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ON THE COINCIDENCE OF SPACES WITH MARTINGALE MIXED NORM WITH THE SPACE OF BOUNDED RANDOM VARIABLES IN THE CASE OF IRREGULAR STOCHASTIC BASES

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Abstract. For spaces $L_{\mathbf{p}}$ of random variables with martingale mixed norm the theorem of the coincidence of $L_{\mathbf{p}}$ with the space L_∞ is generalized. Namely, in [1] a criteria for such a coincidence was obtained in the case of a stochastic basis with a regular filtration, and in the present paper, a transition was made to irregular ones. It is a much more complicated problem. In particular, a family of irregular filtrations was found, for which a criterion for the coincidence of $L_{\mathbf{p}}$ and L_∞ was obtained.

1. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space, where Ω is an arbitrary set, $\mathcal{A}$ is a σ-algebra on Ω, and P is a probability measure on $(\Omega, \mathcal{A})$. Fix a filtration (i.e. increasing sequence of σ-algebras) $\mathbf{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ such that $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{A}$ is the σ-algebra generated by this filtration. A triplet $(\Omega, \mathbf{F}, P)$ is called stochastic basis (s.b.).

Let f be an integrable random variable (r.v.) on $(\Omega, \mathcal{A}, P)$ and $f_n := E^P\left[f|\mathcal{F}_n\right]$ (i.e. $\mathbf{f} = (f_n, \mathbf{F}, P)$ is a martingale). Let $\mathbf{p} = (p_1, p_2, \ldots, p_n, \ldots)$ be an infinite dimensional vector $(1 \le p_k \le \infty, k = 1, 2, \ldots)$. Martingale mixed norm for r.v. f is defined as follows:

$$\|f\|_{\mathbf{p}} := \sup_n \left\| \cdots \left\| \left\| f_n \right\|_{p_n, \mathcal{F}_{n-1}} \right\|_{p_{n-1}, \mathcal{F}_{n-2}} \cdots \right\|_{p_1, \mathcal{F}_0}, \tag{1.1}$$

where the expression $g_{p,\mathcal{F}}$ is equel to $\left(E[|g|^p \middle| \mathcal{F}] \right)^{1/p}$ if $p < \infty$ and $\lim\limits_{r \uparrow \infty} \uparrow \|g\|_{r,\mathcal{F}}$ if $p = \infty$. In what follows, we will also use the following notations:

$$\mathbf{p}^n = (p_1, p_2, \ldots, p_n)$$

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and

$$\|g\|_{\mathbf{p}^n} = \left\| \dots \left\| \|g\|_{p_n,\mathcal{F}_{n-1}} \right\|_{p_{n-1},\mathcal{F}_{n-2}} \dots \right\|_{p_1,\mathcal{F}_0}.$$

If $\|f\|_{\mathbf{p}} < \infty$, we say that $f \in L_{\mathbf{p}}$. Remark that if $\forall k\,(1 \le k < \infty)$ $p_k = p$, then $\|f\|_{\mathbf{p}} = L_p := L_p(\Omega, \mathcal{A}, P)$.

Throughout this paper, we will assume that $\forall n\,(1 \le n < \infty)$ σ-algebra $\mathcal{F}_n$ is finite and generated by the partition $\mathcal{D}_n$ of Ω into atoms of strictly positive probability. Remark that $\mathcal{D}_0 = \{\Omega\}$. We assume that $\forall n \ge 0$ the transition from time n to time $n+1$ cannot be trivial, i.e. $|\mathcal{D}_n| \ge n+1$, where $|\mathcal{D}_n|$ denotes the number of elements of the set $\mathcal{D}_n$. This means that in the transition from n to $n+1$ at least one atom from $\mathcal{D}_n$ is split up. When moving from the point in time n to the point $n+1$ an atom $A_n \in \mathcal{D}_n$ is split into exactly m atoms of $\mathcal{D}_{n+1}$, we call m branch index of A_n.

Let $A_n \in \mathcal{D}_n$ and $A_{n-1} \in \mathcal{D}_{n-1}$ be such that $A_n \subset A_{n-1}$ (it is clear that $A_0 = \Omega$). Denote $A'_n := A_{n-1} \setminus A_n$ and $\mathcal{D}'_n := \mathcal{D}_n \setminus \{A_n \in \mathcal{D}_n | A'_n = \emptyset\}$. Let us introduce the following two sequences, which will play an important role in what follows:

$$c_n := \min_{A_n \in \mathcal{D}'_n} \frac{P(A_n)}{P(A_{n-1})} = \min_{A_n \in \mathcal{D}_n} \frac{P(A_n)}{P(A_{n-1})},\ n = 1, 2, \dots, \tag{1.2}$$

and

$$c'_n := \min_{A_n \in \mathcal{D}'_n} \frac{P(A'_n)}{P(A_{n-1})},\ n = 1, 2, \dots. \tag{1.3}$$

Formulas (1.2) and (1.3) are understood in the following sense. Numbers c_n and c'_n are calculated for a fixed n, and the same index n is fixed on the right-hand side of these formulas as the index at $\mathcal{D}'$ and $\mathcal{D}$. In this case, the minima are taken under the condition that A_n runs through the entire set of atoms $\mathcal{D}'_n$.

Proposition 1.1. *For any $n \ge 1$ the inequalities*

$$0 < c_n \le c'_n < 1 \tag{1.4}$$

and

$$c_n \le 1 - c'_n \tag{1.5}$$

are fulfilled.

If each atom from $\bigcup_{n=0}^{\infty} \mathcal{D}_n$ has an branch index not exceeding 2, then $\forall n \ge 1$ $c_n = c'_n$

Proof. We prove only formula (1.5). Let the expression (1.3) on an event $\tilde{A}'_n \in \mathcal{D}_n$ achieve the equality $c'_n = \frac{P(\tilde{A}'_n)}{P(\tilde{A}_{n-1})}$. Then

$$c'_n = \frac{P(\tilde{A}'_n)}{P(\tilde{A}_{n-1})} = \frac{P(\tilde{A}_{n-1}) - P(\tilde{A}_n)}{P(\tilde{A}_{n-1})} = 1 - \frac{P(\tilde{A}_n)}{P(\tilde{A}_{n-1})} \le 1 - c_n$$

(c.f. (1.2)). From here we get (1.5) □

Definition 1.2. We say that a s.b. $(\Omega, \mathbf{F}, P)$ is forked if $\forall n \ge 1$ $\mathcal{D}_n = \mathcal{D}'_n$, and locally forked if there exists a chain of embedded atoms $A_0 \supset A_1 \dots \supset A_n \supset \dots$ such that $\forall n \ge 1$ $A_n \in \mathcal{D}'_n$.

It is clear that if a s.b. $(\Omega, \mathbf{F}, P)$ is forked, then it is locally forked.

The following definition is borrowed from [2].

Definition 1.3. We say that a s.b. $(\Omega, \mathbf{F}, P)$ is regular if there exists an independent on n constant c, $0 < c < 1$, such that $\forall n \geq 1$, for each atom $A_n \in \mathcal{D}_n$, and for the atom $A_{n-1} \in \mathcal{D}_{n-1}$ containing A_n the following inequality is fulfilled:

$$P(A_n) \geq c \cdot P(A_{n-1}).$$

Otherwise, s.b. is called irregular.

In the case of a locally forked regular stochastic basis, the criteria for the coincidence of $L_{\mathbf{p}}$ and L_∞ obtained in [1] (see also [3]) is presented in this paper by Corollary 3.1. The main result of this paper is Corollary 3.3, which follows from Theorems 2.1 and 2.2.

2. Theorems on the coincidence of $L_{\mathbf{p}}$ and L_∞

Theorem 2.1. *If*

$$\prod_{k=1}^{\infty} c_k^{\frac{1}{p_k}} > 0, \tag{2.1}$$

then $L_{\mathbf{p}} = L_\infty$.

Proof. Prove first that $\forall n = 1, 2, \dots$ and for any event $A \in \mathcal{F}_n$

$$\|I_A\|_{\mathbf{p}} \geq \prod_{k=1}^{n} c_k^{1/p_k},\ 1 \leq \mathbf{p} \leq \infty. \tag{2.2}$$

Consider an atom $A_n \in \mathcal{D}_n$ such that $A_n \subset A$ and the atom $A_{n-1} \in \mathcal{D}_{n-1}$ such that $A_n \subset A_{n-1}$. We have: $\|I_A\|_{\mathbf{p}} \geq \|I_{A_n}\|_{\mathbf{p}} = \|I_{A_n}\|_{\mathbf{p}^n}$. Now we get for $p_n < \infty$:

$$\|I_{A_n}\|_{p_n, \mathcal{F}_{n-1}} = (E[I_{A_n} \mid \mathcal{F}_{n-1}])^{1/p_n} = \left(\frac{P(A_n)}{P(A_{n-1})} \cdot I_{A_{n-1}}\right)^{1/p_n} \geq c_n^{1/p_n} \cdot I_{A_{n-1}}.$$

It is clear, that the obtained inequality is valid for $p_n = \infty$ too. Therefore, (2.2) follows from the formula (1.1).

Now let the condition of theorem 2.1 be satisfy, $f \in L_{\mathbf{p}}$, but $f \notin L_\infty$. Denote by A_n such atom in $\mathcal{D}_n$, for which $\|f_n\|_\infty = |a_n|$, where a_n is the value of f_n on A_n. Therefore, $\sup_n |a_n| = \infty$. We get:

$$\|f\|_{\mathbf{p}} = \sup_n \|f_n\|_{\mathbf{p}} \geq \sup_n |a_n| \cdot \|I_{A_n}\|_{\mathbf{p}} \geq \sup_n \left(|a_n| \cdot \prod_{k=1}^{n} c_k^{1/p_k}\right) = \infty.$$

The contradiction shows that $L_{\mathbf{p}} = L_\infty$. □

Theorem 2.2. *Let s.b.* $(\Omega, \mathbf{F}, P)$ *is locally forked. Then from* $L_{\mathbf{p}} = L_\infty$ *it follows*

$$\prod_{k=1}^{\infty} (1 - c'_k)^{\frac{1}{p_k}} > 0. \tag{2.3}$$

Proof. Consider the chain $A_0 \supset A_1 \cdots \supset A_n \supset \dots$ from Definition 1.2. Prove first that

$$\|I_{A_n}\|_{\mathbf{p}} \le \prod_{k=1}^{n} (1 - c'_k)^{1/p_k},\ 1 \le \mathbf{p} \le \infty. \tag{2.4}$$

We obtain if $p_n < \infty$:

$$\|I_{A_n}\|_{p_n, \mathcal{F}_{n-1}} = \left(\frac{P(A_n)}{P(A_{n-1})} \cdot I_{A_{n-1}} \right)^{1/r_n} = \left(\frac{P(A_{n-1}) - P(A'_n)}{P(A_{n-1})} \cdot I_{A_{n-1}} \right)^{1/r_n}$$
$$\le (1 - c'_n)^{1/r_n} \cdot I_{A_{n-1}}.$$

The obtained inequality is valid for $p_n = \infty$ too. Inequality (2.4) follows from the formula (1.1).

Suppose now that $L_{\mathbf{p}} = L_\infty$. Then there exists a strictly positive number d such that for any r.v. $f \in L_\infty$ the inequality $\|f\|_\infty \le d \|f\|_{\mathbf{p}}$ is true. Applying this inequality for r.v. $f = I_{A_n}$ and taking into account the inequality (2.4), we obtain the inequality:

$$1 = \|I_{A_n}\|_\infty \le d \|I_{A_n}\|_{\mathbf{p}} \le d \cdot \prod_{k=1}^{n} (1 - c'_n)^{1/p_k}.$$

It follows from it the inequality we need. □

3. Corollaries and examples

Corollary 3.1. *Let s.b.* $(\Omega, \mathbf{F}, P)$ *is locally forked and regular. Equality* $L_{\mathbf{p}} = L_\infty$ *holds if and only if* $\sum_{k=1}^{\infty} 1/r_k < \infty$.

Proof. Denote $c = \inf_{n \ge 1} c_n > 0$ and $c' = \inf_{n \ge 1} c'_n$. It is obvious that the regularity of s.b. $(\Omega, \mathbf{F}, P)$ is equivalent to the inequality $c > 0$.

Let $\sum_{k=1}^{\infty} 1/r_k < \infty$. Since $\prod_{k=1}^{\infty} c_k^{1/p_k} \ge \prod_{k=1}^{\infty} c^{1/p_k} > 0$, it follows from Theorem 2.1 that $L_{\mathbf{p}} = L_\infty$.

Let $L_{\mathbf{p}} = L_\infty$. We obtain from the inequality $1 - c \ge 1 - c'_n$ and Theorem 2.2 that $\prod_{k=1}^{\infty} (1 - c)^{1/p_k} \ge \prod_{k=1}^{\infty} (1 - c'_k)^{1/p_k} > 0$. This implies the inequality $\sum_{k=1}^{\infty} 1/r_k < \infty$. □

Corollary 3.2. *Let s.b.* $(\Omega, \mathbf{F}, P)$ *is locally forked. If* $\forall n \ge 1$ $c_n = 1 - c'_n$*, then* $L_{\mathbf{p}} = L_\infty$ *if and only if the inequality* (2.1) *holds.*

Proof. The proof follows from Theorems 2.1 and 2.2. □

Corollary 3.3. *Let* $(\Omega, \mathbf{F}, P)$ *is locally forked. If* $\lim\limits_{n \to \infty} c_n = 0$ *and also* $\forall n \ge 1$ *and for two positive constants* c *and* d $0 < c \le \frac{1 - c'_n}{c_n} \le d$*, then* $L_{\mathbf{p}} = L_\infty$ *if and only if the inequality* (2.1) *holds.*

Proof. From the relations $\lim\limits_{n \to \infty} c_n = 0$ and $0 < c \le \frac{1 - c'_n}{c_n} \le d$ it easily follows that $\ln(1 - c'_n) \sim \ln c_n$ as $n \to \infty$. Hence inequalities (2.1) and (2.3) are equivalent. This implies the required assertion. □

Corollary 3.4. *Let a locally forked s.b.* $(\Omega, \mathbf{F}, P)$ *is such that each atom from* $\bigcup_{n=0}^{\infty} \mathcal{D}_n$ *has a branch index not exceeding 2. If* $\lim\limits_{n\to\infty} c_n = 0$*, then*

$$L_{\mathbf{p}} = L_\infty \Rightarrow \sum_{k=1}^{\infty} \frac{c_k}{p_k} < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{|\ln c_k|}{p_k} < \infty \Rightarrow L_{\mathbf{p}} = L_\infty$$

.

Proof. From Proposition 1.1 we obtain equality $c'_n = c_n \, \forall n \geq 1$. Applying Theorems 2.2 and 2.1 we have:

$$L_{\mathbf{p}} = L_\infty \Rightarrow \prod_{k=1}^{\infty} (1 - c'_k)^{\frac{1}{p_k}} > 0 \Leftrightarrow \prod_{k=1}^{\infty} (1 - c_k)^{\frac{1}{p_k}} > 0 \Leftrightarrow \sum_{k=1}^{\infty} \frac{c_k}{p_k} < \infty,$$

$$\sum_{k=1}^{\infty} \frac{|\ln c_k|}{p_k} < \infty \Leftrightarrow \prod_{k=1}^{\infty} c_k^{\frac{1}{p_k}} > 0 \Rightarrow L_{\mathbf{p}} = L_\infty.$$

Corollary 3.4 shows that, generally speaking, Theorems 2.1 and 2.2 provide a fairly large gap between necessary and sufficient conditions for the coincidence of spaces $L_{\mathbf{p}}$ and L_∞. □

Example 3.5. Let $(m_n)_{n=1}^{\infty}$, where $m_n \geq 2 \; \forall n \leq 1$, be a sequence of natural numbers. Suppose that $\forall n \geq 1$ each atom from $\mathcal{D}_{n-1}$ has a branch index m_n and if $A_n \in \mathcal{D}_n$ and $A_{n-1} \in \mathcal{D}_{n-1}$ are such that $A_n \subset A_{n-1}$, then $P(A_n) = \frac{P(A_{n-1})}{m_n}$. In this case $c_n = \frac{1}{m_n}$, $c'_n = \frac{m_n - 1}{m_n}$ and hence $c_n = 1 - c'_n$. From Corollary 3.2 it easily follows that $L_{\mathbf{p}} = L_\infty$ if and only if $\sum_{k=1}^{\infty} \frac{\ln(m_k)}{p_k} < \infty$. Remark that if $\sup_{k \leq 1} m_k = \infty$ we have here irregular s.b.

Example 3.6. In the scheme considered in Example 3.5, change a little the data. For all $n \geq 1$ choose arbitrarily an atom $A_{n-1} \in \mathcal{D}_{n-1}$ and two atoms $\tilde{A}_n$ and $\hat{A}_n$ from $\mathcal{D}_n$ such that $\tilde{A}_n \subset A_{n-1}$ and $\hat{A}_n \subset A_{n-1}$. Let us put $P(\tilde{A}_n) = \frac{P(A_{n-1})}{2m_n}$, $P(\hat{A}_n) = \frac{3P(A_{n-1})}{2m_n}$ and the probabilities of all other atoms are determined as in Example 3.5. We have $c_n = \frac{1}{2m_n}$ and $1 - c'_n = \frac{3}{2m_n}$. The conditions of Corollary 3.3 are satisfied and hence $L_{\mathbf{p}} = L_\infty$ if and only if $\sum_{k=1}^{\infty} \frac{\ln(2m_k)}{p_k} < \infty$ $\Leftrightarrow \sum_{k=1}^{\infty} \frac{\ln(m_k)}{p_k} < \infty$.

4. Conclusion

This article is made within the framework of the topic related to the general question posed by Professor E.M. Semenov: in what cases do martingale spaces $L_{\mathbf{p}}$ with mixed norm coincide with the classical spaces L_r, $1 \leq r \leq \infty$? At present, only partial answers to this question are known in cases $r = 1$ and $r = \infty$ (c.f. this article and works [1], [3]). To solve this problem in a more general form, it seems that completely new ideas and methods are needed.

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