |u+v|^{2(p+1)} (u+v) + b_1 |v|^p \ v \ |u|^{p+2}, \end{cases}$$

where $a_1, b_1 > 0$ and p a real number to be specified later.

In the absence of delay term, several papers has been investigated our problem in the case one equation and $M(s) = 1$ (see [5], [8], [9], [21], and references therein). In the other side a lot of papers has been studied Kirchhoff equation without viscoelastic term (see [1],[3]).

In [6], Mezouar and Boulaaras showed the global existence and decay properties of solutions for the following viscoelastic Kirchhoff equation:

$$\begin{aligned} & |u_t|^l u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds \\ & +\alpha u + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) \\ = \ & 0. \end{aligned}$$

When the observe vibrations's string is composed two different material the model given is constructed by two equations for this some authors has developed the works to systems as in [7] when the authors established the global existence and exponential decay of solutions for generalized coupled Kirchhoff system with a delay term varies in time .

In [14], Pişkin considered the following system of viscoelastic wave equations :

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)\,d\tau + u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)\,d\tau + v_t = f_2(u,v). \end{cases} \tag{1.2}$$

He obtained the global nonexistence of solutions for the problem (1.2).

The coupled nonlinear Klein-Gordon system has been studied firstly with weak damping terms by Pişkin in [13] when he proved the blow up of solutions. After that, Pişkin, in [15], established the decay estimates of the solution by using Nakao's inequality and he gave a finite time's blow up of solution for a negative initial energy.

Recently, Rahmoune et al. [20] considered, in $\Omega \times R_+$, also the same system with strong damping and distributed delay terms given as follow

$$\begin{cases} u_{tt} + m_1 u^2 - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, u_t(x,t-\varrho)\,d\varrho = f_1(u,v), \\ v_{tt} + m_2 v^2 - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)\,ds + \mu_3 v_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|\, v_t(x,t-\varrho)\,d\varrho = f_2(u,v), \end{cases} \tag{1.3}$$

where m_1, m_2, ω_1, $\omega_2 > 0$. They investigated the exponential growth of solutions for the problem (1.3) under suitable conditions. In [16], we considered the growth of solutions for the problem (1.1). Recently, some other authors studied related hyperbolic type equations (see [2, 17, 18, 19, 22, 23, 24, 25, 26, 27]).

Motivated by previous works and in the presence of both Kirchhoff term $M\left(\|\nabla u\|^2\right)$ and viscoelastic terms with distributed delay terms in our system, we prove that solutions of the system (1.1) blows up in finite time by the similar way of [12].

This paper is structured as follows: In section 2, we set some assumptions and results given as lemmas need its later. The section 3 specified to prove the blow up result for the system (1.1).

2. Preliminaries

Throughout this work we adoptee $\|.\|_p$ as a notation of L_p- norms. We start by setting theses assumptions:

(A1) For $i=1,2$, h_i are positive differentiable decreasing functions over R_+ with total mass

$$\int_0^\infty h_i(s)\,ds = 1-l_i < 1. \tag{2.1}$$

(A2) There exist constants $\xi_i > 0$ such that

$$h_i'(t) \leq -\xi_i h_i(t),\ t \geq 0. \tag{2.2}$$

(A3) μ_2, $\mu_4 : [\tau_1,\tau_2] \to R$ and for all $\delta > \frac{1}{2}$,

$$\begin{cases} \left(\delta+\frac{1}{2}\right)\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho < \mu_1, \\ \left(\delta+\frac{1}{2}\right)\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,d\varrho < \mu_3. \end{cases} \tag{2.3}$$

The next lemma show that there exists a function F in relationship with f_1 and f_2 as follow

Lemma 2.1. [12]

$$\begin{aligned} F(u,v) &= \frac{1}{2(p+2)}\left[uf_1(u,v)+vf_2(u,v)\right] \\ &= \frac{1}{2(p+2)}\left[a_1|u+v|^{2(p+2)}+2b_1|uv|^{p+2}\right] \\ &\geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v),\ \frac{\partial F}{\partial v} = f_2(u,v),$$

In all calculs we will take $a_1=b_1=1$ for convenience.
The norm of the function F is equivalent to $\left(\|u\|^{2(p+2)}+\|v\|^{2(p+2)}\right)$ as it appears in the following lemma.

Lemma 2.2. [11] *There exist* c_0, $c_1 > 0$ *such that*

$$\frac{c_0}{2(p+2)}\left(|u|^{2(p+2)}+|v|^{2(p+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(p+2)}\left(|u|^{2(p+2)}+|v|^{2(p+2)}\right). \tag{2.4}$$

In order to compute the convolution term in our problem, we will need to the next lemma.

Lemma 2.3. [11] *For $\phi, \psi \in C^1(R_+, R)$ we have*

$$\int_\Omega \phi*\psi\psi_t \, dx = -\frac{1}{2}\phi(t)\|\psi(t)\|^2 + \frac{1}{2}(\phi' \circ \psi)(t) - \frac{1}{2}\frac{d}{dt}\Big[(\phi\circ\psi)(t) - \Big(\int_0^t \phi(s)\, ds\Big)\|\psi\|^2\Big]$$

where

$$(\phi \circ \psi)(t) = \int_0^t \phi(t-s)\|\psi(t)-\psi(s)\|^2 \, ds.$$

3. Blow up results

We begin this section, as in [10], by define on $\Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$ the new following functions:

$$y(x,\rho,\varrho,t) = u_t(x, t-\varrho\rho),$$
$$z(x,\rho,\varrho,t) = v_t(x, t-\varrho\rho),$$

therefore,

$$\begin{cases} \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \\ y(x,0,\varrho,t) = u_t(x,t), \end{cases}$$

and

$$\begin{cases} \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \\ z(x,0,\varrho,t) = v_t(x,t). \end{cases}$$

Thus, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t h_1(t-s)\Delta u(s)\, ds + \mu_1 u_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y(x,1,\varrho,t)\, d\varrho = f_1(u,v), \qquad x\in\Omega,\ t\geq 0, \\ v_{tt} - M\left(\|\nabla v\|^2\right)\Delta v + \int_0^t h_2(t-s)\Delta v(s)\, ds + \mu_3 v_t \\ \quad + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|\, z(x,1,\varrho,t)\, d\varrho = f_2(u,v), \qquad x\in\Omega,\ t\geq 0, \\ \quad \varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = 0, \\ \quad \varrho z_t(x,\rho,\varrho,t) + z_\rho(x,\rho,\varrho,t) = 0, \end{cases} \tag{3.1}$$

with the initial and boundary conditions

$$\begin{cases} u(x,t) = 0,\ v(x,t) = 0,\ x\in\partial\Omega, \\ y(x,\rho,\varrho,0) = f_0(x,\varrho\rho),\ z(x,\rho,\varrho,0) = k_0(x,\varrho\rho), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \end{cases}$$

The functional space defined as

$$\begin{aligned} \hat{H} &= H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)) \\ &\quad \times L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)). \end{aligned}$$

Theorem 3.1. [20] *Under (2.1), (2.2), (2.3) hold and*

$$\begin{cases} -1 < p < \frac{4-n}{n-2},\ n\geq 3, \\ p\geq -1,\ n = 1,2. \end{cases} \tag{3.2}$$

The problem (3.1) has a unique solution in $C\left([0,T]\,; \hat{H}\right)$ for some $T>0$ and for any initial data, $(u_0, u_1, v_0, v_1, f_0, k_0) \in \hat{H}$,

We define the energy functional as follows:

Lemma 3.2. *Assume that (2.1), (2.2), (2.3) and (3.2) hold, then the energy associate to the solution (u,v,y,z) of (3.1) given as follow*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}+\frac{1}{2(\gamma+1)}\|\nabla v\|_2^{2(\gamma+1)} \\ &+\frac{l_1}{2}\|\nabla u\|_2^2+\frac{l_2}{2}\|\nabla v\|_2^2+\frac{1}{2}(h_1 o\nabla u)+\frac{1}{2}(h_2 o\nabla v)+\frac{1}{2}M(y,z) \\ &-\|F(u,v)\|_1 \end{aligned}$$

is nonincreasing and

$$\begin{aligned} E'(t) &\leq -c_3\left\{\|u_t\|_2^2+\|v_t\|_2^2+\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,\|y(x,1,\varrho,t)\|_2^2\,d\varrho\right. \\ &\left.\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,\|z(x,1,\varrho,t)\|_2^2\,d\varrho\right\} \end{aligned} \tag{3.3}$$

where

$$M(y,z)=\int_0^1\int_{\tau_1}^{\tau_2}\varrho\left\{|\mu_2(\varrho)|\,\|y(x,\rho,\varrho,t)\|_2^2+|\mu_4(\varrho)|\,\|z(x,\rho,\varrho,t)\|_2^2\right\}d\varrho d\rho$$

and

$$\|F(u,v)\|_1=\frac{1}{2(p+2)}\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{(p+2)}^{(p+2)}\right].$$

Proof. We multiplying the first two equations in (3.1) successively by u_t, v_t , after integrating over Ω and recalling lemma 2.3, we get

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|v_t\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}+\frac{1}{2(\gamma+1)}\|\nabla v\|_2^{2(\gamma+1)}\right. \\ &+\frac{1}{2}\left(1-\int_0^t h_1(s)\,ds\right)\|\nabla u\|_2^2+\frac{1}{2}\left(1-\int_0^t h_2(s)\,ds\right)\|\nabla v\|_2^2 \\ &\left.+\frac{1}{2}(h_1 o\nabla u)+\frac{1}{2}(h_2 o\nabla v)-\|F(u,v)\|_1\right) \\ = &-\mu_1\|u_t\|_2^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,y(x,1,\varrho,t)\,d\varrho dx \\ &-\mu_3\|v_t\|_2^2-\int_\Omega v_t\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,z(x,1,\varrho,t)\,d\varrho dx \\ &+\frac{1}{2}(h_1' o\nabla u)-\frac{1}{2}h_1(t)\|\nabla u\|_2^2 \\ &+\frac{1}{2}(h_2' o\nabla v)-\frac{1}{2}h_2(t)\|\nabla v\|_2^2, \end{aligned} \tag{3.4}$$

by recalling the initial and boundary conditions in (3.1), we obtain

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\varrho\left|\mu_2\left(\varrho\right)\right|\left\|y\left(x,\rho,\varrho,t\right)\right\|_2^2 d\varrho d\rho \\ = & -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2\left|\mu_2\left(\varrho\right)\right|yy_\rho d\varrho d\rho dx \\ = & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|d\varrho\right)\left\|u_t\right\|_2^2 \\ & -\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\left\|y\left(x,1,\varrho,t\right)\right\|_2^2 d\varrho \end{aligned} \tag{3.5}$$

and similarly we obtain

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\varrho\left|\mu_4\left(\varrho\right)\right|\left\|z\left(x,\rho,\varrho,t\right)\right\|_2^2 d\varrho d\rho \\ = & -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2\left|\mu_4\left(\varrho\right)\right|zz_\rho d\varrho d\rho dx \\ = & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_4\left(\varrho\right)\right|d\varrho\right)\left\|v_t\right\|_2^2 \\ & -\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_4\left(\varrho\right)\right|\left\|z\left(x,1,\varrho,t\right)\right\|_1^2 d\varrho, \end{aligned} \tag{3.6}$$

then by combining (3.4)-(3.6) and recalling (2.1), we find

$$\begin{aligned} \frac{d}{dt}E\left(t\right) \leq & -\mu_1\left\|u_t\right\|_2^2-\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|u_t y\left(x,1,\varrho,t\right)d\varrho dx+\frac{1}{2}\left(h_1' o\nabla u\right) \\ & -\frac{1}{2}h_1\left(t\right)\left\|\nabla u\right\|_2^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|d\varrho\right)\left\|u_t\right\|_2^2 \\ & -\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_2\left(\varrho\right)\right|\left\|y\left(x,1,\varrho,t\right)\right\|_2^2 d\varrho \\ & -\mu_3\left\|v_t\right\|_2^2-\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_4\left(\varrho\right)\right|v_t z\left(x,1,\varrho,t\right)d\varrho dx+\frac{1}{2}\left(h_2' o\nabla v\right) \\ & -\frac{1}{2}h_2\left(t\right)\left\|\nabla v\right\|_2^2+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_4\left(\varrho\right)\right|d\varrho\right)\left\|v_t\right\|_2^2 \\ & -\frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu_4\left(\varrho\right)\right|\left\|z\left(x,1,\varrho,t\right)\right\|_2^2 d\varrho. \end{aligned} \tag{3.7}$$

Thus (3.3) is established from Young's inequality and substituting (2.1), (2.2) and (2.3) in (3.7). □

Theorem 3.3. *Assume that (2.1)-(2.3), (3.2) hold and furthermore $E(0) < 0$, then the solution of problem (3.1) blows up in finite time.*

Proof. From lemma 3.2, we get

$$E\left(t\right)\leq E\left(0\right)<0, \tag{3.8}$$

then the functional defined as

$$H(t) = -E(t) \tag{3.9}$$

have the following properties

(1)

$$\begin{aligned} H'(t) &\geq c_3 \left(\|u_t\|_2^2 + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, \|y(x,1,\varrho,t)\|_2^2 \, d\varrho \right. \\ &\left. + \|v_t\|_2^2 + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \, \|z(x,1,\varrho,t)\|_2^2 \, d\varrho \right). \end{aligned}$$

Hence,

$$H'(t) \geq c_3 \max\{\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, \|y(x,1,\varrho,t)\|_2^2 d\varrho, \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \, \|z(x,1,\varrho,t)\|_2^2 d\varrho\} \geq 0 \tag{3.10}$$

thus from (3.8) H is a positive.

(2) H is bounded by

$$H(0) \leq H(t) \leq \|F(u,v)\|_1 \leq \frac{c_1}{2(p+2)} \left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right] \tag{3.11}$$

The last inequality is getting from lemme 2.2.

For a nonnegative constant

$$\alpha < \frac{2p+2}{4(p+2)} < 1 \tag{3.12}$$

we set

$$K(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega (uu_t + vv_t) \, dx + \frac{\varepsilon}{2} \left(\mu_1 \|u\|_2^2 + \mu_3 \|v\|_2^2 \right) \tag{3.13}$$

where ε is a non negative constant will be given later.

A derivation of (3.13) gives

$$\begin{aligned} K'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_\Omega (uu_{tt} + vv_{tt}) \, dx \\ &+ \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) + \varepsilon \int_\Omega (uu_t + vv_t) \, dx. \end{aligned}$$

To compute the second term in the previous equation, we multiply the first two equations on (3.1) respectively by u, v so

$$\begin{aligned} K'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) - \varepsilon \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) \\ &- \varepsilon \int_\Omega \|\nabla u\|_2^{2\gamma} |\nabla u|^2 \, dx - \varepsilon \int_\Omega \|\nabla v\|_2^{2\gamma} |\nabla v|^2 \, dx \\ &+ \varepsilon \int_\Omega \nabla u \int_0^t h_1(t-s) \nabla u(s) \, ds dx + \varepsilon \int_\Omega \nabla v \int_0^t h_2(t-s) \nabla v(s) \, ds dx \\ &- \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, uy(x,1,\varrho,t) \, d\varrho dx - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \, vz(x,1,\varrho,t) \, d\varrho dx \\ &+ \frac{\varepsilon}{2(p+2)} \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2 \|uv\|_{(p+2)}^{(p+2)} \right]. \end{aligned} \tag{3.14}$$

Now, we use Young's inequality to estimate the following terms in $K'(t)$

$$
\begin{aligned}
& \varepsilon \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, uy(x,1,\varrho,t) \, d\varrho dx \\
\leq \; & \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, d\varrho \right) \|u\|_2^2 \right. \\
& \left. + \frac{1}{4\delta_1} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, \|y(x,1,\varrho,t)\|_2^2 \, d\varrho \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& \varepsilon \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \, vz(x,1,\varrho,t) \, d\varrho dx \\
\leq \; & \varepsilon \left\{ \delta_2 \left(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \, d\varrho \right) \|v\|_2^2 \right. \\
& \left. + \frac{1}{4\delta_2} \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \, \|z(x,1,\varrho,t)\|_2^2 \, d\varrho \right\}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \varepsilon \int_0^t h_1(t-s) \, ds \int_{\Omega} \nabla u \, \nabla u(s) \, dx ds \\
= \; & \varepsilon \int_0^t h_1(t-s) \, ds \int_{\Omega} \nabla u \, (\nabla u(s) - \nabla u(t)) \, dx ds \\
& + \varepsilon \left(\int_0^t h_1(s) \, ds \right) \|\nabla u\|_2^2 \\
\geq \; & \frac{\varepsilon}{2} \left(\int_0^t h_1(s) \, ds \right) \|\nabla u\|_2^2 - \frac{\varepsilon}{2} (h_1 o \nabla u),
\end{aligned}
$$

similarly

$$
\begin{aligned}
& \varepsilon \int_0^t h_2(t-s) \, ds \int_{\Omega} \nabla v \, \nabla v(s) \, dx ds \\
= \; & \varepsilon \int_0^t h_2(t-s) \, ds \int_{\Omega} \nabla v \, (\nabla v(s) - \nabla v(t)) \, dx ds \\
& + \varepsilon \left(\int_0^t h_2(s) \, ds \right) \|\nabla v\|_2^2 \\
\geq \; & \frac{\varepsilon}{2} \left(\int_0^t h_2(s) \, ds \right) \|\nabla v\|_2^2 - \frac{\varepsilon}{2} (h_2 o \nabla v).
\end{aligned}
$$

From (3.14), we obtain

$$\begin{aligned} K'(t) \geq\ & (1-\alpha) H^{-\alpha} H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) - \varepsilon \left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)} \right) \\ & -\varepsilon \left(\left(1 - \frac{1}{2}\int_0^t h_1(s)\,ds\right) \|\nabla u\|_2^2 + \left(1 - \frac{1}{2}\int_0^t h_2(s)\,ds\right) \|\nabla v\|_2^2 \right) \\ & -\varepsilon\delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,d\varrho \right) \|u\|_2^2 - \varepsilon\delta_2 \left(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|\,d\varrho \right) \|v\|_2^2 \\ & -\frac{\varepsilon}{2}(h_1 o \nabla u) - \frac{\varepsilon}{4\delta_1} \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|y(x,1,\varrho,t)\|_2^2\,d\varrho \\ & -\frac{\varepsilon}{2}(h_2 o \nabla v) - \frac{\varepsilon}{4\delta_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| \|z(x,1,\varrho,t)\|_2^2\,d\varrho \\ & +\varepsilon \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{(p+2)}^{(p+2)} \right]. \end{aligned} \tag{3.15}$$

Therefore, we take $\delta_1 = \delta_2 = \frac{H^{-\alpha}(t)}{2c_3\kappa}$ and combining (3.10) in (3.15), we find

$$\begin{aligned} K'(t) \geq\ & (1-\alpha) H^{-\alpha} H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) - \varepsilon \left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)} \right) \\ & -\varepsilon \left(1 - \frac{1}{2}\int_0^t h_1(s)\,ds\right) \|\nabla u\|_2^2 + \varepsilon \left(1 - \frac{1}{2}\int_0^t h_2(s)\,ds\right) \|\nabla v\|_2^2 \\ & -\varepsilon \frac{H^\alpha(t)}{2c_3\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\,d\varrho \right) \|u\|_2^2 - \frac{\varepsilon}{2}(h_1 o \nabla u) \\ & -\varepsilon \frac{H^\alpha(t)}{2c_3\kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)|\,d\varrho \right) \|v\|_2^2 - \frac{\varepsilon}{2}(h_2 o \nabla v) \\ & +\varepsilon \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{(p+2)}^{(p+2)} \right]. \end{aligned} \tag{3.16}$$

We multiply (3.9) by $\varepsilon 2(p+2)(1-a)$ with $0 < a < 1$, we obtain

$$\begin{aligned} & \varepsilon \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)} \right] \\ =\ & \varepsilon a \left[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{2(p+2)}^{2(p+2)} \right] + \varepsilon 2(p+2)(1-a) H(t) \\ & +\varepsilon (p+2)(1-a) \left[\left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) + \frac{1}{(\gamma+1)} \left(\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)} \right) \right] \\ & +\varepsilon (p+2)(1-a) \left[l_1 \|\nabla u\|_2^2 + l_2 \|\nabla v\|_2^2 + (h_1 o \nabla u) + (h_2 o \nabla v) + M(y,z) \right]. \end{aligned}$$

By substituting in (3.16), we have

$$\begin{aligned} K'(t) \geq & \left[(1-\alpha)-\varepsilon\kappa\right] H^{-\alpha} H'(t) + \varepsilon\left[(p+2)(1-a)+1\right]\left(\|u_t\|_2^2+\|v_t\|_2^2\right) \\ & +\varepsilon\left[\frac{(p+2)(1-a)}{(\gamma+1)}-1\right]\left(\|\nabla u\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\right) \\ & +\varepsilon\left[(p+2)(1-a)l_1-\left(1-\frac{1}{2}\int_0^t h_1(s)\,ds\right)\right]\|\nabla u\|_2^2 \\ & +\varepsilon\left[(p+2)(1-a)l_2-\left(1-\frac{1}{2}\int_0^t h_2(s)\,ds\right)\right]\|\nabla v\|_2^2 \\ & -\varepsilon\frac{H^\alpha(t)}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)\|u\|_2^2-\varepsilon\frac{H^\alpha(t)}{2c_3\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|\,d\varrho\right)\|v\|_2^2 \\ & +\varepsilon(p+2)(1-a)M(y,z)+\varepsilon\left[(p+2)(1-a)-\frac{1}{2}\right]((h_1 o\nabla u)+(h_2 o\nabla v)) \\ & +\varepsilon a\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{(p+2)}^{(p+2)}\right]+\varepsilon 2(p+2)(1-a)H(t). \end{aligned} \tag{3.17}$$

From (3.11) and (3.12), we can find two positive constants c_4, c_5 such that

$$H^\alpha(t)\|u\|^2 \leq c_4\left(\|u\|_{2(p+2)}^{2\alpha(p+2)+2}+\|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2\right), \tag{3.18}$$

and

$$H^\alpha(t)\|v\|^2 \leq c_5\left(\|v\|_{2(p+2)}^{2\alpha(p+2)+2}+\|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2\right). \tag{3.19}$$

By applying the algebraic inequality,

$$B^\theta \leq (B+1) \leq \left(1+\frac{1}{b}\right)(B+b),\ \forall B>0,\ 0<\theta<1,\ b>0,$$

with $B=\|.\|_{2(p+2)}^{2(p+2)}$, $b=H(0)$ and $\theta=2\alpha(p+2)+2$ which is less than 1 from (3.12), we get

$$\|u\|_{2(p+2)}^{2\alpha(p+2)+2} \leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+H(0)\right) \leq d\left(\|u\|_{2(p+2)}^{2(p+2)}+H(t)\right), \tag{3.20}$$

and

$$\|v\|_{2(p+2)}^{2\alpha(p+2)+2} \leq d\left(\|v\|_{2(p+2)}^{2(p+2)}+H(0)\right) \leq d\left(\|v\|_{2(p+2)}^{2(p+2)}+H(t)\right). \tag{3.21}$$

Also, since the function $F(x)=x^\gamma$ is convex for all positive γ we arrive at

$$\begin{aligned} \|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2 & \leq \left(\|v\|_{2(p+2)}+\|u\|_2\right)^{2(p+2)} \\ & \leq c_6\left(\|v\|_{2(p+2)}^{2(p+2)}+\|u\|_2^{2(p+2)}\right) \\ & \leq c_7\left(\|v\|_{2(p+2)}^{2(p+2)}+\|u\|_{2(p+2)}^{2(p+2)}\right), \end{aligned} \tag{3.22}$$

similarly

$$\|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2 \leq c_8\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right), \tag{3.23}$$

by substituting (3.20)-(3.21) and (3.22)-(3.23) in (3.18)-(3.19) respectively, we get

$$H^{\alpha}(t)\|u\|_2^2 \leq c_9\left(\|v\|_{2(p+2)}^{2(p+2)}+\|u\|_{2(p+2)}^{2(p+2)}\right)+c_9 H(t), \tag{3.24}$$

$$H^{\alpha}(t)\|v\|_2^2 \leq c_{10}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right)+c_{10} H(t). \tag{3.25}$$

Combining (3.24),(3.25) into (3.17) and by using (2.4) with the fact that h_1, h_2 are a positive functions, we find

$$\begin{aligned}
K'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa] H^{-\alpha} H'(t)+\varepsilon[(p+2)(1-a)+1]\left(\|u_t\|_2^2+\|v_t\|_2^2\right) \\
& +\varepsilon\left[\frac{(p+2)(1-a)}{(\gamma+1)}-1\right]\left(\|\nabla u\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\right) \\
& +\varepsilon[(p+2)(1-a) l_1-1]\|\nabla u\|_2^2 \\
& +\varepsilon[(p+2)(1-a) l_2-1]\|\nabla v\|_2^2 \\
& +\varepsilon(p+2)(1-a) M(y, z) \\
& +\varepsilon\left[(p+2)(1-a)-\frac{1}{2}\right]((h_1 o \nabla u)+(h_2 o \nabla v)) \\
& +\varepsilon\left(a c_0-\frac{\lambda_1+\lambda_2}{2 c_3 \kappa}\right)\left[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right] \\
& +\varepsilon\left(2(p+2)(1-a)-\frac{\lambda_1+\lambda_2}{2 c_3 \kappa}\right) H(t)
\end{aligned} \tag{3.26}$$

where $\lambda_1=c_9 \int_{\tau_1}^{\tau_2}|\mu_2(\varrho)| d\varrho$, $\lambda_2=c_{10} \int_{\tau_1}^{\tau_2}|\mu_4(\varrho)| d\varrho$.

In this case, we take $a>0$ small enough such that

$$(p+2)(1-a)\left(\min\left\{\frac{1}{\gamma+1}, l_1, l_2\right\}\right)-1>0,$$

after we choose κ so large such that

$$a c_0-\frac{\lambda_1+\lambda_2}{2 c_3 \kappa}>0$$

and

$$2(p+2)(1-a)-\frac{\lambda_1+\lambda_2}{2 c_3 \kappa}>0.$$

When κ and a be fixed, we take ε small enough, such that

$$(1-\alpha)-\varepsilon\kappa>0.$$

Then we conclude that , for $\beta>0$, the estimate (3.26) becomes

$$\begin{aligned}
K'(t) \geq\ & \beta\left\{H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\right. \\
& \left.+(h_1 o \nabla u)+(h_2 o \nabla v)+M(y, z)+\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right\}.
\end{aligned} \tag{3.27}$$

From (2.4), for $\beta_1>0$, we obtain

$$\begin{aligned}
K'(t) \geq\ & \beta_1\left\{H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}+\|\nabla v\|_2^{2(\gamma+1)}\right. \\
& \left.+(h_1 o \nabla u)+(h_2 o \nabla v)+M(y, z)+\left[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{(p+2)}^{(p+2)}\right]\right\},
\end{aligned}$$

thus

$$K(t) \geq K(0) > 0 \text{ on } R_+.$$

Applying Hölder and Young inequalities, we have

$$\left| \int_\Omega (uu_t + vv_t)\, dx \right|^{1/(1-\alpha)} \geq C \left[\|u\|_{2(p+2)}^{\mu/(1-\alpha)} + \|u_t\|_2^{s/(1-\alpha)} + \|v\|_{2(p+2)}^{\mu/(1-\alpha)} + \|v_t\|_2^{s/(1-\alpha)} \right]$$

where $(1/\mu) + (1/s) = 1$. We put $s = 2(1-\alpha)$, hence

$$\mu/(1-\alpha) = 2/(1-2\alpha) \leq 2(p+2).$$

By using again (3) with $\theta = 2(1-2\alpha)$, we have

$$\|u\|_{2(p+2)}^{2/(1-2\alpha)} \leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + H(t) \right),$$

and

$$\|v\|_{2(p+2)}^{2/(1-2\alpha)} \leq d\left(\|v\|_{2(p+2)}^{2(p+2)} + H(t) \right), \ \forall t \geq 0.$$

Hence,

$$\left| \int_\Omega (uu_t + vv_t)\, dx \right|^{1/(1-\alpha)} \geq c_{12} \left[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + H(t) \right].$$

As a result,

$$\begin{aligned} K^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon \int_\Omega (uu_t + vv_t)\, dx + \frac{\varepsilon}{2}\left(\mu_1 \|u\|_2^2 + \mu_3 \|v\|_2^2 \right) \right)^{1/(1-\alpha)} \\ &\leq C \left\{ H(t) + \left| \int_\Omega (uu_t + vv_t)\, dx \right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)} + \|v\|_2^{2/(1-\alpha)} \right\}. \\ &\leq c \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_2^2 + \|v\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right]. \end{aligned} \tag{3.28}$$

By recalling (3.27), (3.28) leads

$$K'(t) \geq \lambda K^{1/(1-\alpha)}(t), \tag{3.29}$$

with $\lambda > 0$, this quantity depends on β and c. A simple integration of (3.29) gives

$$K^{\alpha/(1-\alpha)}(t) \geq \frac{1}{K^{-\alpha/(1-\alpha)}(0) - \lambda(\alpha/(1-\alpha))\, t}.$$

Thus, $K(t)$ in a situation of blow up when the time approach to

$$T^* = \frac{1-\alpha}{\lambda \alpha K^{\alpha/(1-\alpha)}(0)}.$$

The proof is completed. □

Acknowledgment. The authors thank the referees for their useful remarks and suggestions. The third author would like to express her sincere thanks to **D.G.R.S.D.T** for its support.

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