Global and Stochastic Analysis
Vol. 10 No. 2 (July-December, 2023)

Received: 15th April 2023 *Revised: 20th June 2023* *Accepted: 25th July 2023*

ON THE GENERALIZED MIXED FRACTIONAL BROWNIAN MOTION TIME CHANGED BY INVERSE α-STABLE SUBORDINATOR

EZZEDINE MLIKI AND MARAM ALWOHAIBI

Abstract. Time-changed stochastic processes have attracted much attention and wide interest due to their extensive applications, especially in financial time series, biology and physics. This paper pays attention to a fractional stochastic process, defined by taking linear combinations of a finite number of independent fractional Brownian motions with different Hurst indices called the generalized mixed fractional Brownian motion, which is a Gaussian process with stationary increments exhibit long range dependence property controlled by the Hurst indices. We prove that under some condition on the Hurst indices the generalized mixed fractional Brownian motion time changed by inverse α-stable subordinator is of a long-range dependence property. As application we deduce that the mixed fractional Brownian motion of Hurst index H has long range dependence property for all $H > \frac{1}{2}$.

1. Introduction

Fractional Brownian motion introduced by Mandelbrot and Ness [17] is a self-similar process with stationary increments. The process has long-range dependence controlled by the Hurst index. Fractional Brownian motion is a subclass of the fractional stable process (see [19]). Both fractional Brownian motion and fractional stable process have been applied in areas such as data communication, hydrology, and finance. In particular, the fractional stable process has been applied to the modeling of financial time series having long-range dependence (see [20]). The fractional Brownian motion (fBm) $B^H = \{B_t^H,\, t \geq 0\}$ with Hirst index $H \in (0, 1)$, is a centered Gaussian process with covariance function

$$\mathrm{Cov}(B_t^H,\, B_s^H) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \quad s, t \geq 0,$$

where H is a real number in $(0, 1)$, called the Hurst index. The case $H = \frac{1}{2}$ corresponds to the Brownian motion.

An extension of the fBm was introduced by Cheridito [5], called the mixed fractional Brownian motion (mfBm) which is a linear combination between a Brownian motion and an independent fractional Brownian motion of Hurst index H, with stationary increments exhibit a long-range dependent for $H > \frac{1}{2}$. A mfBm of parameters a_1, a_2 and H is a process $M^H(a_1, a_2) = \{M_t^H(a_1, a_2),\ t \geq 0\}$, defined on

2000 *Mathematics Subject Classification.* Primary 60G22; Secondary 60G20; 60G18.
Key words and phrases. Fractional Braownian motion, Generalized mixed fractional Brownian motion; Long-range dependence; Inverse α-stable subordinator.

some probability space $(\Omega, \mathcal{F}, P)$ by

$$M_t^H(a_1, a_2) = a_1 B_t + a_2 B_t^H, \quad t \geq 0,$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion and $B^H = \left\{B_t^H, t \geq 0\right\}$ is an independent fractional Brownian motion of Hurst index $H \in (0, 1)$. We refer also to [3, 5, 8, 9] for further information on mfBm process.

C. Elnouty [8] propose a generalisation of the mfBm called fractional mixed fractional Brownian motion (fmfBm) of parameters a_1, a_2 and Hirsh index $H = (H_1, H_2)$. A fmfBm is a process $N^H(a_1, a_2) = \{N_t^H(a_1, a_2),\ t \geq 0\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$N_t^H(a_1, a_2) = a_1 B_t^{H_1} + a_2 B_t^{H_2}, \quad t \geq 0,$$

where $B^{H_i} = \{B_t^{H_i},\ t \geq 0\}$ are independent fractional Brownian motion of Hurst index $H_i \in (0, 1)$ for $i = 1, 2$. Also the fmfBm was study by Miao, Y et al. [31].

The fractional mixed fractional Brownian motion has been further generalized by Thäle in 2009 [38] to the generalized mixed fractional Brownian motion. A generalized mixed fractional Brownian motion (gmfBm) of parameter $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0, 1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$ is a stochastic process $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$Z_t^{H,a} = \sum_{k=1}^{n} a_k B_t^{H_k},$$

where $B_t^{H_i}$ are independent fractional Brownian motions of Hurst indices $H_k \in (0, 1)$. The gmfBm is a centered Gaussian process has stationary and independent increments with long range dependence property when there exists some k with $H_k > \frac{1}{2}$. The gmfBm can be used in modeling internet traffic using self-similar processes, see [7] also for an underlying modeled in the gmfBm market, satisfy the following stochastic differential equation:

$$dS_t = aS_t dt + bS_t dZ_t$$

where a and b stand for the standard deviation of the stock return and the volatility, see [39].

Note that the gmfBm model is a generalization of all the fBm models considered in the literature. Such a generalized model degenerates to the single fBm model with $n = 1$, the Bm model with $n = 1$ and $H_1 = \frac{1}{2}$, the mfBm model with $n = 2$ and $H_1 = \frac{1}{2}$ and the fmfBm when $n = 2$. For a detailed survey on the properties of the gmfBm, we refer to [11, 12, 38].

The time-changed generalized mixed fractional Brownian motion is defined as

$$L_\beta^{H,a} = \{L_{\beta_t}^{H,a},\ t \geq 0\} = \{Z_{\beta_t}^{H,a},\ t \geq 0\},$$

where the parent process $T^{H,a}$ is a gmfBm with parameters $H = (H_1, H_2, ..., H_n)$, $a = (a_1, a_2, ..., a_n)$ and the internal process is the subordinator $\beta = \{\beta_t,\ t \geq 0\}$ assumed to be independent of $B_t^{H_k}$, for $k = 1, 2, ..., n$. If $H = (\frac{1}{2}, 0, ..., 0)$ and $a = (1, 0, ..., 0)$, the process $L_\beta^{H,a}$ is called subordinated Brownian motion, it was investigated in [26, 32]. Also, the process $L_\beta^{H,\alpha}$, for $H = (H_1, 0, ..., 0)$ and $a =$

$(1, 0, .., 0)$ is called subordinated fractional Brownian motion it was investigated in [22, 23].

Time-changed process is constructed by taking superposition of tow independent stochastic systems. The evolution of time in external process is replaced by a non-decreasing stochastic process, called subordinator. The resulting time-changed process very often retain important properties of the external process, however certain characteristics might change. This idea of subordination was introduced by Bochner [4] and was explored in many papers (e.g. [1, 2, 33, 37]). In recent years, the interest in subordinators and their hitting-times has grown fast and has reached an increasingly large audience also outside the probability community, involving many areas of mathematics. The importance of Time-changed processes arise in several applications in very different fields, among others: they are scaling limit of continuous time random walks, they are useful to model anomalous diffusion and fractional kinetics they appear in economics and mathematical finance and recently also in neuronal modelling.

In the case $H = (\frac{1}{2}, H_2, 0..., 0)$ and $a = (a_1, a_2, 0, ..., 0)$, The time-changed mixed fractional Brownian motion has been discussed in [10] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This model is the process

$$X_t^{H,a} = X_0^{H,a} \exp\{\mu\beta_t + \sigma M_{\beta_t}^{H,a}\},$$

where μ is the rate of the return and σ is the volatility and β_t is the α-inverse stable subordinator.

The time-changed processes have found many interesting applications, for example in [10, 18, 21, 30, 33, 35].

In this work, our main aim is to discuss the main properties of the time-changed generalized mixed fractional Brownian motion by inverse α-stable subordinator paying attention to the long range dependence property. As application we deduce that (1) the fractional mixed fractional Brownian motion the time-changed by inverse α-stable subordinator exhibits long range dependence property for each $H_1 < H_2$, considered in [30] (2) the mixed fractional Brownian motion has long range dependence property for each $H > \frac{1}{2}$.

The paper is organized as follow. First we review the necessary background of inverse α-stable subordinator, second we recall some properties of the gmfBm. Next we study the long-range dependence property of the time changed generalized mixed fractional Brownian motion by the inverse α-stable subordinator. Finally we deduce some results of known fractional processes.

2. Preliminaries

In this section we review the necessary background of inverse α-stable subordinator and we recall some properties of the gmfBm. Also we recall briefly the commonly used definitions of long range dependence, based on the correlation function of a process.

We begin by defining the inverse α-stable subordinator.

Definition 2.1. The inverse α-stable subordinator $T^{\alpha} = \{T_t^{\alpha},\ t \geq 0\}$ is defined in the following way

$$T_t^{\alpha} = inf\{r > 0,\ \eta_r^{\alpha} \geq t\}, \tag{2.1}$$

where $\eta^{\alpha} = \{\eta_r^{\alpha},\ r \geq 0\}$ is the α-stable subordinator [34, 36] with Laplace transform

$$E(e^{-u\eta_r^{\alpha}}) = e^{-ru^{\alpha}}, \quad \alpha \in (0,1).$$

The inverse α-stable subordinator is a non-decreasing Lévy process, starting from zero, has a stationary and independent increments with α-self similar. Specially, when $\alpha \uparrow 1$, T_t^{α} reduces to the physical time t.

Definition 2.2. A generalized mixed fractional Brownian motion of parameter $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0,1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$ is a stochastic process $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ defined on some probability space by

$$Z_t^{H,a} = \sum_{k=1}^{n} a_k B_t^{H_k}, \tag{2.2}$$

where $B_t^{H_i}$ are independent fractional Brownian motions of Hurst index $H_k \in (0,1)$ for $k = 1, 2, ..., n$ and $a_1, a_2, ..., a_n$ are real coefficients.

Below We collect some properties of the gmfBm. For proofs and additional information on the importance of this process see [38] references therein.

Proposition 2.3. *The gmfBm $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ is a centered Gaussian process with variance $\sum_{k=1}^{n} a_k^2 t^{2H_k}$ and covariance function*

$$Cov(B_t^H,\ B_s^H) = \frac{1}{2}\sum_{k=1}^{n} a_k^2 [t^{2H_k} + s^{2H_k} - |t-s|^{2H_k}], \quad s,t \geq 0, \tag{2.3}$$

the gmfBm has stationary increments and they are uncorrelated if and only if $H_k = \frac{1}{2}$ for all k. $Z^{H,a}$ is also $(c_1, ..., c_n; H_1, ..., H_n)$ self-similar in the sense that

$$\sum_{k=1}^{n} a_k c_k^{-H_k} B_{c_k t}^{H_k} = \sum_{k=1}^{n} a_k B_t^{H_k}$$

in law. $Z^{H,a}$ is neither a Markov process nor a semi-martingale, unless $H_k = \frac{1}{2}$ for all k.

Proposition 2.4. *$Z^{H,a}$ exhibits a long-range dependence if and only if there exists k with $H_k > \frac{1}{2}$.*

Lemma 2.5. *Let T^{α} be an inverse α-stable subordinator with index $\alpha \in (0,1)$. From* [25, 27]*, we know that*

$$E(T_t^{\alpha}) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} \quad \text{and} \quad E((T_t^{\alpha})^n) = \frac{t^{n\alpha} n!}{\Gamma(n\alpha+1)}.$$

Lemma 2.6. *Let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$ and B^H be a fBm. Then, by α-self-similar and non-decreasing sample path of T_t^α, we have*

$$E(B^H_{T_t^\alpha})^2 = \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}.$$

Proof. See [21, 27]. □

Notation 2.7. Let X and Y be two centered random variables defined on the same probability space. Let

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{E(X^2)E(Y^2)}}, \tag{2.4}$$

denote the correlation coefficient between X and Y.

Now we discuss the long range dependent behavior of $L^{H_1 H_2}_{T^\alpha}(a,b)$.

Definition 2.8. A finite variance stationary process $\{X_t,\ t \geq 0\}$ is said to have long range dependence property [6], if $\sum_{k=0}^\infty \gamma_k = \infty$, where

$$\gamma_k = Cov(X_k, X_{k+1}).$$

In the following definition we give the equivalent definition for a non-stationary process $\{X_t,\ t \geq 0\}$.

Definition 2.9. Let $s > 0$ be fixed and $t > s$. The process $\{X_t,\ t \geq 0\}$ is said to have long range dependence property (LRD) if

$$Corr(X_t, X_s) \sim c(s)t^{-d}, \quad as \quad t \to \infty,$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$.

An equivalent definition given in [16].
Let $0 < s < t$ and s be fixed. Assume a stochastic process $\{X_t,\ t \geq 0\}$ has the correlation function $Corr(X_s, X_t)$ that satisfies

$$c_1(s)t^{-d} \leq Corr(X_s, X_t) \leq c_2(s)t^{-d}$$

for large $t, d > 0, c_1(s) > 0$ and $c_2(s) > 0$.

That is,

$$lim_{t\to\infty} \frac{Corr(X_s, X_t)}{t^{-d}} = c(s)$$

for some $c(s) > 0$ and $d > 0$. We say $\{X_t,\ t \geq 0\}$ has the long-range dependence property if $d \in (0,1)$ and has the short-range dependence property if $d \in (1,2)$.

Proposition 2.10. *The inverse α-stable subordinator with index $\alpha \in (0,1)$ has long-range dependence property.*

Proof. First we compute the covariance function using independent increment property of subordinator. For $0 < s < t$, we have

$$\begin{aligned} Cov[T_s^\alpha, T_t^\alpha] &= Cov[T_s^\alpha, (T_t^\alpha - T_s^\alpha) - T_s^\alpha] \\ &= Cov[T_s^\alpha, (T_t^\alpha - T_s^\alpha)] + Cov[T_s^\alpha, T_s^\alpha] \\ &= Var[T_s^\alpha] \\ &= c(\alpha)s^{2\alpha} \end{aligned}$$

Thus the correlation function is given by

$$\begin{aligned} Corr[T_s^\alpha, T_t^\alpha] &= \frac{Cov[T_s^\alpha, T_t^\alpha]}{Var[T_s^\alpha]^{\frac{1}{2}} Var[T_t^\alpha]^{\frac{1}{2}}} \\ &= \frac{Var[T_t^\alpha]^{\frac{1}{2}}}{Var[T_s^\alpha]^{\frac{1}{2}}} \\ &= s^\alpha t^{-\alpha} \end{aligned}$$

Hence,

$$lim_{t\to\infty} \frac{Corr[T_s^\alpha, T_t^\alpha]}{t^{-\alpha}} = s^\alpha$$

Therefore, the inverse α-stable subordinator $\{T_t^\alpha,\ t \geq 0\}$ has LRD property. $\square$

3. LRD of gmfBm time changed by inverse α-stable subordinator

In this section we will discuss the long range dependence properties of the generalized mixed fractional Brownian motion time changed by inverse α-stable subordinator

Definition 3.1. Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0, 1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$. Let T^α be an inverse α-stable subordinator with index $\alpha \in (0, 1)$. The subordinated of $Z^{H,a}$ by means of T^α is the process $L_{T^\alpha}^{H,a} = \{L_{T_t^\alpha}^{H,a},\ t \geq 0\}$ defined by:

$$L_{T_t^\alpha}^{H,a} = Z_{T_t^\alpha}^{H,a} = \sum_{i=1}^{n} a_i B_{T_t^\alpha}^{H_i}, \tag{3.1}$$

where the subordinator T_t^α is assumed to be independent of B^{H_k} for all k.

Remark 3.2. When $\alpha \uparrow 1$, the processes $B_{T_t^\alpha}^{H}$ degenerate to B_t^H.

Lemma 3.3. *$L_{T^\alpha}^{H,a}$ is not a stationary process.*

The main result can be stated as follows.

Theorem 3.4. *Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be the generalized mixed Brownian motion of parameters $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0, 1)$, $a_k \in$ $\mathbf{R}$, $n \in \mathbf{N}^*$ with $H_k < H_n$ for $k = 1, 2, .., n-1$. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0, 1)$ assumed to be independent of all fBm's B^{H_k} with Hurst indices H_k. Then the time-changed generalized mixed fractional Brownian motion by means of T^α has long range dependence property for every Hurst indices satisfying $0 < 2\alpha H_k - \alpha H_n < 1$.*

Proof. Let $n \in \mathbf{N}^*$. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0, 1)$ assumed to be independent of all fBm's. Let $L_{T^\alpha}^{H,a}$ be the time-changed generalized mixed Brownian motion by means of the inverse α-stable subordinator T^α with index $\alpha \in (0, 1)$. The process $L_{T^\alpha}^{H,a}$ is not stationary hence Definition 2.9 will be used to establish the long range dependence property.

Step 1: Let $s \leq t$. The covariance function of $L^{H,a}_{T^\alpha_t}$ and $L^{H,a}_{T^\alpha_s}$ is defined by

$$Cov(L^{H,a}_{T^\alpha_t}, L^{H,a}_{T^\alpha_s}) = E(L^{H,a}_{T^\alpha_t} L^{H,a}_{T^\alpha_s}) - E(L^{H,a}_{T^\alpha_t})E(L^{H,a}_{T^\alpha_s})$$

by observing that $E[L^{H,a}_{T^\alpha_t}] = 0$, $t \geq 0$. Then

$$\begin{aligned}
Cov(L^{H,a}_{T^\alpha_t}, L^{H,a}_{T^\alpha_s}) &= E(L^{H,a}_{T^\alpha_t} L^{H,a}_{T^\alpha_s}) \\
&= \frac{1}{2}E\left[(L^{H,a}_{T^\alpha_t})^2 + (L^{H,a}_{T^\alpha_s})^2 - (L^{H,a}_{T^\alpha_t} - L^{H,a}_{T^\alpha_s})^2\right] \\
&= \frac{1}{2}E[(Z^{H,a}_{T^\alpha_t})^2 + (Z^{H,a}_{T^\alpha_s})^2 - (Z^{H,a}_{T^\alpha_t} - Z^{H,a}_{T^\alpha_s})^2] \\
&= \frac{1}{2}E[(\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_t})^2 + (\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_s})^2] \\
&\quad -\frac{1}{2}E[(\sum_{k=1}^{n} a_k (B^{H_k}_{T^\alpha_t} - B^{H_k}_{T^\alpha_s}))^2]
\end{aligned}$$

Since B^{H_k} has stationary increments, then we have

$$\begin{aligned}
Cov(L^{H,a}_{T^\alpha_t}, L^{H,a}_{T^\alpha_s}) &= \frac{1}{2}E[(\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_t})^2 + (\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_s})^2] - \frac{1}{2}E[(\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_{t-s}})^2] \\
&= \frac{1}{2}E[(\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_t})^2 + 2\sum_{k\neq l}^{n} a_k a_l B^{H_k}_{T^\alpha_t} B^{H_l}_{T^\alpha_t}] \\
&\quad +\frac{1}{2}E[(\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_s})^2 + 2(\sum_{k\neq l}^{n} a_k a_l B^{H_k}_{T^\alpha_s} B^{H_l}_{T^\alpha_s})] \\
&\quad -\frac{1}{2}E[(\sum_{k=1}^{n} a_k B^{H_k}_{T^\alpha_{t-s}})^2 + 2\sum_{k\neq l}^{n} a_k b_l B^{H_k}_{T^\alpha_{t-s}} B^{H_l}_{T^\alpha_{t-s}})].
\end{aligned}$$

By the independence of the fBms' $B^{H_k}_t$ for $k = 1, ..., n$ and their independence of the T^α, we get

$$\begin{aligned}
E[B^{H_k}_{T^\alpha_t} B^{H_l}_{T^\alpha_t}] &= E[E(B^{H_k}_r B^{H_l}_r | T^\alpha_t)] \\
&= \int E[B^{H_k}_r B^{H_l}_r] f_{T^\alpha_t}(dr) \\
&= 0
\end{aligned}$$

where $f_{T^\alpha_t}(.)$ is the distribution function of T^α_t.
Thus

$$E(L^{H,a}_{T^\alpha_t} L^{H,a}_{T^\alpha_s}) = \sum_{k=1}^{n} \frac{a_k^2}{2}\left[E(B^{H_k}_{T^\alpha_t})^2 + E(B^{H_k}_{T^\alpha_s})^2 - E(B^{H_k}_{T^\alpha_{t-s}})^2\right].$$

Using Lemma 2.6 we get

$$\begin{aligned} E(L^{H,a}_{T^{\alpha}_t} L^{H,a}_{T^{\alpha}_s}) &= \sum_{k=1}^{n} \frac{a_k^2}{2}[(\frac{t^{\alpha}}{\Gamma(\alpha+1)})^{2H_k} + (\frac{s^{\alpha}}{\Gamma(\alpha+1)})^{2H_k} - (\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)})^{2H_k}] \\ &= \sum_{k=1}^{n} \frac{a_k^2 \left[t^{2\alpha H_k} + s^{2\alpha H_k} - (t-s)^{2\alpha H_k}\right]}{2[\Gamma(\alpha+1)]^{2H_k}}. \end{aligned}$$

Hence for $t > s$, we have

$$E(L^{H,a}_{T^{\alpha}_t} L^{H,a}_{T^{\alpha}_s}) = \sum_{k=1}^{n} \frac{a_k^2 \left[t^{2\alpha H_k} + s^{2\alpha H_k} - (t-s)^{2\alpha H_k}\right]}{2[\Gamma(\alpha+1)]^{2H_k}}. \tag{3.2}$$

Step 2: Let s be fixed. Then by Taylor's expansion we have for large t

$$\begin{aligned} E(L^{H,a}_{T^{\alpha}_t} L^{H,a}_{T^{\alpha}_s}) &\sim \sum_{k=1}^{n} \frac{a_k^2}{2[\Gamma(\alpha+1)]^{2H_k}} t^{2\alpha H_k} [2\alpha H_k \frac{s}{t} + s^{2\alpha H_k} t^{-2\alpha H_k} + O(t^{-2})] \\ &\sim \sum_{k=1}^{n} \frac{a_k^2 t^{2\alpha H_k}}{2[\Gamma(\alpha+1)]^{2H_k}} [2\alpha H_k \frac{s}{t} + (\frac{s}{t})^{2\alpha H_k} + O(t^{-2})] \\ &\sim \sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}. \end{aligned}$$

Then for fixed s and large t, $L^{H,a}_{T^{\alpha}_t}$ satisfies

$$E(L^{H,a}_{T^{\alpha}_t} L^{H,a}_{T^{\alpha}_s}) \sim \sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}. \tag{3.3}$$

Step 3: Let $H_k < H_n$ for $k = 1, 2, .., n-1$. Using Eqs. (2.4), (3.3) and by Taylor's expansion we get, as $t \to \infty$

$$\begin{aligned} Corr(L^{H,a}_{T^{\alpha}_t}, L^{H,a}_{T^{\alpha}_s}) &\sim \frac{\sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}}{\left[\sum_{k=1}^{n} \frac{a_k^2 \alpha}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k}\right]^{\frac{1}{2}} [E(L_s^{T^{\alpha}})^2]^{\frac{1}{2}}} \\ &= \frac{\sum_{k=1}^{n-1} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1} + \frac{a_n^2 \alpha s}{(\Gamma(\alpha+1))^{2H_n}} t^{2\alpha H_n - 1}}{\frac{|a_n| \alpha^{\frac{1}{2}} t^{\alpha H_n}}{(\Gamma(\alpha+1))^{H_n}} \left[\sum_{k=1}^{n-1} \frac{a_k^2 t^{2\alpha H_k - 2\alpha H_n}}{2a_n^2 (\Gamma(\alpha+1))^{1-2H_n}} + 1\right]^{\frac{1}{2}} [E(L_s^{T^{\alpha}})^2]^{\frac{1}{2}}} \\ &\sim \frac{a_k^2 \alpha^{\frac{1}{2}} s t^{2\alpha H_k - \alpha H_n - 1}}{|a_n| \Gamma(\alpha+1)^{2H_k - H_n} [E(L_s^{T^{\alpha}})^2]^{\frac{1}{2}}} + \frac{|a_n| \alpha^{\frac{1}{2}} s t^{\alpha H_n - 1}}{\Gamma(\alpha+1)^{H_n} [E(L_s^{T^{\alpha}})^2]^{\frac{1}{2}}}. \end{aligned}$$

Hence, for $k = 1, 2, ..., n$, we have

$$Corr(L^{H,a}_{T^{\alpha}_t}, L^{H,a}_{T^{\alpha}_s}) \sim c_1(s) t^{2\alpha H_k - \alpha H_n - 1} + c_2(s) t^{\alpha H_n - 1}. \tag{3.4}$$

Then the correlation function decays like $t^{-(1-(2\alpha H_k - \alpha H_n))} + t^{-(1-\alpha H_n)}$. Then the time-changed process $L^{H,a}_{T^{\alpha}}$ exhibits long range dependence property for all Hurst indices satisfying $0 < 2\alpha H_k - \alpha H_n < 1$ for $i = 1, ..., n$. $\square$

Remark 3.5. Theorem 3.4 extends our result (2023), [30] for fmfBm to gmfBm and it gives a sufficient condition for the long-range dependence property depending on the Hurst indices $H_1, ..., H_n$.

4. Applications

As application we deduce the long range dependence properties for some known fractional models.

Corollary 4.1. *Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of all fBm's B^{H_k}. Let $L_{T^\alpha}^{H,a}$ be the time-changed generalized mixed Brownian motion by means of the inverse α-stable subordinator T^α with index $\alpha \in (0,1)$. Let s be fixed and let $t > s$. Then*

$$E[(L_{T_t^\alpha}^{H,a} - L_{T_s^\alpha}^{H,a})^2] = \sum_{k=1}^{n} \frac{a_k^2 t^{2\alpha H_k}}{[\Gamma(\alpha+1)]^{H_k}} + \sum_{k=1}^{n} \frac{a_k^2 s^{2\alpha H_k}}{[\Gamma(\alpha+1)]^{H_k}} - \sum_{k=1}^{n} \frac{a_k^2 \left[t^{2\alpha H_k} + s^{2\alpha H_k} - (t-s)^{2\alpha H_k}\right]}{[\Gamma(\alpha+1)]^{2H_k}}. \tag{4.1}$$

Proof. Let s be fixed and let $t > s$. We have

$$E[(L_{T_t^\alpha}^{H,a} - L_{T_s^\alpha}^{H,a})^2] = E[(L_{T_t^\alpha}^{H,a})^2] + E[(L_{T_s^\alpha}^{H,a})^2] - 2E[L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}]. \tag{4.2}$$

Since $E[L_{T_t^\alpha}^{H,a}] = 0,\ t \geq 0$ and

$$Cov[L_{T_t^\alpha}^{H,a}, L_{T_s^\alpha}^{H,a}] = E[L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}]$$

Then using Eq. 3.2 and 4.2 Eq. 4.1 holds and we have

$$Var(L_{T_t^\alpha}^{H,a}) = \sum_{k=1}^{n} \frac{a_k^2 t^{2\alpha H_k}}{[\Gamma(\alpha+1)]^{H_k}}$$

Also if $t \to \infty$, we obtain

$$E[(L_{T_t^\alpha}^{H,a} - L_{T_s^\alpha}^{H,a})^2] \sim \sum_{k=1}^{n} \frac{a_k^2 \alpha}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k} + \sum_{k=1}^{n} \frac{a_k^2 \alpha}{(\Gamma(\alpha+1))^{2H_k}} s^{2\alpha H_k} - 2\sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}.$$

$\square$

Remark 4.2. If (n=1) i.e $a_1 = 1$ and $a_2 = ... = a_n = 0$ in Eqs. (3.3) and (3.4) we get

$$E(L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}) = E(B_{T_t^\alpha}^{H_1} B_{T_s^\alpha}^{H_1}) \sim \frac{\alpha s t^{2\alpha H_1 - 1}}{(\Gamma(\alpha+1))^{2H_1}}, \quad as \quad t \to \infty,$$

$$Corr(L_{T_t^\alpha}^{H,a}, L_{T_s^\alpha}^{H,a}) = Corr(B_{T_t^\alpha}^{H_1}, B_{T_s^\alpha}^{H_1}) \sim \frac{\alpha^{\frac{1}{2}} s t^{\alpha H_1 - 1}}{(\Gamma(\alpha+1))^{H_1} \sqrt{E(B_{T_s^\alpha}^{H_1})^2}}, \quad as\ t \to \infty.$$

Hence we obtain the following result

Corollary 4.3. *The fractional Brownian motion time changed by inverse α-stable subordinator with index $\alpha \in (0,1)$ is of long range dependence for all Hurst index $H \in (0,1)$.*

Similar result as Corollary 4.3 was obtained in [22] ([23]) in the case of fractional Brownian motion time changed by tempered stable subordinator (gamma subordinator).

The case $n = 2$ and $H_1 = \frac{1}{2}$ we obtain the following result proved in [1]

Corollary 4.4. *The mixed fractional Brownian motion time-changed by inverse α-stable subordinator has long range dependence property for every $H \in (0,1)$.*

The case $n = 2$ we obtain the following result proved in [30]

Corollary 4.5. *The fractional mixed fractional Brownian motion time changed by inverse α-stable subordinator has long range dependence for every $H_1 < H_2$.*

As application to the original process we obtain the following. .
The case of mfBm when $n = 2$ and $H_1 = \frac{1}{2}$. Assume $H = H_2$. Let $H > \frac{1}{2}$. When $\alpha \uparrow 1$, in Eqs. (3.3) and (3.4) we have

$$lim_{\alpha\to 1} Corr(L^{H,a}_{T^{\alpha}_t}, L^{H,a}_{T^{\alpha}_s}) = c_1(s)t^{-H} + c_2(s)t^{H-1}, \quad as \quad t \to \infty.$$

Hence using Remark 3.2 and corollary 4 we can see that

Corollary 4.6. *The mixed fractional Brownian motion of parameters a_1, a_2 and H has long range dependence property for all $H > \frac{1}{2}$.*

Acknowledgments:
I would like to thank Professor Mohamed Majdoub for the careful reading of the paper and the helpful comments which considerably improved the paper.

Conflicts of Interest:
The authors declare no conflict of interest.

References

[1] Alajmi, S. and Mliki, E.: On the mixed fractional Brownian motion time changed by inverse α-stable subordinator, *Applied Mathematical Sciences*, **14**, (2020), 755-763.

[2] Alajmi, S. and Mliki, E.: On the long range dependence of time-changed mixed fractional Brownian motion, *arkiv:2102.10180v1* 18 Fev (2021).

[3] Alajmi, S. and Mliki, E.: Mixed generalized fractional Brownian motion, *Journal of Stochastic Analysis* **2**, (2021), 2-14.

[4] Bochner, S.: Diffusion equation and stochastic processes,*Proc. Nat. Acad. Sci. U. S. A,* **35**, (1949), 368 - 370.

[5] Cheridito, P.: Mixed fractional Brownian motion, *Bernoulli 7,* **2**, (2001), 913 - 934.

[6] Cont, R. and Tankov, P.: Financial modeling with gump processes, *Chapman and Hall/CRC,* London, (2003).

[7] Drakakis, K. and Radulovic, D. A.: Discretized version of the self-similar model for internet traffic, *Appl. Math. Sci.* **2** (2008), 2743-2756.

[8] El-Nouty, C.: The fractional mixed fractional Brownian motion, *Statist Prob Lett,* **65**, (2003), 111 - 120.

[9] Filatova, D.: Mixed fractional Brownian motion: some related questions for computer network traffic modeling, *International Conference on Signals and Electronic Systems*, (2008), 393-396.
[10] Guo, Z. and Yuan, H.: Pricing European option under the time-changed mixed Brownian-fractional Brownian model, *Physica A: Statistical Mechanics and its Applications,* **406**, (2014), 73 - 79.
[11] Herry, P. S.: Generalized mixed fractional Brownian motion as a generalized white noise functional, *Journal Mat. Stat.*, **11**, (2011), 10-19.
[12] Herry, P. S. and Boby, G.: Self-Intersection local times of generalized mixed fractional Brownian motion as white noise distributions, *Journal of Physics: Conference Series*, **855**, (2017), 012050
[13] Hmissi, M. and Mliki, E.: On exit law for subordinated semigroups by means of C^1-subordinators, *Comment. Math. Univ. Carolinae.,* **4**, (2010), 605 - 617.
[14] Hmissi, M., Mejri, H and Mliki, E.: On the fractional powers of semidynamical systems, *Grazer Mathematishe Berichte*, **351**, (2007), 66-78.
[15] Mejri, H. and Mliki, E.: On the abstract exit equation, *Grazer Mathematishe Berichte*, **354**, (2009), 84-98.
[16] Maheshwari, A. and Vellaisamy, P.: On the long-range dependence of fractional Poisson and negative binomial processes, *Journal of Applied Probability* **53** (4) (2016) 989-1000.
[17] Mandelbrot, B.B. and Ness, J.W.V.: Fractional Brownian motions, fractional noises and applications, *SIAM Review* **10** (4) (1968) 422–437.
[18] Melnikov, A. and Mishura, Y.: On pricing in financial markets with long-range dependence, *Math Finan Econ,* **5**, (2011), 29-46.
[19] Samorodnitsky, G. and Taqqu, M.S.: Stable non-Gaussian random processes, *Chapman and Hall, CRC*, 1994.
[20] Sun, W. Rachev, S.T and Fabozzi, F.J.: Long-range dependence, fractal processes, and intra-daily data, *Handbook on Information Technology in Finance*, 2008, 543–585.
[21] Hui, G., Liang, J.R and Zhang, Y.: The time changed geometric fractional Brownian motion and option pricing with transaction costs, *Physica A* , **391**, (2012), 3971 - 3977.
[22] Kumar, A., Gajda, J. and Wylomańska, A.: Fractional Brownian Motion Delayed by Tempered and Inverse Tempered Stable Subordinators, *Methodol Comput Appl Probab,* **21**, (2019), 185 - 202.
[23] Kumar, A., Wylomańska, A., Polozański, R. and Sundar, S.: Fractional Brownian motion time-changed by gamma and inverse gammam process, *Pysica A,* **468**, (2017), 648 - 667.
[24] Lin, S.: Pricing currency in the mixed fractional Brownian motion, *Physica A: Statistical Mechanics and its Applications,* **392**, (2013), 3441 - 3458.
[25] Magdziarz, M.: Stochastic representation of sub-diffusion processes with time-dependent drift, *Stochastic Process. App.,* **119**, (2009), 3238 - 3252.
[26] Magdziarz, M.: Stochastic path properties of sub-diffusion, a martingale approach, *Stoch. Models,* **26**, (2010), 256 - 271.
[27] Magdziarz, M.: Black-Scholes formula in sub-diffusive regime, *J. Stat. Phys.,* **3**, (2009), 553 - 564.
[28] Mejri, H. and Mliki, E.: On the abstract subordinated exit equation, *Abstract and Applied Analysis*, **2010**, (2010).
[29] Majdoub, M. and Mliki, E.: Well-posedness for Hardy-Hénon parabolic equations with fractional Brownian noise, *Analysis and Mathematical Physics*, **11**, (2021), 1-12.
[30] Mliki, E.: On the fractional mixed fractional Brownian motion time Changed by Inverse α-Stable Subordinator, *Global Stochastic Analysis*, **10**, (2023).
[31] Miao Y., Ren. W. and Ren, Z.: On the fractional mixed Brownian motion, *App. Math. Sci.* **35**, (2008), 1729-1938.
[32] Nane, E.: Laws of the iterated logarithm for a class of iterated processes, *Statist. Probab. Lett.,* **79**, (2009), 1744 - 1751.
[33] Önalan, Ö.: Time-changed generalized mixed fractional Brownian motion and application to arithmetic average Asian option pricing, *International Journal of Applied Mathematical Research,* **6**, (2017), 85 - 92.

[34] Sato, K.: Lévy Process and infinitely divisible distributions, *Cambridge University Press, Cambridge,* (1999).

[35] Shokrollahi, F.: The evaluation of geometric Asian power options under time changed mixed fractional Brownian motion, *Journal of Computational and Applied Mathematics,* **344**, (2018), 716 - 724.

[36] Sokolov, M.: Lévy flights from a continuous-time process, *Phys. Rev.,* **63**, (2003), 469 - 474.

[37] Teuerle, M.: Wyłomańska, A. and Sikora, G.: *Modeling anomalous diffusion by a subordinated fractional Levy-stable process*, J Stat Mech: Theory Exp, (2013), 5-16.

[38] Thäle, C.: Further remarks on mixed fractional Brownian motion, *Applied Mathematical Sciences,* **3**, (2009), 1885 - 1901.

[39] Wenting, C., Bowen Y., Guanghua L. C., Ying Z.: Numerically pricing American options under the generalized mixed fractional Brownian motion model, *Physica A*, (2015), 1-10.

[40] Zhidong, G. and Hongjun, Y.: Pricing European option under the time changed mixed Brownian fractional Brownian model, *Physica A,* **406**, (2014), 73 - 79.

E. Mliki, M. Alwohaibi: Department of Mathematics, College of science, Basic and Applied Scientific Research Center, Imam Abdulrahman Bin Faisal University, P.O. Box 1982, Dammam, 31441, Saudi Arabia

Email address: ermliki@iau.edu.sa

Email address: malwohaibi@iau.edu.sa