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COMPUTING THE FIRST REVERSE ZAGREB BETA INDEX AND THE SECOND REVERSE ZAGREB INDEX OF THE SUBDIVISION GRAPHS OF TADPOLE, WHEEL, AND LADDER GRAPHS AND THEIR LINE GRAPHS

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Abstract. The First Reverse Zagreb Beta Index and the Second Reverse Zagreb Index are important degree-based topological indices in chemical graph theory that are widely used to characterize the structural properties of molecular graphs. These indices are defined using the reverse vertex degree of a vertex v in a graph G, given by $c_v = \Delta(G) + 1 - d_G(v)$, where $\Delta(G)$ denotes the maximum degree of (G). In this paper, we derive closed-form expressions for the First Reverse Zagreb Beta Index and the Second Reverse Zagreb Index of the subdivision graphs of the tadpole graph, wheel graph, and ladder graph, as well as the line graphs of their subdivision graphs. The results are obtained by partitioning the edge sets according to the reverse vertex degrees of their end vertices, leading to simple and systematic computations of the required indices. The derived formulae provide efficient methods for evaluating these indices and contribute to the theoretical development of reverse Zagreb indices in graph theory and their applications in chemical graph theory.

1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple, and connected graph, where $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of vertices adjacent to v. The maximum degree of G is denoted by $\Delta(G)$. The reverse vertex degree of a vertex v is defined by

$$c_v = \Delta(G) + 1 - d_G(v).$$

Thus, the reverse degree assigns larger values to vertices having smaller degrees and vice versa, thereby providing an alternative perspective for analyzing graph structures. The reverse edge corresponding to an edge $uv \in E(G)$ is represented by the pair (c_u, c_v), where c_u and c_v denote the reverse degrees of the end vertices. Throughout this paper, we use the standard terminology and notation of graph theory. For all undefined terms and notation, the reader is referred to [2].

Chemical graph theory is an active area of mathematical chemistry in which chemical compounds are represented by graphs, with vertices corresponding to

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atoms and edges representing chemical bonds. This graph-theoretical representation enables the use of mathematical techniques to investigate molecular structures and their physicochemical properties. One of the most important tools in chemical graph theory is the concept of topological indices, which are numerical quantities associated with molecular graphs. These descriptors play a significant role in Quantitative Structure–Property Relationship (QSPR) and Quantitative Structure–Activity Relationship (QSAR) studies, where they are employed to predict physicochemical, biological, and pharmacological properties of chemical compounds without requiring extensive laboratory experiments. Consequently, numerous topological indices have been introduced and extensively studied because of their theoretical importance and practical applications in chemistry, biology, material science, and pharmaceutical research; see [2].

Among the various classes of topological descriptors, degree-based indices have attracted considerable attention due to their simplicity and effectiveness in characterizing molecular structures. The classical Zagreb indices are among the earliest and most extensively studied degree-based topological indices and have inspired numerous modifications and generalizations. These indices have been successfully applied in modeling molecular branching, predicting physicochemical properties, and investigating structural characteristics of chemical compounds.

Motivated by the concept of reverse vertex degree, Ediz [1, 4, 5] introduced the First Reverse Zagreb Beta Index and the Second Reverse Zagreb Index, which are defined using reverse vertex degrees rather than ordinary vertex degrees. These indices are given by

$$CM_1(G) = \sum_{uv \in E(G)} (c_u + c_v)$$

and

$$CM_2(G) = \sum_{uv \in E(G)} c_u c_v,$$

respectively.

Unlike the classical Zagreb indices, the reverse Zagreb indices emphasize the contribution of vertices with smaller degrees by assigning them larger reverse degree values. As a result, these indices provide complementary structural information and have become useful tools for investigating graph invariants and molecular descriptors. In recent years, reverse Zagreb indices have received increasing attention owing to their interesting mathematical properties and potential applications in chemical graph theory.

The computation of reverse Zagreb indices for standard graph families and graph operations has become an important direction of research. Explicit formulae for these indices not only simplify computations but also provide deeper insight into the structural behavior of different graph classes. In particular, subdivision and line graph operations significantly modify the degree distribution of a graph and therefore influence the corresponding reverse degree-based topological indices. Studying these graph transformations contributes to a better understanding of the relationship between graph operations and reverse Zagreb descriptors.

In this paper, we investigate the First Reverse Zagreb Beta Index and the Second Reverse Zagreb Index for the subdivision graphs of the tadpole graph, wheel graph, and ladder graph, together with the line graphs of their subdivision graphs. The required formulae are obtained by partitioning the edge sets according to the reverse degrees of their end vertices, leading to systematic and straightforward computations. The derived closed-form expressions provide efficient methods for evaluating these indices and enrich the theoretical study of reverse Zagreb indices. Furthermore, the obtained results may serve as a foundation for investigating reverse Zagreb indices of other graph classes and graph operations arising in graph theory and mathematical chemistry.

1.1. Tadpole Graph. Let C_n be a cycle on n vertices ($n \geq 3$) and P_k be a path on k vertices ($k \geq 1$). The *tadpole graph*, denoted by $T_{n,k}$, is the graph obtained by joining one vertex of the cycle C_n to an end vertex of the path P_k by a single edge.

Theorem 1.1. *For a subdivision tadpole graph $S(T_{n,k})$, $(n \geq 3, k \geq 1)$*

$$\begin{aligned} C_1(S(T_{n,k})) &= 8k + 8n - 2 \\ C_2(S(T_{n,k})) &= 8k + 8n - 4. \end{aligned}$$

Proof. A subdivision graph of a tadpole graph $S(T_{n,k})$ has $2n + 2k$ vertices and $2n + 2k$ edges with $\Delta(T_{n,k}) = 3$. The edge partition of the subdivision graph of a tadpole graph $S(T_{n,k})$ based on the reverse vertex degree is given in table 2. By definition of $CM_1(G)$, $CM_2(G)$ and from table 2, for $(n \geq 3, k \geq 1)$, we get

(c_u, c_v) where $uv \in E(S(T_{n,k}))$	(1, 2)	(2, 2)	(2 3)
No. of edges	3	2n+2k-4	1

TABLE 1. Edge partition of $S(T_{n,k})$ using reverse vertex degree

$$\begin{aligned} C_1(S(T_{n,k})) &= 3 \cdot (1 + 2) + (2n + 2k - 4) \cdot (2 + 2) + 1 \cdot (2 + 3) \\ &= 8k + 8n - 2 \end{aligned}$$

$$\begin{aligned} C_2(S(T_{n,k})) &= 3 \cdot (1 \cdot 2) + (2n + 2k - 4) \cdot (2 \cdot 2) + 1 \cdot (2 \cdot 3) \\ &= 8k + 8n - 4. \end{aligned}$$

□

Theorem 1.2. *For a line graph of a subdivision graph of a tadpole graph $L(T_{n,k})$, $(n \geq 3, k \geq 1)$*

$$\begin{aligned} C_1(L(S(T_{n,k}))) &= 8k + 8n - 4 \\ C_2(L(S(T_{n,k}))) &= 8k + 8n - 9. \end{aligned}$$

(c_u, c_v) where $uv \in E(L(S(T_{n,k})))$	(1, 1)	(1, 2)	(2, 2)	(2, 3)
No. of edges	3	3	2n+2k-6	1

TABLE 2. Edge partition of $L(T_{n,k})$ using reverse vertex degree

Proof. A line graph of a subdivision graph of a tadpole graph $L(S(T_{n,k}))$ has $2(n+k)$ vertices and $2n+2k+1$ edges with $\Delta(T_{n,k}) = 3$. The edge partition of the line graph of a subdivision graph of a tadpole graph $L(S(T_{n,k}))$ based on the reverse vertex degree is given in table 4. By definition of $CM_1(G)$, $CM_2(G)$ and from table 4, for $(n \geq 3, k \geq 1)$, we get

$$\begin{aligned} C_1(L(S(T_{n,k}))) &= 3 \cdot (1+1) + 3 \cdot (1+2) + (n+k-6) \cdot (2+2) + 1 \cdot (2+3) \\ &= 8k + 8n - 4 \end{aligned}$$

$$\begin{aligned} C_2(L(S(T_{n,k}))) &= 3 \cdot (1 \cdot 1) + 3 \cdot (1 \cdot 2) + (n+k-6) \cdot (2 \cdot 2) + 1 \cdot (2 \cdot 3) \\ &= 8k + 8n - 9. \end{aligned}$$

□

1.2. Wheel Graph. Let C_{n-1} be a cycle on $n-1$ vertices $(n \geq 4)$. The *wheel graph*, denoted by W_n, is the graph obtained by adding a new vertex, called the *hub*, and joining it to every vertex of the cycle C_{n-1}. Equivalently, W_n is formed by connecting the hub vertex to all vertices of C_{n-1}.

Theorem 1.3. *For a subdivision graph of wheel graph $S(W_n)$, $(n \geq 4)$*

$$CM_1(S(W_n)) = 6n^2 + 10n - 16$$

$$CM_2(S(W_n)) = 2n^3 + 10n^2 + 4n - 16.$$

Proof. A subdivision graph of wheel graph $S(W_n)$ has $3n-2$ vertices and $4n-4$ edges with $\Delta(S(W_n)) = n-1$. The edge partition of the subdivision graph of wheel graph $S(W_n)$ based on the reverse vertex degree is given in table 6. By definition of $CM_1(G)$, $CM_2(G)$ and from table 6, for $(n \geq 4)$, we get

(c_u, c_v) where $uv \in E(S(W_n))$	(1, n+2)	(n+3, n+2)
No. of edges	2(n-1)	2(n-1)

TABLE 3. Edge partition of W_n using reverse vertex degree

$$\begin{aligned} C_1(S(W_n)) &= (n-1) \cdot (1 + (n-3)) + (n-1) \cdot ((n-3) + (n-3)) \\ &= 6n^2 + 10n - 16 \end{aligned}$$

$$\begin{aligned} C_2(S(W_n)) &= (n-1) \cdot (1 \cdot (n-3)) + (n-1) \cdot ((n-3) \cdot (n-3)) \\ &= 2n^3 + 10n^2 + 4n - 16. \end{aligned}$$

□

Theorem 1.4. *For a line graph of subdivision graph of wheel graph $L(S(W_n))$, $(n \geq 4)$*

$$\begin{aligned} CM_1(L(S(W_n))) &= 10n^2 - 38n + 28 \\ CM_2(L(S(W_n))) &= 4n^3 - \frac{53n^2}{2} + \frac{109n}{2} - 32. \end{aligned}$$

Proof. A line graph of subdivision graph of wheel graph $L(S(W_n))$ has $4n-4$ vertices and $\frac{(n-1)(n+8)}{2}$ edges with $\Delta(L(S(W_n))) = n-1$. The edge partition of the line graph of subdivision graph of wheel graph $L(S(W_n))$ based on the reverse vertex degree is given in table 8. By definition of $CM_1(G)$, $CM_2(G)$ and from table 8, for $(n \geq 4)$, we get

(c_u, c_v) where $uv \in E(L(S(W_n)))$	(1, 1)	(1, n-3)	(n-3, n-3)
No. of edges	$\frac{(n-1)(n-2)}{2}$	(n-1)	(4n-4)

TABLE 4. Edge partition of $L(S(W_n))$ using reverse vertex degree

$$\begin{aligned} C_1(L(S(W_n))) &= \frac{(n-1)(n-2)}{2} \cdot (1+1) + (n-1) \cdot (1 + (n-3)) + (4n-4) \cdot ((n-3) + (n-3)) \\ &= 10n^2 - 38n + 28 \\ C_2(L(S(W_n))) &= \frac{(n-1)(n-2)}{2} \cdot (1 \cdot 1) + (n-1) \cdot (1 \cdot (n-3)) + (4n-4) \cdot ((n-3) \cdot (n-3)) \\ &= 4n^3 - \frac{53n^2}{2} + \frac{109n}{2} - 32. \end{aligned}$$

□

1.3. Ladder Graph. Let P_n be a path on n vertices $(n \geq 2)$. The *ladder graph*, denoted by L_n, is the graph obtained by joining the corresponding vertices of two disjoint copies of the path P_n by edges. Equivalently, the ladder graph L_n is the Cartesian product of the path graphs P_n and P_2, that is,

$$L_n = P_n \square P_2.$$

Theorem 1.5. *For a subdivision graph of ladder graph $S(L_n)$, $(n \geq 2)$*

$$\begin{aligned} CM_1(L_n) &= 18n - 4 \\ CM_2(L_n) &= 12n + 8. \end{aligned}$$

Proof. A subdivision graph of ladder graph $S(L_n)$ has $5n-2$ vertices and $6n-4$ edges with $\Delta(L_n) = 3$. The edge partition of the subdivision graph of ladder graph $S(L_n)$ based on the reverse vertex degree is given in table 10. By definition of $CM_1(G)$, $CM_2(G)$ and from table 10, for $(n \geq 2)$, we get

(c_u, c_v) where $uv \in E(S(L_n))$	(2, 2)	(1, 2)
No. of edges	8	6n-12

TABLE 5. Edge partition of $S(L_n)$ using reverse vertex degree

$$\begin{aligned} C_1(S(L_n)) &= 8 \cdot (2+2) + (6n-12) \cdot (1+2) \\ &= 18n - 4 \end{aligned}$$

$$\begin{aligned} C_2(S(L_n)) &= 8 \cdot (2 \cdot 2) + (6n-12) \cdot (1 \cdot 2) \\ &= 12n + 8. \end{aligned}$$

□

Theorem 1.6. *For a line graph of subdivision graph of ladder graph $L(S(L_n))$, $(n \geq 2)$*

$$CM_1(L(S(L_n))) = 18n - 4$$

$$CM_2(L(S(L_n))) = 9n + 12.$$

Proof. A line graph of subdivision graph of ladder graph $L(S(L_n))$ has $6n-4$ vertices and $9n-10$ edges with $\Delta(L(S(L_n))) = 3$. The edge partition of the line graph of subdivision graph of ladder graph $L(S(L_n))$ based on the reverse vertex degree is given in table 12. By definition of $CM_1(G)$, $CM_2(G)$ and from table 12, for $(n \geq 2)$, we get

(c_u, c_v) where $uv \in E(L(S(L_n))$	(3, 2)	(2, 1)	(1, 1)
No. of edges	4	8	6n-20

TABLE 6. Edge partition of $L(S(L_n))$ using reverse vertex degree

$$\begin{aligned} C_1(L(S(L_n))) &= 6 \cdot (2+2) + 4 \cdot (2+1) + (9n-20) \cdot (1+1) \\ &= 18n - 4 \end{aligned}$$

$$\begin{aligned} C_2(S(L_n)) &= 6 \cdot (2 \cdot 2) + 4 \cdot (2 \cdot 1) + (9n-20) \cdot (1 \cdot 1) \\ &= 9n + 12. \end{aligned}$$

□

2. Conclusion

In this paper, we derived explicit formulae for the First Reverse Zagreb Beta Index and the Second Reverse Zagreb Index of the tadpole graph, wheel graph, ladder graph, and their subdivision graphs. The computations were carried out by partitioning the edge sets according to the reverse vertex degrees of their end vertices, resulting in simple and systematic derivations of these indices. The obtained formulae provide an efficient method for computing the First Reverse Zagreb Beta Index and the Second Reverse Zagreb Index and contribute to the study of reverse Zagreb indices in graph theory and mathematical chemistry. Furthermore, the results presented in this work may serve as a foundation for investigating these indices for other graph families and graph operations in future research.

References

[1] S. Ediz, *Maximal graphs of the first reverse Zagreb beta index*, TWMS J. App. Eng. Math., **8**(1a) (2018), 306-310.

[2] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, (1972).

[3] S. M. Hosamani, V. Lokesha, I. M. Cangul, and K. M. Devendraiah, *On certain topological indices of the Derived graphs of Subdivision graphs*, TWMS J. App. Eng. Math., **6**(2) (2016), 324–332.

[4] V. R. Kulli, *On the Product Connectivity Reverse Index of Silicate and Hexagonal Networks*, International Journal of Mathematics And its Applications, **5** (2017), 175–179.

[5] V. R. Kulli, *Revan Indices of Oxide and Honeycomb Networks*, International Journal of Mathematics And its Applications, **5**(4) (2017), 663–667.

[6] P. S. Ranjini, V. Lokesha, and I. N. Cangul, *On the Zagreb indices of the line graphs of the subdivision graphs*, Appl. Math. Comput., **218** (2011), 699–702.

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