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ONE DIMENSIONAL OCTONION FOURIER TRANSFORM

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Abstract. The main purpose of this paper is to study a one dimensional octonion Fourier transform. We prove some of its properties, including the Plancherel's and Parseval's theorems and the inversion formula. Applying these properties we show an important inversion formula for a three dimensional octonion Fourier Transform.

1. Introduction

The classical signal theory deals with real or complex-valued time series (or images). However, in some practical applications, signals are represented by more abstract structures, e.g. hypercomplex algebras such as quaternions and octonions [6, 7, 9, 13] . Working with quaternion and Clifford algebras has allowed to generalize concepts of symmetry, phase, analytic signal, holomorphic function and also the Fourier Transform. The latter is known for its multiple applications to higher dimensions [5].

In 2011 Hahn and Snopek [8] introduced the Octonion Fourier Transform of a $3-$dimensional real signal. They described how analytic signals can alternatively be defined using inverse Fourier transforms of their spectra. This transform was presented as a case of the most general hypercomplex Fourier Transform with imaginary units satisfying the multiplication rules of the Cayley-Dickson algebra [12]. Then, in 2017, Błaszczyk and Snopek [3], described the main properties of this transform, such as symmetry properties, and the octonion analogues of Parseval and Plancherel Theorems. They explain how the last one is true for the transform of a real-valued function, but is not true, in general, for octonion-valued functions due to the non-associativity presented in the octonion algebra. In 2020 Błaszczyk [2] proved that the Octonion Fourier Transform is well-defined for octonion-valued functions and almost all well-known properties of classical (complex) Fourier transform (e.g. argument scaling, modulation and shift theorems) have their direct equivalents in the octonion setup.

Following these precedents and the fact that the kernel in the transform can be considered in different ways (in view of the non$-$commutativity and non-associativity of octonions), in this paper we defined a $1-$dimensional Octonion Fourier Transform for octonion-valued functions. This transform uses a pure unit octonion in the exponent of the transform kernel. It allowed us to prove some properties similar to that given in the complex context, such as a Parseval

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and Plancherel theorems for octonion-valued functions. We show some examples and give its principal algebraic properties with an application to a given three-dimensional octonion Fourier transform.

The paper is organized as follows. In section 2 we recall the octonion algebra, its basic properties and an estimate which will be used in the last part of section 3. In section 3 we focus on to obtain and prove some important properties of the one-dimensional Octonion Fourier Transform, e.g., convolution, differentiation, inversion formula, Parseval and Plancherel's Theorems. Then, in section 4 we give an application to a 3-dimensional Octonion Fourier Transform, proving its inversion formula. Finally, in section 5 we give some concluding remarks.

2. Preliminaries and Notation

2.1. Octonions. Most of the properties given in this subsection can be found in reference [1]. Octonions is a Cayley-Dickson algebra [10]. An elementary way to construct the octonions is to give their multiplication table. The octonions $\mathbb{O}$ are an 8-dimensional algebra with basis:

$$e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7.$$

An arbitrary element $o \in \mathbb{O}$ can be represented as

$$o = o_0 + o_1 e_1 + o_2 e_2 + o_3 e_3 + o_4 e_4 + o_5 e_5 + o_6 e_6 + o_7 e_7$$

where $o_0, \cdots, o_7 \in \mathbb{R}$. The multiplication between two basis elements is given in the Table 1, which describes the result of multiplying the element in the $i-$th row by the element in the $j-$th column:

$\bullet$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

TABLE 1. Octonion Multiplication Table

In this table we can see some algebraic properties of the octonions:

i) $e_1, \ldots, e_7$ are square roots of -1.
ii) e_i and e_j anticommute when $i \neq j$ and $i, j = 1, \ldots, 7$:

$$e_i e_j = -e_j e_i.$$

iii) $e_0 = 1$ is the unit element of the algebra.

iv) the index cycling identity holds:

$$e_i e_j = e_k \Longrightarrow e_{i+1} e_{j+1} = e_{k+1},$$

where we think the indices as living in $\mathbb{Z}_7$, and

v) the index doubling identity holds:

$$e_i e_j = e_k \Longrightarrow e_{2i} e_{2j} = e_{2k}.$$

Using the table, it is an easy task to see that this algebra is not associative. In fact, we have

$$(e_1 e_2) e_5 = e_3 e_5 = -e_6, \quad \text{but} \quad e_1(e_2 e_5) = e_1 e_7 = e_6.$$

We also see that the elements $\{1, e_1, e_2, e_3\}$ generate the quaternion algebra, i.e., $\mathbb{H}$ can be seen as a subalgebra of $\mathbb{O}$.

As we saw in Table 1, the octonions are not associative. However, it's well known that $\mathbb{O}$ is an alternative algebra. A theorem by Emil Artin [11] establish that an algebra A is **alternative** iff for all $a, b \in A$ we have

$$(aa)b = a(ab), \quad (ab)a = a(ba), \quad (ba)a = b(aa). \tag{2.1}$$

An important property of an alternative algebra is that the subalgebra generated by any two elements is associative [1].

We will call the part $o_0 e_0 =: Sc\ o$, the **scalar part** of o and $\underline{o} = o - o_0 e_0 =: Vec\ o$, the **vector part** of o. Vector parts are isomorphic to the seven dimensional Euclidean vector space $\mathbb{R}^7$. The whole algebra of octonions is naturally identified with the vector space $\mathbb{R}^8$.

If $o = \underline{o}$ then o is called a **pure octonion**. The subset of all pure octonions is denoted by $Vec\ \mathbb{O}$, while the subset of all scalars will be denoted by $Sc\ \mathbb{O}$. The octonion $\overline{o} = o_0 - \underline{o}$ is called the **conjugate** to o. The mapping $o \longmapsto \overline{o}$ is called **conjugation**.

The **norm** or **absolute value** of a octonion o is defined as

$$|o| := \sqrt{o\overline{o}}. \tag{2.2}$$

If $|o| = 1$ the octonion o is said to be a **unit octonion**. The imaginary octonions of norm one form a $6-sphere$ in the $7-dimensional$ space of imaginary octonions [4].

$$S^6_{\mathbb{O}} := \left\{ o = e_1 r_1 + \cdots + e_7 r_7 |\ r_1^2 + \cdots + r_7^2 = 1 \right\}. \tag{2.3}$$

The **dot product** in octonions can be obtained from the norm by a process known as polarization. Polarizing (2.2) we obtain an inner product on $\mathbb{R}^8$, namely [4]

$$o_1 \odot o_2 = \frac{1}{2}\left(|o_1 + o_2|^2 - |o_1|^2 - |o_2|^2\right) = o_1\overline{o_2} + o_2\overline{o_1} = \overline{o_1}o_2 + \overline{o_2}o_1.$$

We need to define now **the exponential function of an octonion variable**. The exponential of an octonion is defined through the infinite series [3]: For any $o \in \mathbb{O}$,

$$e^o := \sum_{k=0}^{\infty} \frac{o^k}{k!}.$$

It can be shown that if we denote $\mathbf{o} = Vec\ o$, then

$$e^o = e^{Sc\ o}\left(\cos|\mathbf{o}| + \frac{\mathbf{o}}{|\mathbf{o}|}\sin|\mathbf{o}|\right),$$

where $|\cdot|$ is the octonion norm.

Due to the fact that octonions are non-commutative, the relation

$$e^{o_1+o_2} = e^{o_1} \cdot e^{o_2}$$

is not always true. However, this property holds when $o_1 \cdot o_2 = o_2 \cdot o_1$.

Lemma 2.1. *In view of the alternativity property of octonions, we see that, given any two octonions o_1, o_2, the subalgebra generated by $\{1, o_1, o_2\}$ is associative. This means, in particular, that, given $o,\ \mu \in \mathbb{O}$, where μ is a pure unit octonion, and $a, b, c, d, \alpha, \beta \in \mathbb{R}$, we have*

$$[o \cdot (a + b\mu)] \cdot (c + d\mu) = o \cdot [(a + b\mu) \cdot (c + d\mu)]$$

and

$$[(a + b\mu) \cdot o] \cdot (c + d\mu) = (a + b\mu) \cdot [o \cdot (c + d\mu)],$$

which implies

$$(o \cdot e^{\alpha\mu}) \cdot e^{\beta\mu} = o \cdot (e^{\alpha\mu} \cdot e^{\beta\mu}) \quad \text{and} \quad (e^{\alpha\mu} \cdot o) \cdot e^{\beta\mu} = e^{\alpha\mu} \cdot (o \cdot e^{\beta\mu}). \tag{2.4}$$

2.2. An important estimate. By definition, the norm of an octonion $o = o_0 + \sum_{i=1}^{7} o_i e_i$ is given by

$$|o|^2 = \sum_{i=0}^{7} |o_i|^2 .$$

This implies that

$$|o_i| \le |o| .$$

On the other hand, given an integrable function $f : \mathbb{R} \to \mathbb{O}$, since

$$\int_{\mathbb{R}} f(x)dx = \left(\sum_{i=0}^{7} \int_{\mathbb{R}} f_i(x)dx\right) e_i,$$

we see that

$$\begin{aligned}
\left|\int_{\mathbb{R}} f(x)dx\right|^2 &= \sum_{i=0}^{7} \left|\int_{\mathbb{R}} f_i(x)dx\right|^2 \\
&\le \sum_{i=0}^{7} \left(\int_{\mathbb{R}} |f_i(x)|\, dx\right)^2 \\
&\le \sum_{i=0}^{7} \left(\int_{\mathbb{R}} |f(x)| dx\right)^2 \\
&= 8 \left(\int_{\mathbb{R}} |f(x)| dx\right)^2 .
\end{aligned}$$

That is,

$$\left|\int_{\mathbb{R}} f(x)dx\right| \leq 2\sqrt{2}\int_{\mathbb{R}} |f(x)|dx.$$

A similar argument yields

$$\left|\int_{-\infty}^{a} f(x)dx\right| \leq 2\sqrt{2}\int_{-\infty}^{a} |f(x)|dx, \text{ with } a \leq 0 \tag{2.5}$$

and

$$\left|\int_{b}^{\infty} f(x)dx\right| \leq 2\sqrt{2}\int_{b}^{\infty} |f(x)|dx, \text{ with } b \geq 0. \tag{2.6}$$

The same estimate is true for integrals of the form $\displaystyle\int_a^0$ and $\displaystyle\int_0^b$.

3. One dimensional Octonion Fourier Transform

Definition 3.1. Let $f \in L^1(\mathbb{R}; \mathbb{O})$. We define the **One-dimensional (right) Octonion Fourier Transform** of f (**1D−OFT** for short), by

$$\mathcal{F}_{OFT}\{f(x)\}(t) = \widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-\mu t x}dx, \tag{3.1}$$

where $\mu \in \mathbb{O}$ is a pure unit octonion.

Example 3.2. For a positive real number a, consider $f(x) = e^{-ax^2}$. Then,

$$\begin{aligned}
\mathcal{F}_{OFT}\left\{e^{-ax^2}\right\}(t) &= \int_{\mathbb{R}} e^{-ax^2}e^{-\mu t x}dx \\
&= \int_{\mathbb{R}} \exp\left[-a\left(x + \frac{\mu t}{2a}\right)^2 - \frac{t^2}{4a}\right]dx \\
&= e^{-\frac{t^2}{4a}}\int_{\mathbb{R}} e^{-ay^2}dy \\
&= \sqrt{\frac{\pi}{a}}e^{-\frac{t^2}{4a}}.
\end{aligned}$$

Particularly, when $a = \frac{1}{2}$, we have

$$\mathcal{F}_{OFT}\left\{e^{-\frac{x^2}{2}}\right\}(t) = \sqrt{2\pi}e^{-\frac{t^2}{2}}.$$

Example 3.3. Let $f(x) = e^{-a|x|}$, for $a > 0$. Then,

$$\begin{aligned}
\mathcal{F}_{OFT}\left\{e^{-a|x|}\right\}(t) &= \int_{\mathbb{R}} e^{-a|x|}e^{-\mu t x}dx \\
&= \int_{-\infty}^{0} e^{(a-\mu t)x}dx + \int_{0}^{\infty} e^{-(a+\mu t)x}dx \\
&= \frac{1}{a-\mu t} + \frac{1}{a+\mu t}
\end{aligned}$$

That is,

$$\mathcal{F}_{OFT}\left\{e^{-a|x|}\right\}(t)=\frac{2a}{a^2+t^2}.$$

3.1. Algebraic Properties. In this subsection we prove some algebraic properties of the 1D−OFT.

Proposition 3.4. (ℝ−linearity)
The 1D−OFT is ℝ−linear. That is, given $a,b\in\mathbb{R}$ and $f,g\in L^1(\mathbb{R};\mathbb{O})$,

$$\mathcal{F}_{OFT}\left\{(af+bg)(x)\right\}(t)=a\widehat{f}(t)+b\widehat{g}(t). \tag{3.2}$$

Proof.

$$\begin{aligned}\mathcal{F}_{OFT}\left\{(af+bg)(x)\right\}(t) &= \int_{\mathbb{R}}(af(x)+bg(x))e^{-\mu tx}dx\\ &= a\int_{\mathbb{R}}f(x)e^{-\mu tx}dx+b\int_{\mathbb{R}}g(x)e^{-\mu tx}dx\\ &= a\widehat{f}(t)+b\widehat{g}(t).\end{aligned}$$

□

Remark 3.5. Note that, in view of non-associativity, (left) 𝕆−linearity is false in general.

Now, we see the following result.

Proposition 3.6. (Dilation or Scaling Property)
Let a be a positive real number and $f\in L^1(\mathbb{R};\mathbb{O})$. Then,

$$\frac{1}{a}\mathcal{F}_{OFT}\left\{f\left(\frac{x}{a}\right)\right\}(t)=\widehat{f}(at) \tag{3.3}$$

or, analogously,

$$a\mathcal{F}_{OFT}\{f(ax)\}(t)=\widehat{f}\left(\frac{t}{a}\right). \tag{3.4}$$

Proof.

$$\widehat{f}(at)=\int_{\mathbb{R}}f(x)e^{-\mu(at)x}dx=\frac{1}{a}\int_{\mathbb{R}}f\left(\frac{y}{a}\right)e^{-\mu ty}dy=\frac{1}{a}\mathcal{F}_{OFT}\left\{f\left(\frac{x}{a}\right)\right\}(t).$$

□

Proposition 3.7. (Shift Property)
Let $x_0\in\mathbb{R}$ and $f\in L^1(\mathbb{R};\mathbb{O})$. Then,

$$\mathcal{F}_{OFT}\left\{f\left(x-x_0\right)\right\}(t)=\widehat{f}(t)e^{-\mu tx_0}. \tag{3.5}$$

Proof. In view of (2.4) and the fact that $e^{\alpha+\beta} = e^{\alpha}e^{\beta}$ if $\alpha\beta = \beta\alpha$, where $\alpha, \beta \in Vec\, \mathbb{O}$, we have that

$$\begin{aligned}
\mathcal{F}_{OFT}\left\{f\left(x-x_0\right)\right\}(t) &= \int_{\mathbb{R}} f\left(x-x_0\right) e^{-\mu t x} dx \\
&= \int_{\mathbb{R}} f(y) e^{-\mu t(y+x_0)} dy \\
&= \int_{\mathbb{R}} f(y)(e^{-\mu t y} e^{-\mu t x_0}) dy \\
&= \int_{\mathbb{R}} (f(y) e^{-\mu t y}) dy \cdot e^{-\mu t x_0} \\
&= \widehat{f}(t) e^{-\mu t x_0}.
\end{aligned}$$

□

Proposition 3.8. (Modulation Property)
Let $t_0 \in \mathbb{R}$ and $f \in L^1(\mathbb{R}; \mathbb{O})$. Then,

$$\mathcal{F}_{OFT}\left\{f(x)e^{\mu t_0 x}\right\}(t) = \widehat{f}\left(t - t_0\right). \tag{3.6}$$

Proof. Using (2.4), we have that

$$\begin{aligned}
\mathcal{F}_{OFT}\left\{f(x)e^{\mu t_0 x}\right\}(t) &= \int_{\mathbb{R}} (f(x)e^{\mu t_0 x}) e^{-\mu t x} dx \\
&= \int_{\mathbb{R}} f(x)(e^{\mu t_0 x} e^{-\mu t x}) dx \\
&= \int_{\mathbb{R}} f(x) e^{-\mu(t-t_0)x} dx \\
&= \widehat{f}\left(t - t_0\right).
\end{aligned}$$

□

3.2. Differentiation. In this subsection we will prove some properties of the 1D−OFT related with the derivative.

Proposition 3.9. *Let $f, f' \in L^1(\mathbb{R}; \mathbb{O})$, with $f(x) \to 0$ as $|x| \to \infty$, then*

$$\mathcal{F}_{OFT}\left\{f'(x)\right\}(t) = \widehat{f}(t)\mu t. \tag{3.7}$$

Proof.

$$\begin{aligned}
\mathcal{F}_{OFT}\left\{f'(x)\right\}(t) &= \int_{\mathbb{R}} f'(x)e^{-\mu t x}dx \\
&= \int_{-\infty}^{0} f'(x)e^{-\mu t x}dx + \int_{0}^{\infty} f'(x)e^{-\mu t x}dx \\
&= \lim_{b\to-\infty} \left(f(x)e^{-\mu t x}\right)\Big|_b^0 + \int_{-\infty}^{0} f(x)(e^{-\mu t x}\mu t)dx \\
&+ \lim_{b\to\infty} \left(f(x)e^{-\mu t x}\right)\Big|_0^b + \int_{0}^{\infty} f(x)(e^{-\mu t x}\mu t)dx \\
&= \int_{\mathbb{R}} f(x)(e^{-\mu t x}\mu t)dx \\
&= \int_{\mathbb{R}} (f(x)e^{-\mu t x})\mu t dx \\
&= \left(\int_{\mathbb{R}} f(x)e^{-\mu t x}dx\right)\mu t.
\end{aligned}$$

□

Induction proves that

Corollary 3.10. *If $f, f^{(k)} \in L^1(\mathbb{R};\mathbb{O})$, for $k = 1,\ldots,n$, and $f^{(k)}(x) \to 0$, as $|x| \to \infty$, for $k = 1,\ldots,n-1$, then*

$$\mathcal{F}_{OFT}\left\{f^{(n)}(x)\right\}(t) = \widehat{f}(t)(\mu t)^n.$$

Proposition 3.11. *Let $xf(x) \in L^1(\mathbb{R};\mathbb{O})$, then*

$$\mathcal{F}_{OFT}\{xf(x)\}(t) = \frac{d}{dt}\widehat{f}(t)\cdot\mu. \tag{3.8}$$

Proof. From the definition of the 1D−OFT, one see that $\mathcal{F}_{OFT}\left\{f(x)\right\}(t)$ is differentiable, so

$$\begin{aligned}
\frac{d}{dt}\widehat{f}(t) &= \int_{\mathbb{R}} \frac{d}{dt}\left(f(x)e^{-\mu t x}\right)dx \\
&= -\int_{\mathbb{R}} f(x)(e^{-\mu t x}\cdot\mu x)dx \\
&= -\int_{\mathbb{R}} (xf(x)\cdot e^{-\mu t x})\cdot\mu dx \\
&= -\left(\int_{\mathbb{R}} xf(x)e^{-\mu t x}dx\right)\cdot\mu,
\end{aligned}$$

which implies (3.8).

□

From the last proposition we also have that

Corollary 3.12. *If* $x^n f(x) \in L^1(\mathbb{R}; \mathbb{O})$,

$$\mathcal{F}_{OFT}\{x^n f(x)\}(t) = \frac{d^n}{dt^n}\widehat{f}(t)\mu^n.$$

3.3. Convolution. It's well known that one useful tool related with the Fourier transform is the convolution. Here we introduce the concept of convolution in this context and give some of its properties.

Definition 3.13. Let f and g be two integrable, octonion-valued functions. The convolution of f and g, denoted by $(f * g)(x)$, is defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - \xi)\, g(\xi) d\xi. \tag{3.9}$$

Remark 3.14. In view of the non-commutativity of octonions, in general

$$(f * g)(x) \neq (g * f)(x).$$

However, when f and g commutes, we also have commutativity for the convolution. This is true, in particular, if either f or g are real−valued functions.

Proposition 3.15. *If* $a, b \in \mathbb{R}$, *the convolution defined above have the following properties.*

$$af * h = f * ah = a(f * h),$$
$$(af + bg) * h = a(f * h) + b(g * h),$$
$$f * (ag + bh) = a(f * g) + b(f * h),$$

and, provided f, g, h *are octonion-valued functions such that they generate an associative subalgebra of* $\mathbb{O}$,

$$f * (g * h) = (f * g) * h.$$

Proof. It's direct from the definition.

□

Proposition 3.16. (Derivative of a convolution)

Consider f *a differentiable function and* g *an integrable function, both octonion-valued functions, and suppose that* $f * g$ *and* $f' * g$ *are well-defined. Then*

$$(f * g)' = f' * g. \tag{3.10}$$

Proof.

$$(f * g)'(x) = \frac{d}{dx}\int_{\mathbb{R}} f(x-\xi)g(\xi)d\xi = \int_{\mathbb{R}} f'(x-\xi)g(\xi)d\xi = (f' * g)(x).$$

□

Remark 3.17. Since, in general, $(f * g)(x) \neq (g * f)(x)$, we have that

$$(f' * g)(x) \neq (f * g')(x).$$

3.4. Inversion Formula. In this section we will prove the inversion formula for the 1D−OFT. In order to do that, we first prove the following theorem.

Theorem 3.18. *Let $g \in L^2(\mathbb{R};\mathbb{R})$ and $\int_{\mathbb{R}} g(y)dy = 1$. Let*

$$\alpha := \int_{-\infty}^{0} g(y)dy \quad \text{and} \quad \beta := \int_{0}^{\infty} g(y)dy,$$

so that $\alpha+\beta = 1$ and $\alpha = \beta = \frac{1}{2}$ if g is even. Let $f : \mathbb{R} \to \mathbb{O}$ be a piecewise continuous function on $\mathbb{R}$ and suppose either that f is bounded or that g vanishes outside a finite interval so that $f * g$ is well-defined for all x. If $g_\varepsilon(x) = \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}\right)$, then

$$\lim_{\varepsilon \to 0} \left(f * g_\varepsilon\right)(x) = \alpha f(x^+) + \beta f\left(x^-\right) \tag{3.11}$$

for all x, where $f(x^+) = \lim_{\varepsilon\to 0} f(x+\varepsilon)$ and $f(x^-) = \lim_{\varepsilon \to 0} f(x-\varepsilon)$.

Even more, if f is continuous at x, we have

$$\lim_{\varepsilon \to 0} \left(f * g_\varepsilon\right)(x) = f(x). \tag{3.12}$$

Proof. Let's consider the difference

$$\begin{aligned}(f * g_\varepsilon)(x) - \left(\alpha f\left(x^+\right) + \beta f\left(x^-\right)\right) &= \int_{-\infty}^{0} [f(x-y) - f(x^+)] g_\varepsilon(y)dy \\ &+ \int_{0}^{\infty} \left[f(x-y) - f\left(x^-\right)\right] g_\varepsilon(y)dy.\end{aligned} \tag{3.13}$$

So, in order to prove (3.11), we need to check that the previous expression goes to zero as $\varepsilon \to 0$.

Consider the second integral in (3.13). Let $\delta > 0$. We can choose c small enough so that $|f(x-y) - f(x^-)| < \delta$ if $0 < y < c$. Breaking up the integral as $\int_0^c + \int_c^\infty$, we have, first, that

$$\begin{aligned}\left|\int_0^c [f(x-y) - f(x^-)] g_\varepsilon(y)dy\right| &\leq 2\sqrt{2} \int_0^c \left|f(x-y) - f\left(x^-\right)\right| \left|g_\varepsilon(y)\right| dy \\ &\leq 2\sqrt{2}\delta \int_0^c \left|g_\varepsilon(y)\right| dy \\ &= 2\sqrt{2}\delta \int_0^c \left|\frac{1}{\varepsilon} g\left(\frac{y}{\varepsilon}\right)\right| dy \\ &= 2\sqrt{2}\delta \int_0^{c/\varepsilon} |g(y)| dy \\ &\leq 2\sqrt{2}\delta \int_0^{\infty} |g(y)| dy.\end{aligned}$$

For a suitable chosen δ, we can make small enough the last expression.

On the other hand, in order to estimate the integral from c to ∞ we consider the two following cases:
Case 1: f is bounded, say $|f| \leq M$, for some positive M.

In this case,

$$\begin{aligned} \left| \int_c^{\infty} \left[f(x-y) - f\left(x^-\right) \right] g_\varepsilon(y) dy \right| &\leq 2\sqrt{2} \int_c^{\infty} \left| f(x-y) - f(x^-) \right| \left| g_\varepsilon(y) \right| dy \\ &\leq 4\sqrt{2} M \int_c^{\infty} \left| g_\varepsilon(y) \right| dy \\ &\leq 4\sqrt{2} M \int_{c/\varepsilon}^{\infty} |g(y)| dy. \end{aligned}$$

which goes to zero as $\varepsilon \to 0$.

Case 2: g vanishes outside a finite interval, say $g(x) = 0$ for $|x| > R$.

In this case, $g_\varepsilon(x) = 0$ for $|x| > R\varepsilon$ and, in particular, $g_\varepsilon(x) = 0$ for $x > c$ if $c > R\varepsilon$, i.e., if $\varepsilon < \frac{c}{R}$.

So, we have that

$$\int_c^{\infty} [f(x-y) - f(x^-)] g_\varepsilon(y) \to 0$$

as $\varepsilon \to 0$.

A similar argument is valid when one considers the first integral in (3.13). Summarizing, we have that

$$\lim_{\varepsilon \to 0} \left(f * g_\varepsilon \right)(x) = \alpha f\left(x^+\right) + \beta f\left(x^-\right).$$

When f is continuous at x, it's clear from the above result that

$$\lim_{\varepsilon \to 0} \left(f * g_\varepsilon \right)(x) = f(x).$$

□

Remark 3.19. Note that, when g is even, $\alpha = \beta = \frac{1}{2}$, and formula (3.11) becomes equal to

$$\lim_{\varepsilon \to 0} \left(f * g_\varepsilon \right)(x) = \frac{1}{2} \left(f(x^+) + f\left(x^-\right) \right).$$

Theorem 3.20. (Fourier Inversion Formula for the 1D−OFT)
Let f be an integrable and piecewise continuous on $\mathbb{R}$, with values in $\mathbb{O}$, defined at its points of discontinuity as to satisfy $f(x) = \dfrac{1}{2}[f(x^-) + f(x^+)]$ for all x. Then

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(t) e^{-\varepsilon^2 t^2/2} e^{\mu t x} dt, \quad x \in \mathbb{R}. \tag{3.14}$$

Moreover, if $\widehat{f} \in L^1(\mathbb{R};\mathbb{O})$*, then* f *is continuous and*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)e^{\mu t x}dt, \quad x \in \mathbb{R}. \tag{3.15}$$

Proof. Let $\varepsilon > 0$ and consider the function $e^{-\varepsilon^2 t^2/2}$, which decreases rapidly as $t \to \pm\infty$. Let $f \in L^1(\mathbb{R};\mathbb{O})$ and consider the integral

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t x}dt = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(f(y)e^{-\mu t y}\right) e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t x}dydt.$$

By (2.4) and the fact that this double integral is absolutely convergent, we can write

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t x}dt = \frac{1}{2\pi}\int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}} e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t(x-y)}dt\right)dy.$$

On the other hand, if $a = \varepsilon^2/2$, we have that

$$\int_{\mathbb{R}} e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t(x-y)}dt = \int_{\mathbb{R}} e^{-at^2} e^{-\mu t(y-x)}dt = \mathcal{F}_{OFT}\{e^{-at^2}\}(y-x) = \sqrt{\frac{\pi}{a}}e^{-\frac{(y-x)^2}{4a}}.$$

So,

$$\int_{\mathbb{R}} e^{-\varepsilon^2 t^2/2} e^{\mu t(x-y)}dt = \frac{\sqrt{2\pi}}{\varepsilon}e^{-\frac{(x-y)^2}{2\varepsilon^2}},$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t x}dt = \frac{1}{\varepsilon\sqrt{2\pi}}\int_{\mathbb{R}} f(y)e^{-\frac{(x-y)^2}{2\varepsilon^2}}dy = (f * \phi_\varepsilon)(x),$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon\sqrt{2\pi}}e^{-\frac{x^2}{2\varepsilon^2}}$.

Since f is piecewise continuous, Theorem 3.18 and Remark 3.19 implies that

$$\lim_{\varepsilon\to 0}(f * \phi_\varepsilon)(x) = \frac{1}{2}[f(x^-) + f(x^+)]$$

for all x. This proves (3.14). Finally, since $|e^{\mu t x}| = 1$ and $\left|e^{-\frac{\varepsilon^2 t^2}{2}}\right| \leq 1$ for small enough ε,

$$\left|\widehat{f}(t)e^{-\varepsilon^2 \frac{t^2}{2}} e^{\mu t x}\right| = |\widehat{f}(t)|\left|e^{-\varepsilon^2 \frac{t^2}{2}}\right|\left|e^{\mu t x}\right| \leq |\widehat{f}(t)|.$$

If $\widehat{f} \in L^1(\mathbb{R};\mathbb{O})$, the dominated convergence theorem implies that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)e^{\mu t x}dt &= \frac{1}{2\pi}\int_{\mathbb{R}} \lim_{\varepsilon\to 0}\left(\widehat{f}(t)e^{-\frac{\varepsilon^2 t^2}{2}} e^{\mu t x}\right)dt \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)e^{-\frac{\varepsilon^2 t^2}{2}} e^{\mu t x}dt \\ &= \lim_{\varepsilon\to 0}(f * \phi_\varepsilon)(x) \\ &= \frac{1}{2}\left[f(x^-) + f\left(x^+\right)\right].\end{aligned}$$

Since

$$\int_{\mathbb{R}} \widehat{f}(t)e^{\mu tx}dt = \mathcal{F}_{OFT}\{\widehat{f}(t)\}(-x)$$

and the Fourier transform of an integrable function is continuous, we have that f is continuous and, thus,

$$f(x) = \frac{1}{2}\left[f(x^-) + f\left(x^+\right)\right],$$

which proves (3.15).

□

3.5. Plancherel and Parseval's Theorems. Now, we want to establish the Plancherel and Parseval's theorems in this context. We begin with the definition of inner product and norm in $L^2(\mathbb{R};\mathbb{O})$.

Definition 3.21. Let $f, g \in L^2(\mathbb{R};\mathbb{O})$, we define

$$\langle f, g\rangle_{L^2(\mathbb{R};\mathbb{O})} := \int_{\mathbb{R}} f(x)\overline{g(x)}dx \tag{3.16}$$

and

$$\|f\|_{L^2(\mathbb{R};\mathbb{O})} = \langle f, f\rangle^{1/2}_{L^2(\mathbb{R};\mathbb{O})} = \left(\int_{\mathbb{R}} |f(x)|^2\, dx\right)^{1/2}. \tag{3.17}$$

Using the previous definition of inner product, we have the following theorem.

Theorem 3.22. (Plancherel's Theorem)
Let $f, g \in L^2(\mathbb{R};\mathbb{O})$ such that $1, f, g$ and μ generates an associative sub-algebra of $\mathbb{O}$. Then,

$$\langle f, g\rangle_{L^2(\mathbb{R};\mathbb{O})} = \frac{1}{2\pi}\langle \widehat{f}, \widehat{g}\rangle_{L^2(\mathbb{R};\mathbb{O})}. \tag{3.18}$$

Proof.

$$\begin{aligned}
\langle f, g\rangle_{L^2(\mathbb{R};\mathbb{O})} &= \int_{\mathbb{R}} f(x)\overline{g(x)}dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \widehat{f}(t)e^{\mu tx}dt\right)\overline{g(x)}dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)\left(\int_{\mathbb{R}} e^{\mu tx}\overline{g(x)}dx\right)dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)\left(\int_{\mathbb{R}} \overline{g(x)e^{-\mu tx}}dx\right)dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)\left(\overline{\int_{\mathbb{R}} g(x)e^{-\mu tx}dx}\right)dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)\overline{\widehat{g}(t)}dt.
\end{aligned}$$

□

As a corollary, we have the

Corollary 3.23. (Parseval's Theorem)
In the previous formula, if $f = g$, then

$$\|f\|_{L^2(\mathbb{R};\mathbb{O})} = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_{L^2(\mathbb{R};\mathbb{O})}.$$

Remark 3.24. Note that $1, f$ and μ always generate an associative sub-algebra of $\mathbb{O}$. So, Parseval's theorem is always true for all $f \in L^2(\mathbb{R};\mathbb{O})$.

4. Application to the 3−dimensional Octonion Fourier Transform

Although we can apply the 1D−OFT to other topics, the main motivation to define it was the use of this transform to obtain an inversion formula for the 3−dimensional octonion Fourier transform defined below.

Definition 4.1. Let $f \in L^1(\mathbb{R}^3;\mathbb{O})$. The **3−dimensional Octonion Fourier Transform** of f (or 3D−OFT for short) is the function $\mathcal{F}_{OFT}\{f\} : \mathbb{R}^3 \to \mathbb{O}$ defined by

$$\mathcal{F}_{OFT}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \widehat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} f(\boldsymbol{x})e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}d^3\boldsymbol{x}, \tag{4.1}$$

where $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$, $\omega = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \omega_3\boldsymbol{e}_3$, μ is as before and $e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}$ is called the **octonion Fourier kernel**.

Theorem 4.2. (Inversion formula for the 3D−OFT)
Suppose that $f \in L^2(\mathbb{R}^3;\mathbb{O})$ and $\mathcal{F}_{OFT}\{f\} \in L^1(\mathbb{R}^3;\mathbb{O})$. Then the 3D−OFT is invertible with inverse

$$\mathcal{F}^{-1}_{OFT}\left[\mathcal{F}_{OFT}\{f\}\right](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}_{OFT}\{f(\boldsymbol{x})\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}d^3\boldsymbol{\omega}. \tag{4.2}$$

Proof. Let

$$\tilde{\boldsymbol{x}}_{\boldsymbol{1}} = (x_1, \omega_2, \omega_3),\ \tilde{\boldsymbol{x}}_{\boldsymbol{2}} = (\omega_1, x_2, \omega_3),\ \tilde{\boldsymbol{x}}_{\boldsymbol{3}} = (\omega_1, \omega_2, x_3)$$

and

$$\tilde{\boldsymbol{\omega}}_{\boldsymbol{1}} = (\omega_1, x_2, x_3),\ \tilde{\boldsymbol{\omega}}_{\boldsymbol{2}} = (x_1, \omega_2, x_3),\ \tilde{\boldsymbol{\omega}}_{\boldsymbol{3}} = (x_1, x_2, \omega_3).$$

Denote the 1D−OFT with respect to x_k, for $k = 1, 2, 3$, by $\mathcal{F}_{x_k}\{f(\boldsymbol{x})\}$, that is,

$$\mathcal{F}_{x_k}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}) = \int_{\mathbb{R}} f(\boldsymbol{x})e^{-\mu\omega_k x_k}dx_k.$$

Denote also, for $i, j, k \in \{1, 2, 3\}$, and $i \neq j,\ i \neq k,\ j \neq k$,

$$\mathcal{F}_{x_i,x_j}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{x}}_{\boldsymbol{k}}) = \mathcal{F}_{x_i}\left\{\mathcal{F}_{x_j}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{\omega}}_{\boldsymbol{j}})\right\}(\omega_i),$$

and

$$\mathcal{F}_{x_i,x_j,x_k}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \mathcal{F}_{x_i,x_j}\left\{\mathcal{F}_{x_k}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{\omega}}_{\boldsymbol{k}})\right\}(\tilde{\boldsymbol{x}}_{\boldsymbol{k}}).$$

Then, it can be seen that

$$\mathcal{F}_{x_i,x_j}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{x}}_{\boldsymbol{k}}) = \mathcal{F}_{x_j,x_i}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{x}}_{\boldsymbol{k}}),$$

$$\mathcal{F}_{x_i,x_j,x_k}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \mathcal{F}_{x_i}\left\{\mathcal{F}_{x_j,x_k}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{x}}_{\boldsymbol{i}})\right\}(\omega_i)$$

and

$$\mathcal{F}_{x_i,x_j,x_k}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \mathcal{F}_{x_j,x_i,x_k}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}) = \mathcal{F}_{x_i,x_k,x_j}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}),\ \text{etc.}$$

Moreover, we have that

$$\begin{aligned}\mathcal{F}_{OFT}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^3} f(\boldsymbol{x})e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}d^3\boldsymbol{x} \\ &= \int_{\mathbb{R}^3} f(\boldsymbol{x})\left(e^{-\mu\omega_1 x_1}e^{-\mu\omega_2 x_2}e^{-\mu\omega_3 x_3}\right)d^3\boldsymbol{x} \\ &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} f(\boldsymbol{x})e^{-\mu\omega_1 x_1}dx_1\right)\left(e^{-\mu\omega_2 x_2}e^{-\mu\omega_3 x_3}\right)dx_2dx_3 \\ &= \int_{\mathbb{R}^2}\mathcal{F}_{x_1}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{\omega}}_{\mathbf{1}})\left(e^{-\mu\omega_2 x_2}e^{-\mu\omega_3 x_3}\right)dx_2dx_3 \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathcal{F}_{x_1}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{\omega}}_{\mathbf{1}})e^{-\mu\omega_2 x_2}dx_2\right)e^{-\mu\omega_3 x_3}dx_3 \\ &= \int_{\mathbb{R}}\mathcal{F}_{x_2,x_1}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{x}}_{\mathbf{3}})e^{-\mu\omega_3 x_3}dx_3 \\ &= \mathcal{F}_{x_3,x_2,x_1}\{f(\boldsymbol{x})\}(\boldsymbol{\omega}).\end{aligned}$$

Now, inversion formula for the 1D−OFT implies that

$$\begin{aligned}f(\boldsymbol{x}) &= \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}_{x_1}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{\omega}}_{\mathbf{1}})e^{\mu\omega_1 x_1}d\omega_1 \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathcal{F}_{x_2,x_1}\{f(\boldsymbol{x})\}(\tilde{\boldsymbol{x}}_{\mathbf{3}})e^{\mu\omega_2 x_2}d\omega_2\right)e^{\mu\omega_1 x_1}d\omega_1 \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathcal{F}_{x_3,x_2,x_1}\{f(\boldsymbol{x})\}(\boldsymbol{\omega})e^{\mu\omega_3 x_3}d\omega_3\right)(e^{\mu\omega_2 x_2}e^{\mu\omega_1 x_1})\,d\omega_2 d\omega_1 \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}_{x_3,x_2,x_1}\{f(\boldsymbol{x})\}(\boldsymbol{\omega})\,(e^{\mu\omega_3 x_3}e^{\mu\omega_2 x_2}e^{\mu\omega_1 x_1})\,d\omega_3 d\omega_2 d\omega_1 \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathcal{F}_{OFT}\{f(\boldsymbol{x})\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}d^3\boldsymbol{\omega},\end{aligned}$$

as we claimed.

□

5. Concluding Remarks

We have introduced a one dimensional octonion Fourier Transform, proving some of its properties. We use this theory to prove an inversion formula for a three−dimensional octonion Fourier Transform (3D−OFT). In future works we will see that this last inversion formula, together with other properties of the 3D−OFT are important to study a octonion wavelet transform and, in particular, we will be able to give an admissibility condition for octonion wavelets. The theory here developed can be extended and we will present this in some future papers. It must be noted that it is also possible to consider other multidimensional (4D, 5D, etc) Octonion Fourier Transforms and use the $1D-$OFT and the same arguments given here in order to obtain an inversion formula and other properties for these multidimensional OFT.

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