Global and Stochastic Analysis
Vol. 6, No. 1, June (2019), 1-6

MUK PUBLICATIONS
Open Access Publisher

GUIDING POTENTIALS AND PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS ON MANIFOLDS

YURI GLIKLIKH, SERGEI KORNEV AND VALERI OBUKHOVSKII

Abstract. This paper deals with the modification of the theory of guiding functions such that it becomes applicable to the investigation of ordinary differential equations on noncompact finite-dimensional smooth manifolds. The main purpose is to apply this theory to searching for periodic solutions of those equations. We introduce two different constructions of topological index compatible with the problem under consideration and prove the existence of a periodic solution basing on the theory of guiding potentials. All results are formulated in terms of internal topological structures of manifolds, i.e., geometrical notions are used only as a machinery and are not involved into the formulations.

1. Introduction

The method of guiding functions was originally developed by M.A. Krasnosel'-skii and A.I. Perov as one of the tools for solving problems of periodic oscillations and bounded solutions in nonlinear systems (see, e.g., [1, 2, 3, 4]). Being geometrically clear and simple to use in applications, it became one of the most powerful and effective instruments for dealing with periodic problems. In the subsequent years it was generalized and extended in many various directions and found important applications not only in the frameworks of periodic and bounded solutions but also to the investigations of qualitative behavior of solutions including such properties as bifurcations, asymptotic estimates and others (see, e.g., the monograph [5] and the references therein).

It is worth noting that all these numerous extensions and applications were dealing only with systems governed by differential equations and inclusions in finite dimensional linear spaces or in infinite dimensional Hilbert spaces. Meantime it is well known that the problems of qualitative behavior of solutions are of a great importance also for systems given on manifolds which arise from applications in mathematical physics and in other branches of natural sciences (see, e.g., [6]).

Date: Date of Submission May 11, 2018 ; Date of Acceptance October 1, 2018 , Communicated by Yuri I. Sapronov.

2010 *Mathematics Subject Classification.* Primary 34A26; Secondary 34C25 46T99 58C99.

Key words and phrases. smooth manifolds, ordinary differential equations, guiding potentials, periodic solutions.

* The research is supported by the RFBR Grant 18-01-00048 and by the Ministry of Education and Science of the Russian Federation in the frameworks of the project part of the state work quota (Project No 1.3464.2017/4.6).

In the present paper we are trying to extend the method of guiding functions towards the study of the periodic problem for differential equations on smooth noncompact finite dimensional manifolds. We introduce two different constructions of the topological index compatible with the problem under consideration and prove the existence of a periodic solution basing on the theory of guiding potentials. All results are formulated in terms of internal topological structures of manifolds, i.e., geometrical notions are used only as a machinery and are not involved into the formulations.

2. The Topological Index of Maps on a Manifold

Let M be an n-dimensional noncompact manifold and Ω a domain in M homeomorphic to an open ball in $\mathbb{R}^n$. By $\overline{\Omega}$ we denote the closure of Ω, and by $\partial\Omega$ its boundary. We will suppose everywhere that $\overline{\Omega}$ is homeomorphic to a closed ball and hence it is compact.[1]

Let $F : \overline{\Omega} \to M$ be a continuous map which is fixed point free on the boundary $\partial\Omega$ (i.e., $x \neq F(x)$, $\forall x \in \partial\Omega$).

By the Whitney theorem (see, e.g., [7]) the manifold M can be embedded into the Euclidean space $\mathbb{R}^N$ of sufficiently large dimension $N \geq 2n+1$. Let $W \subset \mathbb{R}^N$ be a tubular neighborhood of M and $r\colon W \to M$ a retraction. Let $U \subset W$ be an open set such that $r(U) = \Omega$. Let us extend the map F to $\overline{U}$ as $\overline{F} : \overline{U} \to M \subset \mathbb{R}^N$ by the formula

$$\overline{F}(x) = F(r(x)).$$

By construction, it is clear that the map $\overline{F}$ is fixed point free on the boundary ∂U. This means that for the corresponding vector field $I - \overline{F}$, where $I\colon \mathbb{R}^N \to \mathbb{R}^N$ is the identity, the topological degree (or rotation) $\gamma(I - \overline{F}, \partial U)$ is well defined (see, e.g., [2, 4]).

Definition 2.1. The fixed point index of the map F on $\partial\Omega$ is defined in the following way

$$ind(F, \partial\Omega) := \gamma(I - \overline{F}, \partial U).$$

First of all, let us mention that the above notion is well defined.

Theorem 2.2. *The fixed point index $ind(F, \partial\Omega)$ does not depend on the choice of space $\mathbb{R}^N$, embedding, open set U, and retraction r.*

The proof of Theorem 2.2 (even for an infinite-dimensional case) can be found in [8].

Immediately from the construction it follows that the characteristic, defined above, possesses usual properties, including homotopy invariance. It is also easy to see that its difference from zero implies the existence of at least one fixed point of F in Ω.

For dealing with zeros of tangent and cotangent vector fields inside $\overline{\Omega}$ we have to apply another construction of index.

---

[1]Notice that it does not follow from the homeomorphism of Ω to an open ball.

Let a continuous tangent vector field X having no zero singular points be given on $\partial\Omega$. Since $\overline{\Omega}$ is homeomorphic to a closed ball, there exists a neighbourhood V of $\overline{\Omega}$ that is a chart, i.e., it can be presented as a set in $\mathbb{R}^n$, homeomorphic to an open ball. According to this presentation the tangent vectors of X are becoming vectors in $\mathbb{R}^n$ (are embedded into $\mathbb{R}^n$), and for the field X on $\partial\Omega$ the ordinary topological degree of a vector field is well defined. In order not to confuse it with the index of $\gamma(F, \partial\Omega)$ type, we denote it by $\widehat{\gamma}(X, \partial\Omega)$. Note that the presentation of V as a chart is ambiguously determined, but different versions of such presentation are diffeomorphic to each other and so $\widehat{\gamma}$ does not depend on the choice of such presentation, i.e., it is well defined. Since by the use of a scalar product in $\mathbb{R}^n$ the cotangent vectors (1-forms) can be identified with the tangent ones, the index $\widehat{\gamma}$ is well defined for cotangent vectors also.

Let a continuous map $\Phi : \partial\Omega \to M$ be given.

Definition 2.3. We call Φ admissible on $\overline{\Omega}$, if for every $m \in \overline{\Omega}$ the point $\Phi(m)$ belongs to V.

Definition 2.4. The fixed point index of $\widehat{\gamma}$ type for an admissible map Φ is defined by the formula

$$\widehat{ind}(\Phi, \partial\Omega) := \widehat{\gamma}(I - \Phi, \partial\Omega).$$

Consider the case where the map F mentioned above, is admissible. In this situation, besides index $ind(F, \partial\Omega)$ we can deal with the index $\widehat{ind}(F, \partial\Omega)$.

Theorem 2.5. *Let the mapping F be admissible on $\overline{\Omega}$. Then*

$$ind(F, \partial\Omega) = \widehat{ind}(F, \partial\Omega).$$

Proof. In fact we have to show that $\gamma(I - \overline{F}, \partial U)$ coincides with $\widehat{\gamma}(I - F, \partial\Omega)$. Since $\overline{F}$ sends $\overline{U}$ to V and on V it coincides with F, this fact follows from the principle of a map restriction (see, e.g., [2, 4]). □

3. The Main Result

Recall the following notion.

Definition 3.1. A map from the topological space Y to the topological space Z is called proper, if the preimage of every relatively compact set in Z is relatively compact in Y. In particular, a function $\varphi : M \to \mathbb{R}$ is called proper if the preimage of every bounded subset of $\mathbb{R}$ is relatively compact in M.

Remark 3.2. It is easy to see that a function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is proper if and only if it is coercive, i.e., $\|x\| \to \infty$ implies $|\varphi(x)| \to \infty$.

The replacement of the coercivity property with that of properness allows us not to deal with metric notions on the manifold from the very beginning. The notion of properness is formulated in topological terms, i.e., it does not require the use of additional structures on a manifold.

By the symbol Xf we denote the derivative of a function $f\colon M \to \mathbb{R}$ in the direction of a vector field X. Recall that always

$$Xf = df(X),$$

where $df(X)$ denotes the value of the co-vector df on the vector X. Notice also that if a Riemannian metric is given on M (e.g., the Euclidean metric $(\cdot,\cdot)$ after embedding of M into $\mathbb{R}^N$), then

$$Xf = df(X) = (grad\, f, X).$$

Notice that if a non-autonomous vector field $X(t,m)$ for all t does not equal to zero on $\partial\Omega$, the fields at different values of t are homotopic to each other without zeroes on $\partial\Omega$, i.e., they have the same indices $\widehat{\gamma}$. In pasrticular the fields $X(t,m)$ on $\partial\Omega$ for all t do not equal to zero if for all t on $\partial\Omega$ the relation $X\varphi > 0$ holds for a certain function φ.

The next proposition is a simple generalisation of the theorem on the existence of a solution to the Cauchy problem under the estimates of one-sided type (see, e.g., [3, 6]).

Proposition 3.3. *Let $f : M \to \mathbb{R}$ be a proper smooth positive function and for a vector field X the inequality*

$$Xf < C \tag{3.1}$$

holds at all points for a certain constant $C > 0$. Then every solution of the Cauchy problem for the differential equation

$$x' = X$$

is well defined for all $t \in [0, \infty)$.

Proof. Indeed, from the the theorem on the existence of a local solution to the Cauchy problem it follows that the domain of a solution is an open set. However, inequality (3.1) implies that on every open bounded interval $[0,\varepsilon)$ the values of the function f on the solution are not greater than $C\varepsilon$. Hence, the solution on the open interval $[0,\varepsilon)$ belongs to a relatively compact set in M, i.e., the solution can be prolonged to the closed interval $[0,\varepsilon]$. Thus, the domain of the solution is both open and closed. But the only set in metric topology on $\mathbb{R}$ that is both open and closed, is the entire $\mathbb{R}$. □

We will consider the problem of existence of T-periodic solutions of the differential equation

$$\frac{d}{dt}m(t) = X(t, m(t)). \tag{3.2}$$

Our main result is the following assertion.

Theorem 3.4. *Let $X(t,m)$ be a smooth T-periodic vector field on $\mathbb{R} \times M$, i.e.,*

$$X(t+T, m) = X(t,m), \quad \forall t \in \mathbb{R},\, m \in M$$

and let a smooth proper positive function $\varphi : M \to \mathbb{R}_+$ be such that:

(*i*) *for every* $t \in [0, T]$ *the inequality*

$$X\varphi < \infty, \tag{3.3}$$

holds;

(*ii*) *the relation*

$$X\varphi > 0, \tag{3.4}$$

is fulfilled on $\partial\Omega$*;*

(*iii*) *every point* $x_0 \in \partial\Omega$ *is a* T*-non-recurrence point for solutions of equation (3.2), i.e.,* $x(t) \neq x_0,\ \forall t \in (0, T]$ *for every solution* $x(\cdot)$ *of (3.2) emanating from* x_0;

(*iv*) *for the covector field (1-form)* $d\varphi$ *the relation*

$$\widehat{\gamma}(d\varphi, \partial\Omega) \neq 0. \tag{3.5}$$

holds.

Then there exists a T*-periodic solution of equation (3.2) with an initial condition in* Ω.

Proof. By Proposition 3.3 it follows from (3.3) that all solutions of equation (3.2) exist for $t \in [0, \infty)$. From (3.4) it follows that on $\partial\Omega$ for all t the field X does not equal to zero, i.e., the index $\widehat{\gamma}(X, \partial\Omega)$ is well defined. Recall that the transition from $d\varphi$ to $grad\,\varphi$ is realized by a fiber-wise linear isomorphism of the tangent and cotangent bundles. Hence,

$$\widehat{\gamma}(d\varphi, \partial\Omega) = \widehat{\gamma}(grad\,\varphi, \partial\Omega) \neq 0.$$

In addition, from (3.4) it follows that for every t the inequality

$$(X, grad\,\varphi) > 0$$

holds, i.e., the angle between the vectors X and $grad\,\varphi$ is acute. Thus, the linear homotopy between these vectors has no zeroes on $\partial\Omega$ and so

$$\widehat{\gamma}(X, \partial\Omega) \neq 0.$$

Consider on $\partial\Omega$ the operator of translation $u(T) : \partial\Omega \to M$ along the trajectories of equation (3.2), that sends a point $m \in \partial\Omega$ to the value of the solution of (3.2) with initial condition m at the time T (see [2, 3, 4, 5]). From condition (*iii*) it follows that the index $ind(u(T), \partial\Omega)$ for $u(T)$ is well defined. Consider the family of the operators of translation $u(t)$ for $t \in [0, T]$ along the trajectories of (3.2). Since $\partial\Omega$ is compact, there exists $t^* \in [0, T]$ such that $u(t^*)$ is admissible on $\partial\Omega$. Moreover, condition (*iii*) yields that the operators $u(t)$ for $t \in [0, T]$ form the homotopy without fixed points on $\partial\Omega$ that connects $u(T)$ with $u(t^*)$ implying

$$ind(u(T), \partial\Omega) = ind(u(t^*), \partial\Omega).$$

From Theorem 2.5 we get

$$\widehat{ind}(u(t^*), \partial\Omega) = ind(u(t^*), \partial\Omega).$$

In complete analogy with the proof of [3, Lemma 6.1] it can be shown that the fields $X(0)$ and $u(t^*) - I$ do not admit opposite directions on $\partial\Omega$. This yields

$$\widehat{\gamma}(X, \partial\Omega) = \widehat{ind}(u(t^*), \partial\Omega).$$

From all the above arguments we obtain that

$$ind(u(T), \partial\Omega) \neq 0.$$

Hence, inside Ω there exists a fixed point of $u(T)$, implying that a solution of (3.2) emanating from this point is T-periodic. □

Let us present an example of situation where condition (3.5) is fulfilled.

Proposition 3.5. *Let there exist a vector field Y on $\partial\Omega$, transversal to $\partial\Omega$ and such that $d\varphi(Y) > 0$. Then condition (3.5) is fulfilled.*

Proof. Since Y is transversal to $\partial\Omega$,

$$\widehat{\gamma}(Y, \partial\Omega) = \pm 1.$$

From $d\varphi(Y) > 0$ it follows that

$$(Y, grad\varphi) > 0$$

and as above, by the linear homotopy without zeroes on $\partial\Omega$ we obtain that

$$\widehat{\gamma}(grad\varphi, \partial\Omega) = \pm 1.$$

Hence

$$\widehat{\gamma}(d\varphi, \partial\Omega) = \pm 1.$$

□

References

[1] Krasnoselskii, M. A.; Perov, A. I.: *On a certain principle of existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations.* (Russian) Dokl. Akad. Nauk SSSR 123, 1958, 235–238.

[2] Krasnosel'skii, M. A.: *Topological Methods in the Theory of Nonlinear Integral Equations*, New York, Macmillan, 1964.

[3] Krasnoselskii, M.A.: *The Operator of Translation Along the Trajectories of Differential Equations*, Providence, RI, AMS, 1968.

[4] Krasnosel'skii, M. A. and Zabreiko, P. P.: *Geometrical Methods of Nonlinear Analysis* (Grundlehren der Mathematischen Wissenschaften, 263), Berlin, Springer-Verlag, 1984.

[5] Obukhovskii, V.; Zecca, P.; Loi N.V. and Kornev, S. *Method of Guiding Functions in Problems of Nonlinear Analysis.* Lecture Notes in Mathematics, 2076. Springer, Heidelberg, 2013.

[6] Gliklikh Yu. E.: *Global and stochastic analysis with applications to mathematical physics*, London, Springer-Verlag, 2011.

[7] Hirsch, M.W.: *Differential Topology.* (Graduate Texts in Mathematics, 33), New York, Springer-Verlag, 1994.

[8] Borisovich, Ju. G. and Gliklih, Ju. E.: *Fixed points of mappings of Banach manifolds and some applications.* Nonlinear Anal., 4, no. 1 (1980) 165–192.

Yuri Gliklikh: Professor of Voronezh State University, Voronezh, 394018, Russia
E-mail address: yeg@ math.vsu.ru

Sergei Kornev: Professor of Voronezh State Pedagogical University, Voronezh, 394043, Russia
E-mail address: kornev$_v$*rn@rambler.ru*

Valeri Obukhovskii: Head of Department of Higher Mathematics, Voronezh State Pedagogical University, Voronezh, 394043, Russia
E-mail address: valerio-ob2000@mail.ru