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A BOUNDARY VALUE PROBLEM FOR A CLASS OF MIXED TYPE EQUATIONS

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ABSTRACT. In this paper we study a problem with conditions given on inner characteristics in hyperbolic part of considered domain and on some parts of the line of parabolic degeneracy. With the help of the method of integral equations and the principle of extremum we prove the unique solvability of the investigated problem.

1. Introduction and formulation of a problem

Sweden mathematician Gellerstedt [1] was investigated boundary value problem for the equation $y^m u_{xx} + u_{yy} = 0$, (m is odd number), in which values of sought function are given on two pieces of characteristics and on curve $x^2 + 4/(m+2)^2 y^{m+2} = 1 (y > 0)$ the value of its derivative is given. This problem has applications in transonic gas dynamics [2]. The work [3] is devoted to the studying Gellerstedt problem with data on one family of characteristics and with nonlocal gluing conditions. The unique solvability of the Gellerstedt problem for parabolic-hyperbolic equation of the second kind was investigated in [4]. Due to applications in gas dynamics the interest on studying boundary-value problems for degenerate elliptic and mixed type equations with singular coefficients is growing. Note latest work [5] on this topic, where the Dirichlet problem for three-dimensional elliptic equation with singular coefficients was considered. Nonlocal boundary-value problems for the degenerate elliptic equation and mixed type equations in unbounded domains were studied in the works [6], [7], [8], [9].

Consider the equation

$$u_{xx}(signy) + u_{yy} + (\beta_0/y)u_y = 0 \tag{1.1}$$

in the domain $D = D^+ \cup D^- \cup I$ of complex plane $z = x + iy$, where D^+ is a first quadrant of the plane, D^- is a finite domain in the fourth quadrant of the plane bounded by characteristics OC and BC of Equation (1.1) issuing from points $O(0,0)$, $B(1,0)$, and by the segment OB of the straight line $y = 0$, $I = \{(x,y) : 0 < x < 1, y = 0\}$. In Equation (1.1) β_0 is some real number such that $0 < \beta_0 < 1$.

Introduce the following denotations: $I_0 = \{(x,y) : 0 < y < \infty, x = 0\}$, $I_1 = \{(x,y) : 1 < x < \infty, y = 0\}$, C_0 and C_1 are, correspondingly, points of intersection of characteristics OC and BC with the characteristic issuing from the point $E(c,0)$, where $c \in I$ is an arbitrary fixed number.

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Let $p(x) \in C^1[0,c]$ be a diffeomorphism from the set of points of the segment $[0,c]$ to the set of points of the segment $[c,1]$ such that $p'(x) < 0$, $p(0) = 1$, $p(c) = c$. As an example of such a function consider the linear function $p(x) = 1 - kx$, where $k = \frac{1-c}{c}$.

Problem G In the domain D find a function $u(x,y)$ with the properties:
1)$u(x,y) \in C(\bar{D})$ where $\bar{D} = \bar{D^-} \cup D^+ \cup \bar{I_0} \cup \bar{I_1}$;
2) $u(x,y) \in C^2(D^+)$ and satisfies Equation (1.1) in this domain;
3) $u(x,y)$ is a generalized solution from the class R_1 [10] in the domain D^-;
4) the following relations hold:

$$\lim_{R\to\infty} u(x,y) = 0, R^2 = x^2 + y^2,\ x \geq 0,\ y \geq 0; \tag{1.2}$$

5) $u(x,y)$ satisfies the boundary conditions

$$u(0,y) = \varphi(y),\ y \geq 0, \tag{1.3}$$

$$u(x,0) = \tau_1(x),\ \ x \in \bar{I_1}, \tag{1.4}$$

$$u(x,y)|_{EC_0} = \psi(x),\ (c/2) \leq x \leq c, \tag{1.5}$$

$$u(p(x),0) = \mu u(x,0) + f(x),\ 0 \leq x \leq c, \tag{1.6}$$

and the transmission condition

$$\lim_{y\to+0} y^{\beta_0} u_y = \lim_{y\to-0} (-y)^{\beta_0} u_y, \quad x \in I \setminus \{c\}, \tag{1.7}$$

moreover, for $x = 0$, $x = 1$ and $x = c$ these limits can have singularities of order less than $1 - 2\beta$, where $\beta = \frac{\beta_0}{2}$, $f(x) \in C[0,c] \cap C^{1,\delta_1}(0,c)$, $f(0) = 0$, $f(c) = 0$, $\psi(x) \in C[\frac{c}{2},c] \cap C^{1,\delta_2}(\frac{c}{2},c)$, $\psi(c) = 0$, $\tau_1(x) \in C(\bar{I_1})$, moreover, the function $\tau_1(x)$ near the point $x = 1$ is representable in the form $\tau_1(x) = (1-x)\tilde{\tau_1}(x)$, $\tilde{\tau_1}(x) \in C(\bar{I_1})$ and for sufficiently large x it satisfies the inequality $|\tau_1(x)| \leq \frac{M}{x^\varepsilon}$, ε, M are positive constants, $\tau_1(x)$ satisfies the Hölder condition on any segment $[1,N]$, $N > 1$, $\varphi(y) \in C(\bar{I_0})$, $y^{\frac{\beta_0}{2}}\varphi(y) \in L(0,\infty)$, $\varphi(y)$ satisfies the Hölder condition on any segment $[0,N]$, $N > 0$, $\varphi(\infty) = 0$, $\varphi(0) = 0$.

Note, in the Gellerstedt problem [1], the values of sought function in the hyperbolic part of mixed domain D are given on the characteristics EC_0 and EC_1 :

$$u|_{EC_0} = \psi_1(x),\ u|_{EC_1} = \psi_2(x).$$

In the present work we study new boundary-value problem where characteristics EC_1 is free from the conditions and needed condition of Gellerstedt is replaced by inner boundary condition with local shifting on the parabolic line of degeneracy.

2. Main results

Theorem 2.1. *Let $\varphi(y) \equiv 0$, $\psi(x) \equiv 0$, $f(x) \equiv 0$, $\tau_1(x) \equiv 0$, $0 < \mu < 1$. Then Problem G can have only the trivial solution.*

Proof. It is known that the solution of the modified Cauchy problem $u(x,0) = \tau(x)$, $x \in \bar{I}$, $\lim\limits_{y\to -0}(-y)^{\beta_0}u_y = \nu(x)$, $x \in I$, has the form

$$\begin{aligned} u(x,y) = \gamma_1 \int_0^1 \tau\left(x - y(2t-1)\right) t^{\beta-1}(1-t)^{\beta-1}dt \\ -\gamma_2(-y)^{1-\beta_0}\int_0^1 \nu\left(x - y(2t-1)\right) t^{-\beta}(1-t)^{-\beta}dt, \end{aligned} \tag{2.1}$$

where $\gamma_1 = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)}$, $\gamma_2 = \frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}$, $\Gamma(z)$ is gamma function [10]. By virtue of the formula (2.1), from the boundary condition (1.5), after simple calculations, we obtain

$$\nu(X) = \gamma D^{1-2\beta}_{X,c}\tau(X) + \Psi(X), X \in (0,c), \tag{2.2}$$

$\Psi(X) = -\frac{(c-X)^{\beta}D^{1-\beta}_{X,c}\psi\left(\frac{X+c}{2}\right)}{\gamma_2\left(\frac{1}{2}\right)^{1-2\beta}\Gamma(1-\beta)}$, $\gamma = \frac{2\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(\beta)\Gamma(1-2\beta)}\left(\frac{1}{2}\right)^{2\beta}$, $X = 2x - c$, $D^{l}_{X,c}$ is a fractional differentiation operator in the sense of Riemann-Liouville [10]. The equality (2.2) is the first functional relation between the unknown functions $\tau(x)$ and $\nu(x)$, brought to the interval $(0,c)$ of the axis $y = 0$ from the hyperbolic part D^- mixed domain D. Now let us prove that if $\varphi(y) \equiv 0$, $\psi(x) \equiv 0$, $f(x) \equiv 0$, $\tau_1(x) \equiv 0$, $0 < \mu < 1$, then the solution of the problem G in the domain $D^+ \cup I_0 \cup \bar{I} \cup I_1$ by virtue of (1.2), is identically equal to zero. Let (x_0, y_0) be a point of positive maximum of the function $u(x,y)$ in the domain $\bar{D}^+_R$.

Let D^+_R be a finite domain cut out of the domain D^+ by the arc A_RB_R of the circle $x^2 + y^2 = R^2$, $0 \le x \le R$, $0 \le y \le R$, where A_R and B_R are points with coordinates $(0,R)$ and $(R,0)$ correspondingly. In view of formula (1.2) for any $\varepsilon > 0$ there exists $R_0 = R_0(\varepsilon)$, such that for $R > R_0(\varepsilon)$ the inequality

$$|u(x,y)| < \varepsilon, \ \ (x,y) \in A_RB_R. \tag{2.3}$$

By virtue of the notation $u(x,0) = \tau(x)$, $\quad x \in \bar{I}$, the condition (1.6) rewritten in the form

$$\tau(p(x)) = \mu\tau(x) + f(x), \ x \in [0,c]. \tag{2.4}$$

Hence, for $x = c$ (where $f(x) \equiv 0$) we have $\tau(p(c)) = \mu\tau(c)$. Then by virtue of the equality $p(c) = c$ it follows that $\tau(c)(1-\mu) = 0$, i.e. $\tau(c) = 0$.

According to the Hopf principle [11], the function $u(x,y)$ does not attain its positive maximum and negative minimum at the inner points of the domain $\bar{D}^+_R$. Let $(x_0, 0)$ (where $x_0 \in (0,c)$) be the point of positive maximum (negativ minimum) of the function $u(x,0) = \tau(x)$. Then at this point in the case of a positive maximum (negativ minimum) [7]

$$\nu(x_0) < 0 \ (\nu(x_0) > 0). \tag{2.5}$$

It is well known that at the point of the positive maximum (negative minimum) of the function $\tau(x)$ the fractional differentiation operators satisfy the inequality $D^{1-2\beta}_{x,c}\tau(x)|_{x=x_0}\tau(x) > 0$ $(D^{1-2\beta}_{x,c}\tau(x)|_{x=x_0}\tau(x) < 0)$. Then by virtue of (where $\Psi(x) \equiv 0$), we have

$$\nu(x_0) = \gamma D^{1-2\beta}_{x,c}\tau(x)|_{x=x_0} > 0(\nu(x_0) = \gamma D^{1-2\beta}_{x,c}\tau(x)|_{x=x_0} < 0). \tag{2.6}$$

Inequalities (2.5) and (2.6) contradict the conjugation condition (1.7), whence we deduce that $x_0 \notin (0,c)$. By virtue of $0 < \mu < 1$ from (2.4) (where $f(x) \equiv 0$) it follows that they are also absent in the interval $(c,1)$ of the axis $y = 0$. Consequently, there are no points of positive maximum (negative minimum) of the function $u(x,y)$ on the interval AB. Let $R > R_0$. From the Hopf principle and the previous reasoning, if $(x_0, y_0) \in A_R B_R$, then by virtue of (2.3) we have $|u(x_0,y_0)| < \varepsilon$. Therefore, $|u(x,y)| < \varepsilon$ for any $(x,y) \in \bar{D}_R^+$.

Since $\varepsilon > 0$ is arbitrary, with $R \to +\infty$ we conclude that $u(x,y) \equiv 0$ in the domain $D^+ \cup I_0 \cup \bar{I} \cup I_1$. Hence,

$$\lim_{y\to+0} u(x,y) = 0,\ x \in \bar{I};\ \lim_{y\to+0} y^{\beta_0} u_y = 0, x \in I. \tag{2.7}$$

Taking into account (2.7), due to the continuity of the solution in the domain $\bar{D}_R^+$ and the conjugation condition (1.7),restoring the sought function $u(x,y)$ in the domain D^- as a solution of the modified Cauchy problem with homogeneous data, we obtain $u(x,y) \equiv 0$ in the domain $\bar{D}^-$. □

Theorem 2.2. *Let the conditions* $\mu k^{\frac{1}{2}-3\alpha} sin(\alpha\pi) < 1$, *where* $\alpha = (1-2\beta)/4$, $\beta_0 > \frac{1}{3}$, $p(x) = 1 - kx$. *Then the solution of the problem G exists.*

Proof. The solution of the Dirichlet problem in the domain D^+ satisfying the conditions (1.2)-(1.4) and the condition $u(x,0) = \tau(x),\ x \in \bar{I}$, can be represented in the form

$$\begin{aligned} u(x,y) &= k_2 y^{1-\beta_0} \int_0^1 \tau(t) \left(\left((t-x)^2 + y^2\right)^{\beta-1} - \left((t+x)^2 + y^2\right)^{\beta-1} \right) dt \\ &+ k_2 y^{1-\beta_0} \int_1^\infty \tau_1(t) \left(\left((t-x)^2 + y^2\right)^{\beta-1} - \left((t+x)^2 + y^2\right)^{\beta-1} \right) dt \\ &+ y^{\frac{1-\beta_0}{2}} \int_0^\infty t^{\frac{1+\beta_0}{2}} \varphi(t) dt \int_0^\infty s e^{-sx} J_{\frac{1-2\beta}{2}}(st) J_{\frac{1-2\beta}{2}}(sy) ds, \end{aligned} \tag{2.8}$$

where $k_2 = \frac{2^{2\beta}}{\pi} \frac{\Gamma^2(1-\beta)(1-\beta_0)}{\Gamma(2-2\beta)}$, $\beta = \frac{\beta_0}{2}$, $J_\nu(z)$ is the Bessel function of the first kind.

From the formula (2.8), after some calculations, we obtain

$$\begin{aligned} &\nu(x) = -k_2 \int_0^1 \tau'(t) \left[(x-t)|x-t|^{2\beta-2} + (t+x)^{2\beta-1} \right] dt + \Phi_0(x), \\ &x \in (0,1), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \Phi_0(x) = \lim_{y\to+0} y^{\beta_0} \frac{\partial}{\partial y} \left(F_1(x,y) + F_2(x,y) \right) = k_2(1-\beta_0) \int_0^\infty \tau_1(t) \\ \times \left[(t-x)^{2\beta-2} - (t+x)^{2\beta-2} \right] dt \\ + \frac{2}{(2)^{\frac{1-2\beta}{2}} \Gamma(\frac{1}{2} - \beta)} \int_0^\infty \varphi(t) t^{\frac{1+\beta_0}{2}} dt \int_0^\infty s^{\frac{3-2\beta}{2}} e^{-sx} J_{\frac{1-2\beta}{2}}(s) ds. \end{aligned}$$

Equality (2.9) is a functional relation between unknown functions $\tau(x)$ and $\nu(x)$, brought to I from the elliptic part D^+ of the mixed domain D. Note that the relation (2.9) is valid for the entire interval I. Further, dividing the integration interval $(0,1)$ into intervals $(0,c)$ and $(c,1)$, and then in the integrals with the limit $(c,1)$ making the change of variable integration $t=p(s)=1-ks$ and taking into account the equality (2.4), the relation (2.9) is reduced to the form

$$\nu(x)=-k_2\left(\int_0^x \tau'(t)(x-t)^{2\beta-1}dt-\int_x^c \tau'(t)(t-x)^{2\beta-1}dt\right)$$
$$-k_2\left(\int_0^c \tau'(t)(t+x)^{2\beta-1}dt+\mu\int_0^c \tau'(s)\left[(p(s)-x)^{2\beta-1}-(p(s)+x)^{2\beta-1}\right]ds\right)$$
$$+\Phi_1(x),\ x\in(0,c), \tag{2.10}$$

where

$$\Phi_1(x)=-k_2\int_0^c f'(s)\left[(p(s)-x)^{2\beta-1}-(p(s)+x)^{2\beta-1}\right]ds+\Phi_0(x).$$

By virtue of (1.7), excluding the function $\nu(x)$ from (2.2) and (2.10), we obtain

$$\gamma D_{x,c}^{1-2\beta}\tau(x)+\Psi(x)=-k_2\left(\int_0^x \tau'(t)(x-t)^{2\beta-1}dt-\int_x^c \tau'(t)(t-x)^{2\beta-1}dt\right)$$
$$-k_2\left(\int_0^c \tau'(t)(t+x)^{2\beta-1}dt+\mu\int_0^c \tau'(s)\left[(p(s)-x)^{2\beta-1}-(p(s)+x)^{2\beta-1}\right]ds\right)$$
$$+\Phi_1(x),\ x\in(0,c). \tag{2.11}$$

Equality (2.11) can be rewritten in the form

$$-\frac{\gamma}{k_2}D_{x,c}^{1-2\beta}\tau(x)+\Phi_2(x)=\int_0^x \tau'(t)(x-t)^{2\beta-1}dt-\int_x^c \tau'(t)(t-x)^{2\beta-1}dt$$
$$+\int_0^c \tau'(t)(t+x)^{2\beta-1}dt+\mu\int_0^c \tau'(s)\left[(p(s)-x)^{2\beta-1}-(p(s)+x)^{2\beta-1}\right]ds,$$
$$x\in(0,c), \tag{2.12}$$

where $\Phi_2(x)=-k_2\left(\Psi(x)-\Phi_1(x)\right)$. Applying the operator $\Gamma(1-2\beta)D_{x,c}^{2\beta-1}$ to both sides of the equality (2.12) and taking into account that $D_{x,c}^{2\beta-1}D_{x,c}^{1-2\beta}\tau(x)=\tau(x)$, we have

$$-\frac{\gamma}{k_2}\Gamma(1-2\beta)\tau(x)+\Gamma(1-2\beta)D_{x,c}^{2\beta-1}\Phi_2(x)=\Gamma(1-2\beta)D_{x,c}^{2\beta-1}$$
$$\times\left(\int_0^x \tau'(t)(x-t)^{2\beta-1}dt-\int_x^c \tau'(t)(t-x)^{2\beta-1}dt\right)+\Gamma(1-2\beta)D_{x,c}^{2\beta-1}$$
$$\times\left(\int_0^c \tau'(t)(t+x)^{2\beta-1}dt+\mu\int_0^c \tau'(s)\left[(p(s)-x)^{2\beta-1}-(p(s)+x)^{2\beta-1}\right]ds\right)$$
$$,x\in(0,c). \tag{2.13}$$

It is easy to see that

$$\Gamma(1-2\beta)D_{x,c}^{2\beta-1}\int_0^x \tau'(t)(x-t)^{2\beta-1}dt = -\Gamma(2\beta)\Gamma(1-2\beta)cos(2\pi\beta)\tau(x)$$
$$+\int_0^c \left(\frac{c-x}{c-t}\right)^{1-2\beta}\frac{\tau(t)dt}{t-x}, \tag{2.14}$$

$$\Gamma(1-2\beta)D_{x,c}^{2\beta-1}\int_x^c \tau'(t)(t-x)^{2\beta-1}dt = -\Gamma(2\beta)\Gamma(1-2\beta)\tau(x), \tag{2.15}$$

$$\Gamma(1-2\beta)D_{x,c}^{2\beta-1}\int_0^c \tau'(t)(t+x)^{2\beta-1}dt = \int_0^c \left(\frac{c-x}{c+t}\right)^{1-2\beta}\frac{\tau(t)dt}{t+x}, \tag{2.16}$$

$$\begin{gathered}\Gamma(1-2\beta)\mu D_{x,c}^{2\beta-1}\int_0^c \tau'(s)(p(s)-x)^{2\beta-1}ds \\ =\mu\int_0^c \left(\frac{c-x}{p(s)-c}\right)^{1-2\beta}\frac{\tau(s)p'(s)ds}{p(s)-x},\end{gathered} \tag{2.17}$$

$$\begin{gathered}\mu\Gamma(1-2\beta)\mu D_{x,c}^{2\beta-1}\textstyle\int_0^c \tau'(s)(p(s)+x)^{2\beta-1}ds \\ =\mu\displaystyle\int_0^c \left(\frac{c-x}{p(s)+c}\right)^{1-2\beta}\frac{\tau(s)p'(s)ds}{p(s)+x}.\end{gathered} \tag{2.18}$$

Due to (2.14)-(2.18), equality (2.13) can be written in the form

$$\Gamma(1-2\beta)D_{x,c}^{2\beta-1}\Phi_2(x) - \mu\int_0^c \left(\frac{c-x}{p(s)-c}\right)^{1-2\beta}\frac{\tau(s)p'(s)ds}{p(s)-x}$$
$$+\mu\int_0^c \left(\frac{c-x}{p(s)+c}\right)^{1-2\beta}\frac{\tau(s)p'(s)ds}{p(s)+x} + \int\limits_0^c \left(\left(\frac{c-x}{c-t}\right)^{1-2\beta} - \left(\frac{c-x}{c+t}\right)^{1-2\beta}\right)$$
$$\times\frac{\tau(t)dt}{t+x}$$
$$=\frac{\pi(1+sin(\pi\beta))}{cos(\pi\beta)}\tau(x) + \int_0^c \left(\frac{c-x}{c-t}\right)^{1-2\beta}\left(\frac{1}{t-x}+\frac{1}{t+x}\right)\tau(t)dt. \tag{2.19}$$

Equality (2.19) can be rewritten in the form

$$\tau(x) + \lambda\int_0^c \left(\frac{c-x}{c-t}\right)^{1-2\beta}\left(\frac{1}{t-x}+\frac{1}{t+x}\right)\tau(t)dt$$
$$= -\lambda\mu\int_0^c \left(\frac{c-x}{p(s)-c}\right)^{1-2\beta}\frac{\tau(s)p'(s)ds}{p(s)-x}$$
$$+\lambda\mu\int_0^c \left(\frac{c-x}{p(s)+c}\right)^{1-2\beta}\frac{\tau(s)p'(s)ds}{p(s)+x} \tag{2.20}$$
$$+\lambda\int\limits_0^c \left(\left(\frac{c-x}{c-t}\right)^{1-2\beta} - \left(\frac{c-x}{c+t}\right)^{1-2\beta}\right)\frac{\tau(t)dt}{t+x} + \Phi_3(x),\ x\in[0,c],$$

where $\lambda = \frac{cos(\pi\beta)}{\pi(1+sin(\pi\beta))}$, $\Phi_3(x) = \lambda\Gamma(1-2\beta)D_{x,c}^{2\beta-1}\Phi_2(x)$. Equality (2.20) can be written in the form

$$\begin{aligned} &\tau(x) + \lambda \int_0^c \left(\frac{c-x}{c-t}\right)^{1-2\beta} \left(\frac{1}{t-x} + \frac{1}{t+x}\right)\tau(t)dt \\ &= -\lambda\mu \int_0^c \left(\frac{c-x}{p(s)-c}\right)^{1-2\beta} \frac{\tau(s)p'(s)ds}{p(s)-x} + R[\tau] + \Phi_3(x),\ x \in [0,c], \end{aligned} \tag{2.21}$$

where

$$R[\tau] = \lambda\mu \int_0^c \left(\frac{c-x}{p(s)+c}\right)^{1-2\beta} \frac{\tau(s)p'(s)ds}{p(s)+x}$$
$$+\lambda \int_0^c \left(\left(\frac{c-x}{c-t}\right)^{1-2\beta} - \left(\frac{c-x}{c+t}\right)^{1-2\beta}\right) \frac{\tau(t)dt}{t+x}$$

is a regular operator. The first integral operator in the right-hand side of (2.21) is not regular, because the integrand with $x = c$ and $s = c$ has an isolated first-order singularity, so this term in (2.21) is separated. For a time being we assume the right-hand side of (2.21) to be a known function and rewrite the equation as follows

$$\tau(x) + \lambda \int_0^c \left(\frac{c-x}{c-t}\right)^{1-2\beta} \left(\frac{1}{t-x} + \frac{1}{t+x}\right)\tau(t)dt = g_0(x),\ x \in [0,c], \tag{2.22}$$

where

$$g_0(x) = -\lambda\mu \int_0^c \left(\frac{c-x}{p(s)-c}\right)^{1-2\beta} \frac{\tau(s)p'(s)ds}{p(s)-x} + R[\tau] + \Phi_3(x). \tag{2.23}$$

Putting $(c-x)^{2\beta-1}\tau(x) = \rho(x)$, $(c-x)^{2\beta-1}g_0(x) = g_1(x)$, we write Equation (2.22) in the form

$$\rho(x) + \lambda \int_0^c \left(\frac{1}{t-x} + \frac{1}{t+x}\right)\rho(t)dt = g_1(x),\ x \in [0,c]. \tag{2.24}$$

In Equation (2.24) changing variables $t^2 = s$, $x^2 = \xi$, $\rho(x) = \rho(\sqrt{\xi}) = \rho_1(\xi)$, $g_1(x) = g_1(\sqrt{\xi}) = g_2(\xi)$, we obtain the following singular integral equation

$$\rho_1(\xi) + \lambda \int_0^{c^2} \frac{\rho_1(s)ds}{s-\xi} = g_2(\xi),\ \xi \in [0,c^2]. \tag{2.25}$$

We will seek solution of Equation (2.25) in the class of functions satisfying Hölder condition on $(0,c^2)$ and bounded at $\xi = 0$, such that at $\xi = c^2$ they can tend to infinity of order less than $1-2\beta$. The only solution of the Equation (2.25) in the class $h(0)$ is expressed by the formula

$$\begin{aligned} \rho_1(\xi) = &\frac{1+sin(\beta\pi)}{2}g_2(\xi) - \frac{cos(\beta\pi)}{2\pi}\left(\frac{\xi}{c^2-\xi}\right)^{\frac{1}{4}(1-2\beta)} \\ &\times \int_0^{c^2} \frac{g_2(t)dt}{\left(\frac{t}{c^2-t}\right)^{\frac{1}{4}(1-2\beta)}(t-\xi)}. \end{aligned} \tag{2.26}$$

Returning to the previous variables and functions from (2.26), we have

$$\begin{aligned}\tau(x) &= cos^2(\pi\alpha)g_0(x) - \frac{sin(2\pi\alpha)}{2\pi}\\ &\times \int_0^c \left(\frac{x}{t}\right)^{2\alpha}\left(\frac{c+t}{c+x}\right)^{\alpha}\left(\frac{c-x}{c-t}\right)^{3\alpha}\left(\frac{1}{t-x}+\frac{1}{t+x}\right)g_0(t)dt,\end{aligned} \tag{2.27}$$

where $\alpha = (1-2\beta)/4$. Substituting (2.23) to (2.27), we obtain

$$\begin{aligned}\tau(x) &= -\lambda\mu cos^2(\lambda\pi)\int_0^c \left(\frac{c-x}{p(s)-c}\right)^{4\alpha}\frac{p'(s)\tau(s)ds}{p(s)-x}\\ &+\lambda\mu\frac{sin(2\pi\alpha)}{2\pi}\int_0^c \tau(s)p'(s)ds\int_0^c \left(\frac{x}{t}\right)^{2\alpha}\left(\frac{c+t}{c+x}\right)^{\alpha}\left(\frac{c-x}{c-t}\right)^{3\alpha}\\ &\times\left(\frac{1}{t-x}+\frac{1}{t+x}\right)\left(\frac{c-t}{p(s)-c}\right)^{4\alpha}\frac{dt}{p(s)-t}+R_1[\tau]+\Phi_4(x),\end{aligned} \tag{2.28}$$

where

$$R_1[\tau] = cos^2(\pi\alpha)R[\tau] - \frac{sin(2\pi\alpha)}{2\pi}\int_0^c \left(\frac{x}{t}\right)^{2\alpha}\left(\frac{c+t}{c+x}\right)^{\alpha}\left(\frac{c-x}{c-t}\right)^{3\alpha}$$

$$\times\left(\frac{1}{t-x}+\frac{1}{t+x}\right)R[\tau]dt$$

is a regular operator,

$$\Phi_4(x) = cos^2(\pi\alpha)\Phi_3(x) - \frac{sin(2\pi\alpha)}{2\pi}\int_0^c \left(\frac{x}{t}\right)^{2\alpha}\left(\frac{c+t}{c+x}\right)^{\alpha}\left(\frac{c-x}{c-t}\right)^{3\alpha}$$

$$\times\left(\frac{1}{t-x}+\frac{1}{t+x}\right)\Phi_3(t)dt$$

is a known function. We write Equation (2.28) in the form

$$\begin{aligned}\tau(x) &= -\lambda\mu cos^2(\lambda\pi)\int_0^c \left(\frac{c-x}{p(s)-c}\right)^{4\alpha}\frac{p'(s)\tau(s)ds}{p(s)-x}\\ &+\lambda\mu\frac{sin(2\pi\alpha)}{2\pi}\int_0^c \tau(s)p'(s)ds\int_0^c \left(\frac{x}{t}\right)^{2\alpha}\left(\frac{c-x}{c-t}\right)^{3\alpha}\\ &\times\left(\frac{c-t}{p(s)-c}\right)^{4\alpha}\left(\frac{1}{t-x}+\frac{1}{t+x}\right)\frac{dt}{p(s)-t}+R_2[\tau]+\Phi_4(x), x\in(0,c),\end{aligned} \tag{2.29}$$

where

$$R_2[\tau] = R_1[\tau] + \lambda\mu\frac{sin(2\pi\alpha)}{2\pi}\int_0^c \tau(s)p'(s)ds$$

$$\times\int_0^c \left(\frac{x}{t}\right)^{2\alpha}\left(\frac{c-x}{c-t}\right)^{3\alpha}\left(\frac{c-t}{p(s)-c}\right)^{4\alpha}\left[\left(\frac{c+t}{c+x}\right)^{\alpha}-1\right]$$

$$\times\left(\frac{1}{t-x}+\frac{1}{t+x}\right)\frac{dt}{p(s)-t}$$

is a regular operator. Since $p(x) = 1 - kx$, from Equation (2.29) we have

$$\tau(x) = \lambda\mu k^{1-4\alpha} cos^2(\lambda\pi) \int_0^c \left(\frac{c-x}{c-s}\right)^{4\alpha} \frac{\tau(s)ds}{1-ks-x}$$
$$-\lambda\mu k^{1-4\alpha} \frac{sin(2\pi\alpha)}{2\pi} \int_0^c \tau(s) \frac{ds}{(c-s)^{4\alpha}} \int_0^c \left(\frac{x}{t}\right)^{2\alpha} \left(\frac{c-x}{c-t}\right)^{3\alpha} \tag{2.30}$$
$$\times (c-t)^{4\alpha} \left(\frac{1}{t-x} + \frac{1}{t+x}\right) \frac{dt}{1-ks-t} + R_2[\tau] + \Phi_4(x), x \in (0,c).$$

It is easy to verify that the value of the inner integral in (2.30) has the form

$$A(x,s) = \int_0^c \left(\frac{x}{t}\right)^{2\alpha} (c-x)^{3\alpha}(c-t)^{\alpha} \left(\frac{1}{t-x} + \frac{1}{t+x}\right) \frac{dt}{1-ks-t}$$
$$= \frac{(c-x)^{3\alpha} x^{2\alpha}}{1-ks-x} \left(\frac{(c-x)^{\alpha}}{x^{2\alpha}} \pi ctg(\pi\alpha)\right.$$
$$\left.-\frac{c^{\alpha}}{x^{2\alpha}} \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} F(1-2\alpha, -\alpha, 1-\alpha; \frac{c-x}{c})\right) + \frac{(c-x)^{3\alpha} x^{2\alpha}}{1-ks-x} \tag{2.31}$$
$$\times \left(\frac{c^{1-\alpha}}{1-ks} \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} F(1-2\alpha, 1, 1-\alpha; \frac{k(c-s)}{1-ks})\right.$$
$$\left.-\Gamma(\alpha)\Gamma(1-\alpha) \frac{k^{\alpha}(c-s)^{\alpha}}{(1-ks)^{2\alpha}}\right) + B_0(x,s),$$

where

$$B_0(x,s) = \frac{(c-x)^{3\alpha} x^{2\alpha}}{1-ks+x}$$
$$\times \left(\frac{c^{1-\alpha}}{x^{2\alpha}(c+x)^{1-2\alpha}} \frac{\Gamma(1-2\alpha)\Gamma(1+\alpha)}{\Gamma(2-\alpha)} F(1-\alpha, 1-\alpha, 2-\alpha; \frac{c}{c+x})\right)$$
$$+\frac{(c-x)^{3\alpha} x^{2\alpha}}{1-ks+x} \left(\frac{c^{1-\alpha}}{1-ks} \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} F(1-2\alpha, 1, 1-\alpha; \frac{k(c-s)}{1-ks})\right.$$
$$\left.-\Gamma(\alpha)\Gamma(1-\alpha) \frac{k^{\alpha}(c-s)^{\alpha}}{(1-ks)^{2\alpha}}\right),$$

here $B_0(x,s)$ is a continuously differentiable function in the square $[0,c] \times [0,c]$. Substituting (2.31) to (2.30), we get

$$\tau(x) = \lambda\mu k^{1-3\alpha} cos(\pi\alpha) \int_0^c \left(\frac{c-x}{c-s}\right)^{3\alpha} \left(\frac{x}{1-ks}\right)^{2\alpha} \frac{\tau(s)ds}{1-ks-x} \tag{2.32}$$
$$+R_3[\tau] + \Phi_4(x), \; x \in (0,c),$$

where

$$R_3[\tau] = R_2[\tau] + \lambda\mu k^{1-4\alpha} \frac{sin(2\pi\alpha)}{2\pi} \frac{\Gamma(1-2\alpha)\Gamma(\alpha)}{\Gamma(1-\alpha)} \int_0^c \frac{\tau(s)(c-x)^{3\alpha} c^{\alpha}}{(c-s)^{4\alpha}(1-ks-x)}$$
$$\times \left(c^{\alpha} F\left(1-2\alpha, -\alpha, 1-\alpha; \frac{c-x}{c}\right) - x^{2\alpha} \frac{c^{1-\alpha}}{1-ks}\right.$$
$$\left.\times F\left(1-2\alpha, 1, 1-\alpha; \frac{k(c-x)}{1-ks}\right) ds\right) - \lambda\mu k^{1-4\alpha} \frac{sin(2\pi\alpha)}{2\pi} \int_0^c \frac{B_0(x,s)\tau(s)ds}{(c-s)^{4\alpha}}$$

is a regular operator. Equality (2.32) can be written in the form

$$\tau(x) = \lambda\mu k^{1-3\alpha} cos(\pi\alpha) \int_0^c \left(\frac{c-x}{c-s}\right)^{3\alpha} \frac{\tau(s)ds}{1-ks-x} + R_4[\tau] + \Phi_4(x), \quad x \in (0,c), \tag{2.33}$$

where

$$R_4[\tau] = R_3[\tau] + \lambda\mu k^{1-3\alpha} cos(\pi\alpha) \int_0^c \left(\frac{c-x}{c-s}\right)^{3\alpha} \frac{\tau(s)}{1-ks-x} \left[\left(\frac{x}{1-ks}\right)^{2\alpha} - 1\right] ds$$

is a regular operator. The Equation (2.33) can be rewritten as

$$\tau(x) = \lambda\mu k^{1-3\alpha} cos(\pi\alpha) \int_0^c \left(\frac{c-x}{c-s}\right)^{3\alpha} \frac{\tau(s)ds}{(c-s)\left[k+\frac{c-x}{c-s}\right]} + R_4[\tau] + \Phi_4(x), \quad x \in (0,c). \tag{2.34}$$

After changing the variables $x = c - ce^{-\xi}$, $s = c - ce^{-t}$ and introducing the notation

$\rho(\xi) = \tau(c - ce^{-\xi})e^{(3\alpha-\frac{1}{2})\xi}$ Equation (2.34) takes the form

$$\rho(\xi) = \lambda\mu k^{1-3\alpha} cos(\pi\alpha) \int_0^\infty \frac{\tau(t)dt}{ke^{\frac{\xi-t}{2}} + e^{-\frac{\xi-t}{2}}} + R_5[\tau] + \Phi_5(\xi),\ \xi \in (0,\infty), \tag{2.35}$$

where $R_5[\tau] = R_4[\tau]e^{(3\alpha-\frac{1}{2})\xi}$ is a regular operator, $\Phi_5(\xi) = \Phi_4(c - ce^{-\xi})e^{(3\alpha-\frac{1}{2})\xi}$. Note that by virtue of the condition $\beta_0 > \frac{1}{3}$, the inequality $6\alpha - 1 < 0$. Let us introduce the notation

$$N(\xi) = \frac{\lambda\mu k^{1-3\alpha} cos(\alpha\pi)}{ke^{\frac{\xi}{2}} + e^{-\frac{\xi}{2}}}.$$

Then the equation (2.35) can be written as

$$\rho(\xi) = \int_0^\infty N(\xi - t)\rho(t)dt + R_5[\tau] + \Phi_5(\xi),\ \xi \in (0,\infty). \tag{2.36}$$

Equation (2.36) is an integral Wiener-Hopf equation [12], the Fourier transformation turns it into the Riemann boundary -value problem which is solvable in quadratures. Functions $N(\xi)$, $\Phi_5(\xi)$ have exponential decreasing order at infinity, moreover $N(\xi) \in C(0,\infty)$, $\Phi_5(\xi) \in H_{\alpha_1}(0,\infty)$. Therefore, $N(\xi)$, $\Phi_5(\xi) \in L_2 \cap H_{\alpha_1}$. The Fredholm theorems for integral equations of convolution type are valid only in a particular case, when their index equals zero. The index of the Equation (2.36) is the index of the expression $1 - N^\wedge(\xi)$ with the opposite sign, where

$$N^\wedge(\xi) = \int_{-\infty}^\infty e^{i\xi t} N(t)dt = \lambda\mu k^{1-3\alpha} cos(\alpha\pi) \int_{-\infty}^\infty \frac{e^{i\xi t}dt}{ke^{\frac{t}{2}} + e^{-\frac{t}{2}}}. \tag{2.37}$$

With the help of the residue theory, calculating the Fourier integral [6], we obtain

$$\int_{-\infty}^\infty \frac{e^{i\xi t}dt}{ke^{\frac{t}{2}} + e^{-\frac{t}{2}}} = \frac{\pi e^{-i\xi lnk}}{\sqrt{k}ch(\pi\xi)}. \tag{2.38}$$

Substituting (2.38) to (2.37), by virtue of $\lambda = \frac{cos(\beta\pi)}{\pi(1+sin(\beta\pi))}$, $\alpha = (1-2\beta)/4$, we have

$$N^\wedge(\xi) = \mu k^{\frac{1}{2}-3\alpha} sin(\alpha\pi) \frac{e^{-i\xi lnk}}{ch(\pi\xi)}.$$

So far as $\mu k^{\frac{1}{2}-3\alpha} sin(\pi\alpha) < 1$ and since

$$Re(N^\wedge(\xi)) = Re\left(\mu k^{\frac{1}{2}-3\alpha} sin(\alpha\pi)\frac{e^{-i\xi lnk}}{ch(\pi\xi)}\right) =$$

$$= \mu k^{\frac{1}{2}-3\alpha} sin(\pi\alpha)\frac{cos(\xi lnk)}{ch(\pi\xi)} < \mu k^{\frac{1}{2}-3\alpha} sin(\pi\alpha) < 1,$$

we have $Re(1 - N^\wedge(\xi)) > 0$. Therefore, the Equation index (2.36) $\chi = -Jnd(1 - N^\wedge(\xi) = 0$, i.e., the variation of the argument $1 - N^\wedge(\xi))$ on the real axis expressed via complete revolutions is zero [12]. Consequently, the Equation (2.36) is uniquely reduced to a Fredholm integral equation of the second kind, whose unique solvability follows from the uniqueness of the solution to problem G. $\square$

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