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ON THE GENERALIZED MIXED FRACTIONAL BROWNIAN MOTION TIME CHANGED BY INVERSE α-STABLE SUBORDINATOR

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Abstract. Time-changed stochastic processes have attracted much attention and wide interest due to their extensive applications, especially in financial time series, network traffic, biology, and physics. This paper pays attention to a fractional stochastic process, defined by taking linear combinations of a finite number of independent fractional Brownian motions with different Hurst indices called the generalized mixed fractional Brownian motion, which is a Gaussian process with stationary increments exhibit long-range dependence controlled by the Hurst indices. We prove that under some conditions on the Hurst indices, the generalized mixed fractional Brownian motion time changed by inverse α-stable subordinator exhibits long-range dependence property. As application, we deduce that the mixed fractional Brownian motion of Hurst index H has long-range dependence for all $H > 1/2$.

1. Introduction

Fractional Brownian motion (fBm) introduced by Mandelbrot and Ness [12] is a self-similar process with stationary increments. A fBm $B^H = \{B_t^H,\, t \geq 0\}$ with Hirst index $H \in (0,1)$, is a centered Gaussian process with covariance function

$$\mathrm{Cov}(B_t^H,\, B_s^H) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \quad s,t \geq 0,$$

where H is a real number in $(0,1)$, called the Hurst index. The fBm is often used to model phenomena that exhibit long-range dependence property.The case $H = 1/2$ corresponds to the Brownian motion (Bm).

An extension of the fBm was introduced by Cheridito [3], called the mixed fractional Brownian motion (mfBm) which is a linear combination between a Bm and an independent fBm of Hurst index H, with stationary increments exhibiting a long-range dependence for $H > 1/2$. A mfBm of parameters a_1, a_2 and H is a process $M^H(a_1,a_2) = \{M_t^H(a_1,a_2),\ t \geq 0\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$M_t^H(a_1,a_2) = a_1 B_t + a_2 B_t^H, \quad t \geq 0,$$

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where $B = \{B_t, t \geq 0\}$ is a Bm and $B^H = \left\{B_t^H, t \geq 0\right\}$ is an independent fBm of Hurst index $H \in (0,1)$. We refer also to [2, 3, 6] for further information on mfBm process.

C. Elnouty [5] proposes a generalization of the mfBm called fractional mixed fractional Brownian motion (fmfBm) of parameters a_1, a_2, and $H = (H_1, H_2)$. A fmfBm is a process $N^H(a_1, a_2) = \{N_t^H(a_1, a_2),\ t \geq 0\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$ by

$$N_t^H(a_1, a_2) = a_1 B_t^{H_1} + a_2 B_t^{H_2}, \quad t \geq 0,$$

where $B^{H_i} = \{B_t^{H_i},\ t \geq 0\}$ are independent fractional Brownian motion of Hurst index $H_i \in (0,1)$ for $i = 1, 2$. Also, the fmfBm was studied by Miao, Y et al. [**?**].

The fmfBm has been further generalized by Thäle in 2009 [20] to the generalized mixed fractional Brownian motion (gmfBm). A gmfBm of parameter $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0,1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$ is a fractional stochastic process, defined by taking linear combinations of a finite number of independent fBms with different Hurst indices. The gmfBm is a centered Gaussian process that has stationary increments with long-range dependence property when there exists some k with $H_k > 1/2$. The gmfBm has been used in modeling internet traffic using self-similar processes, see [**?**] also for an underlying modeled in the gmfBm market, see [21].

Note that the gmfBm model is a generalization of all the fBm models considered in the literature. Such a generalized model degenerates to the single fBm model with $n = 1$, the Bm model with $n = 1$ and $H_1 = 1/2$, the mfBm model with $n = 2$ and $H_1 = 1/2$ and the fmfBm when $n = 2$. For a detailed survey on the properties of the gmfBm, we refer to [8, 9, 20].

The time-changed generalized mixed fractional Brownian motion is defined as

$$L_\beta^{H,a} = \{L_{\beta_t}^{H,a},\ t \geq 0\} = \{Z_{\beta_t}^{H,a},\ t \geq 0\},$$

where the parent process $T^{H,a}$ is a gmfBm with parameters $H = (H_1, H_2, ..., H_n)$, $a = (a_1, a_2, ..., a_n)$ and the internal process is the subordinator $\beta = \{\beta_t,\ t \geq 0\}$ assumed to be independent of $B_t^{H_k}$, for $k = 1, 2, ..., n$. If $H = (\frac{1}{2}, 0, ..., 0)$ and $a = (1, 0, ..., 0)$, the process $L_\beta^{H,a}$ is called subordinated Bm. Also, the case of the process $L_\beta^{H,\alpha}$, where $H = (H_1, 0, ..., 0)$ and $a = (1, 0, .., 0)$ is called subordinated fBm.

A time-changed process is constructed by taking the superposition of two independent stochastic systems. The evolution of time in the external process is replaced by a non-decreasing stochastic process, called the subordinator. The resulting time-changed process very often retains important properties of the external process, however certain characteristics might change. Time-changed processes are a powerful tool for modeling a wide range of phenomena, including scaling limit of continuous time random walks, they are useful to model anomalous diffusion and fractional kinetics that appear in economics, finance, and recently also in neuronal modeling.

In the case $H = (\frac{1}{2}, H_2, 0..., 0)$ and $a = (a_1, a_2, 0, ..., 0)$, the time-changed mixed fractional Brownian motion has been discussed in [7] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets.

This model is the process

$$X_t^{H,a} = X_0^{H,a} \exp\{\mu\beta_t + \sigma M_{\beta_t}^{H,a}\},$$

where μ is the rate of the return and σ is the volatility and β_t is the α-inverse stable subordinator.

The time-changed processes have found many interesting applications, for example in [1, 7, 15, 18].

In this work, we aims to discuss the main properties of the time-changed gmfBm by inverse α-stable subordinator paying attention to the long-range dependence property. This process can be used to model phenomena that exhibit long-range dependence, even when the underlying process is not itself long-range dependent. The fact that the time-changed gmfBm by α-stable subordinator exhibits long-range dependence is significant because it means that this process can be used to model phenomena that exhibit long-range dependence, even when the underlying process is not itself long-range dependent. This opens up a wide range of new possibilities for modeling real-world data.

This paper extends the results in [15] for fmfBm to gmfBm and it gives a sufficient condition for the long-range dependence property depending on the Hurst indices $H_1, ..., H_n$.

The remainder of this paper is organized as follows. First, we review the necessary background of the inverse α-stable subordinator. Second, we recall some properties of the gmfBm. Next, we study the long-range dependence property of the time changed generalized mixed fractional Brownian motion by the inverse α-stable subordinator. Finally, we deduce some results of known fractional processes.

2. Preliminaries

In this section we review the necessary background of inverse α-stable subordinator and we recall some properties of the gmfBm. Also, we recall briefly the commonly used definitions of long range dependence, based on the correlation function of a process.

We begin by defining the mgfBm.

Definition 2.1. A generalized mixed fractional Brownian motion of parameter $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0, 1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$ is a stochastic process $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ defined on some probability space by

$$Z_t^{H,a} = \sum_{k=1}^{n} a_k B_t^{H_k}, \tag{2.1}$$

where $B_t^{H_i}$ are independent fractional Brownian motions of Hurst index $H_k \in (0, 1)$ for $k = 1, 2, ..., n$ and $a_1, a_2, ..., a_n$ are real coefficients.

Below we collect some properties of the gmfBm. For proofs and additional information on the importance of this process see [20] and the references therein.

Lemma 2.2. *The gmfBm* $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ *is a centered Gaussian process with variance* $\sum_{k=1}^{n} a_k^2 t^{2H_k}$ *and covariance function*

$$Cov(B_t^H,\, B_s^H) = \frac{1}{2}\sum_{k=1}^{n} a_k^2[t^{2H_k} + s^{2H_k} - |t-s|^{2H_k}], \quad s,t \geq 0, \tag{2.2}$$

and it has stationary increments. $Z^{H,a}$ *is also* $(c_1, ..., c_n; H_1, ..., H_n)$*-self-similar in the sense that*

$$\sum_{k=1}^{n} a_k c_k^{-H_k} B_{c_k t}^{H_k} = \sum_{k=1}^{n} a_k B_t^{H_k}$$

in law. $Z^{H,a}$ *is neither a Markov process nor a semi-martingale, unless* $H_k = 1/2$ *for all* k.

Proposition 2.3. $Z^{H,a}$ *has the LRD if and only if there exists* k *with* $H_k > 1/2$.

The mgfBm is a versatile stochastic process with a wide range of applications. It is used in finance, hydrology, physics, network science, and other fields. For example, mgfBm has been used to model the prices of financial assets, the dynamics of physical systems, and the behavior of networks.

Now, we define the inverse α-stable subordinator.

Definition 2.4. The inverse α-stable subordinator $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ is defined in the following way

$$T_t^\alpha = inf\{r > 0,\ \eta_r^\alpha \geq t\}, \tag{2.3}$$

where $\eta^\alpha = \{\eta_r^\alpha,\ r \geq 0\}$ is the α-stable subordinator [17, 19] with Laplace transform

$$E(e^{-u\eta_r^\alpha}) = e^{-ru^\alpha}, \quad \alpha \in (0,1).$$

The inverse α-stable subordinator is a non-decreasing Lévy process, starting from zero, and has stationary and independent increments with α-self similar. Specially, when $\alpha \uparrow 1$, T_t^α reduces to the physical time t.

Lemma 2.5. *Let* T^α *be an inverse* α*-stable subordinator with index* $\alpha \in (0,1)$. *From* [13, 14], *we know that*

$$E(T_t^\alpha) = \frac{t^\alpha}{\Gamma(\alpha+1)} \quad and \quad E((T_t^\alpha)^n) = \frac{t^{n\alpha} n!}{\Gamma(n\alpha+1)}.$$

Lemma 2.6. *Let* T^α *be an inverse* α*-stable subordinator with index* $\alpha \in (0,1)$ *and* B^H *be a fBm. Then, by* α*-self-similar and non-decreasing sample path of* T_t^α, *we have*

$$E(B_{T_t^\alpha}^H)^2 = \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}.$$

Proof. See [**?**, 14]. □

Notation 2.7. Let X and Y be two centered random variables defined on the same probability space. Let

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{E(X^2)E(Y^2)}}, \tag{2.4}$$

denote the correlation coefficient between X and Y.

Now, we discuss the long-range dependent behavior of the inverse α-stable subordinator.

Definition 2.8. A finite variance stationary process $\{X_t,\ t \geq 0\}$ is said to have long-range dependence property [4], if $\sum_{k=0}^{\infty} \gamma_k = \infty$, where

$$\gamma_k = Cov(X_k, X_{k+1}).$$

In the following definition we give the equivalent definition for a non-stationary process $\{X_t,\ t \geq 0\}$.

Definition 2.9. Let $s > 0$ be fixed and $t > s$. The process $\{X_t,\ t \geq 0\}$ is said to have long-range dependence property (LRD) if

$$Corr(X_t, X_s) \sim c(s)t^{-d}, \quad as \quad t \to \infty,$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$.

An equivalent definition is given in [11].
Let $0 < s < t$ and s be fixed. Assume a stochastic process $\{X_t,\ t \geq 0\}$ has the correlation function $Corr(X_s, X_t)$ that satisfies

$$c_1(s)t^{-d} \leq Corr(X_s, X_t) \leq c_2(s)t^{-d}$$

for large $t, d > 0, c_1(s) > 0$ and $c_2(s) > 0$.

That is,

$$lim_{t\to\infty} \frac{Corr(X_s, X_t)}{t^{-d}} = c(s)$$

for some $c(s) > 0$ and $d > 0$. We say $\{X_t,\ t \geq 0\}$ has the long-range dependence property (LRD for short) if $d \in (0,1)$.

Long-range dependence is a common feature of many real-world processes, such as financial time series, hydrological data, telecommunications, and network traffic. Long-range dependence is a well-known challenge in financial forecasting, as it makes it difficult to predict the future values of financial assets based on their past values.

Proposition 2.10. *The inverse α-stable subordinator with index $\alpha \in (0,1)$ has the LRD property.*

Proof. First, we compute the covariance function using the independent increment property of the subordinator. For $0 < s < t$, we have

$$\begin{aligned} Cov[T_s^\alpha, T_t^\alpha] &= Cov[T_s^\alpha, (T_t^\alpha - T_s^\alpha) - T_s^\alpha] \\ &= Cov[T_s^\alpha, (T_t^\alpha - T_s^\alpha)] + Cov[T_s^\alpha, T_s^\alpha] \\ &= Var[T_s^\alpha] \\ &= c(\alpha)s^{2\alpha} \end{aligned}$$

Thus the correlation function is given by

$$\begin{aligned} Corr[T_s^\alpha, T_t^\alpha] &= \frac{Cov[T_s^\alpha, T_t^\alpha]}{Var[T_s^\alpha]^{\frac{1}{2}} Var[T_t^\alpha]^{\frac{1}{2}}} \\ &= \frac{Var[T_s^\alpha]^{\frac{1}{2}}}{Var[T_t^\alpha]^{\frac{1}{2}}} \\ &= s^\alpha t^{-\alpha} \end{aligned}$$

Hence,

$$lim_{t\to\infty} \frac{Corr[T_s^\alpha, T_t^\alpha]}{t^{-\alpha}} = s^\alpha.$$

Therefore, the inverse α-stable subordinator has the LRD property. □

3. LRD of gmfBm time changed by inverse α-stable subordinator

In this section we will discuss the LRD propert of the generalized mixed fractional Brownian motion time changed by inverse α-stable subordinator.

Definition 3.1. Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be a gmfBm of parameters $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0,1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$. Let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$. The subordinated of $Z^{H,a}$ by means of T^α is the process $L_{T^\alpha}^{H,a} = \{L_{T_t^\alpha}^{H,a},\ t \geq 0\}$ defined by:

$$L_{T_t^\alpha}^{H,a} = Z_{T_t^\alpha}^{H,a} = \sum_{i=1}^{n} a_i B_{T_t^\alpha}^{H_i}, \tag{3.1}$$

where the subordinator T_t^α is assumed to be independent of B^{H_k} for all k.

Remark 3.2. When $\alpha \uparrow 1$, the processes $B_{T_t^\alpha}^H$ degenerate to B_t^H.

Lemma 3.3. *$L_{T^\alpha}^{H,a}$ is not a stationary process.*

The main result can be stated as follows

Theorem 3.4. *Let $Z^{H,a} = \{Z_t^{H,a},\ t \geq 0\}$ be the gmfBm of parameters $H = (H_1, H_2, ..., H_n)$ and $a = (a_1, a_2, ..., a_n)$, $H_k \in (0,1)$, $a_k \in \mathbf{R}$, $n \in \mathbf{N}^*$ with $H_k < H_n$ for $k = 1, 2, .., n-1$. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of all fBm's B^{H_k} with Hurst indices H_k. Then the time-changed gmfBm by means of T^α exhibits LRD property for every Hurst indices satisfying $0 < 2\alpha H_k - \alpha H_n < 1$.*

Proof. Let $n \in \mathbf{N}^*$. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of all fBm's. Let $L_{T^\alpha}^{H,a}$ be the time-changed gmfBm by means of the inverse α-stable subordinator T^α. The process $L_{T^\alpha}^{H,a}$ is not stationary hence Definition 2.9 will be used to establish the LRD property.

Step 1: Let $s \leq t$. The covariance function of $L_{T_t^\alpha}^{H,a}$ and $L_{T_s^\alpha}^{H,a}$ is defined by

$$Cov(L_{T_t^\alpha}^{H,a}, L_{T_s^\alpha}^{H,a}) = E(L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}) - E(L_{T_t^\alpha}^{H,a})E(L_{T_s^\alpha}^{H,a})$$

by observing that $E[L^{H,a}_{T^\alpha_t}]=0$, $t\geq 0$. Then

$$\begin{aligned} Cov(L^{H,a}_{T^\alpha_t},L^{H,a}_{T^\alpha_s}) &= E(L^{H,a}_{T^\alpha_t}L^{H,a}_{T^\alpha_s}) \\ &= \frac{1}{2}E\left[(L^{H,a}_{T^\alpha_t})^2+(L^{H,a}_{T^\alpha_s})^2-(L^{H,a}_{T^\alpha_t}-L^{H,a}_{T^\alpha_s})^2\right] \\ &= \frac{1}{2}E[(Z^{H,a}_{T^\alpha_t})^2+(Z^{H,a}_{T^\alpha_s})^2-(Z^{H,a}_{T^\alpha_t}-Z^{H,a}_{T^\alpha_s})^2] \\ &= \frac{1}{2}E[(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_t})^2+(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_s})^2] \\ &\quad -\frac{1}{2}E[(\sum_{k=1}^n a_k(B^{H_k}_{T^\alpha_t}-B^{H_k}_{T^\alpha_s}))^2]. \end{aligned}$$

Since B^{H_k} has stationary increments, then we have

$$\begin{aligned} Cov(L^{H,a}_{T^\alpha_t},L^{H,a}_{T^\alpha_s}) &= \frac{1}{2}E[(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_t})^2+(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_s})^2]-\frac{1}{2}E[(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_{t-s}})^2] \\ &= \frac{1}{2}E[(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_t})^2+2\sum_{k\neq l}^n a_ka_lB^{H_k}_{T^\alpha_t}B^{H_l}_{T^\alpha_t}] \\ &\quad +\frac{1}{2}E[(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_s})^2+2(\sum_{k\neq l}^n a_ka_lB^{H_k}_{T^\alpha_s}B^{H_l}_{T^\alpha_s})] \\ &\quad -\frac{1}{2}E[(\sum_{k=1}^n a_kB^{H_k}_{T^\alpha_{t-s}})^2+2\sum_{k\neq l}^n a_kb_lB^{H_k}_{T^\alpha_{t-s}}B^{H_l}_{T^\alpha_{t-s}})]. \end{aligned}$$

By the independence of the fBms' $B^{H_k}_t$ for $k=1,...,n$ and their independence of the T^α, we get

$$\begin{aligned} E[B^{H_k}_{T^\alpha_t}B^{H_l}_{T^\alpha_t}] &= E[E(B^{H_k}_rB^{H_l}_r|T^\alpha_t)] \\ &= \int E[B^{H_k}_rB^{H_l}_r]f_{T^\alpha_t}(dr) \\ &= 0, \end{aligned}$$

where $f_{T^\alpha_t}(.)$ is the distribution function of T^α_t.
Thus

$$E(L^{H,a}_{T^\alpha_t}L^{H,a}_{T^\alpha_s}) = \sum_{k=1}^n\frac{a_k^2}{2}\left[E(B^{H_k}_{T^\alpha_t})^2+E(B^{H_k}_{T^\alpha_s})^2-E(B^{H_k}_{T^\alpha_{t-s}})^2\right].$$

Using Lemma 2.6 we get

$$\begin{aligned} E(L^{H,a}_{T^\alpha_t}L^{H,a}_{T^\alpha_s}) &= \sum_{k=1}^n\frac{a_k^2}{2}[(\frac{t^\alpha}{\Gamma(\alpha+1)})^{2H_k}+(\frac{s^\alpha}{\Gamma(\alpha+1)})^{2H_k}-(\frac{(t-s)^\alpha}{\Gamma(\alpha+1)})^{2H_k}] \\ &= \sum_{k=1}^n\frac{a_k^2\left[t^{2\alpha H_k}+s^{2\alpha H_k}-(t-s)^{2\alpha H_k}\right]}{2[\Gamma(\alpha+1)]^{2H_k}}. \end{aligned}$$

Hence for $t > s$, we have

$$E(L^{H,a}_{T^\alpha_t} L^{H,a}_{T^\alpha_s}) = \sum_{k=1}^{n} \frac{a_k^2 \left[t^{2\alpha H_k} + s^{2\alpha H_k} - (t-s)^{2\alpha H_k}\right]}{2[\Gamma(\alpha+1)]^{2H_k}}. \tag{3.2}$$

Step 2: Let s be fixed. Then by Taylor's expansion we have for large t

$$\begin{aligned} E(L^{H,a}_{T^\alpha_t} L^{H,a}_{T^\alpha_s}) &\sim \sum_{k=1}^{n} \frac{a_k^2}{2[\Gamma(\alpha+1)]^{2H_k}} t^{2\alpha H_k}[2\alpha H_k \frac{s}{t} + s^{2\alpha H_k} t^{-2\alpha H_k} + O(t^{-2})] \\ &\sim \sum_{k=1}^{n} \frac{a_k^2 t^{2\alpha H_k}}{2[\Gamma(\alpha+1)]^{2H_k}} [2\alpha H_k \frac{s}{t} + (\frac{s}{t})^{2\alpha H_k} + O(t^{-2})] \\ &\sim \sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}. \end{aligned}$$

Then for fixed s and large t, $L^{H,a}_{T^\alpha_t}$ satisfies

$$E(L^{H,a}_{T^\alpha_t} L^{H,a}_{T^\alpha_s}) \sim \sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}. \tag{3.3}$$

Step 3: Let $H_k < H_n$ for $k = 1, 2, .., n-1$. Using Eqs. (2.4), (3.3) and by Taylor's expansion we get, as $t \to \infty$

$$\begin{aligned} Corr(L^{H,a}_{T^\alpha_t}, L^{H,a}_{T^\alpha_s}) &\sim \frac{\sum_{k=1}^{n} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}}{\left[\sum_{k=1}^{n} \frac{a_k^2 \alpha}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k}\right]^{\frac{1}{2}} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} \\ &= \frac{\sum_{k=1}^{n-1} \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1} + \frac{a_n^2 \alpha s}{(\Gamma(\alpha+1))^{2H_n}} t^{2\alpha H_n - 1}}{\frac{|a_n| \alpha^{\frac{1}{2}} t^{\alpha H_n}}{(\Gamma(\alpha+1))^{H_n}} \left[\sum_{k=1}^{n-1} \frac{a_k^2 t^{2\alpha H_k - 2\alpha H_n}}{2a_n^2 (\Gamma(\alpha+1))^{1-2H_n}} + 1\right]^{\frac{1}{2}} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} \\ &\sim \frac{a_k^2 \alpha^{\frac{1}{2}} s t^{2\alpha H_k - \alpha H_n - 1}}{|a_n| \Gamma(\alpha+1)^{2H_k - H_n} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} + \frac{|a_n| \alpha^{\frac{1}{2}} s t^{\alpha H_n - 1}}{\Gamma(\alpha+1)^{H_n} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}}. \end{aligned}$$

Hence, for $k = 1, 2, ..., n$, we have

$$Corr(L^{H,a}_{T^\alpha_t}, L^{H,a}_{T^\alpha_s}) \sim c_1(s) t^{2\alpha H_k - \alpha H_n - 1} + c_2(s) t^{\alpha H_n - 1}. \tag{3.4}$$

Thus the time-changed process $L^{H,a}_{T^\alpha}$ exhibits LRD property for all Hurst indices satisfying $0 < 2\alpha H_k - \alpha H_n < 1$ for $i = 1, ..., n$. □

Remark 3.5. Theorem 3.4 extends the results in [15] for fmfBm to gmfBm and it gives a sufficient condition for the LRD property depending on the Hurst indices $H_1, ..., H_n$.

4. Applications

As application we deduce the LRD properties for some known fractional models.

Corollary 4.1. *Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of all fBm's B^{H_k}. Let $L_{T^\alpha}^{H,a}$ be the time-changed gmfBm by means of the inverse α-stable subordinator T^α. Let s be fixed and let $t > s$. Then*

$$E[(L_{T_t^\alpha}^{H,a} - L_{T_s^\alpha}^{H,a})^2] = \sum_{k=1}^n \frac{a_k^2 t^{2\alpha H_k}}{[\Gamma(\alpha+1)]^{H_k}} + \sum_{k=1}^n \frac{a_k^2 s^{2\alpha H_k}}{[\Gamma(\alpha+1)]^{H_k}} - \sum_{k=1}^n \frac{a_k^2 \left[t^{2\alpha H_k} + s^{2\alpha H_k} - (t-s)^{2\alpha H_k}\right]}{[\Gamma(\alpha+1)]^{2H_k}}. \quad (4.1)$$

Proof. Let s be fixed and let $t > s$. We have

$$E[(L_{T_t^\alpha}^{H,a} - L_{T_s^\alpha}^{H,a})^2] = E[(L_{T_t^\alpha}^{H,a})^2] + E[(L_{T_s^\alpha}^{H,a})^2] - 2E[L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}]. \quad (4.2)$$

Since $E[L_{T_t^\alpha}^{H,a}] = 0$, $t \geq 0$ and

$$Cov[L_{T_t^\alpha}^{H,a}, L_{T_s^\alpha}^{H,a}] = E[L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}].$$

Then using Eq. 3.2 and 4.2 Eq. 4.1 holds and we have

$$Var(L_{T_t^\alpha}^{H,a}) = \sum_{k=1}^n \frac{a_k^2 t^{2\alpha H_k}}{[\Gamma(\alpha+1)]^{H_k}}.$$

Also if $t \to \infty$, we obtain

$$E[(L_{T_t^\alpha}^{H,a} - L_{T_s^\alpha}^{H,a})^2] \sim \sum_{k=1}^n \frac{a_k^2 \alpha}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k} + \sum_{k=1}^n \frac{a_k^2 \alpha}{(\Gamma(\alpha+1))^{2H_k}} s^{2\alpha H_k} - 2\sum_{k=1}^n \frac{a_k^2 \alpha s}{(\Gamma(\alpha+1))^{2H_k}} t^{2\alpha H_k - 1}.$$

□

Remark 4.2. If (n=1) i.e $a_1 = 1$ and $a_2 = ... = a_n = 0$ in Eqs. (3.3) and (3.4) we get

$$E(L_{T_t^\alpha}^{H,a} L_{T_s^\alpha}^{H,a}) = E(B_{T_t^\alpha}^{H_1} B_{T_s^\alpha}^{H_1}) \sim \frac{\alpha s t^{2\alpha H_1 - 1}}{(\Gamma(\alpha+1))^{2H_1}}, \quad as \quad t \to \infty,$$

$$Corr(L_{T_t^\alpha}^{H,a}, L_{T_s^\alpha}^{H,a}) = Corr(B_{T_t^\alpha}^{H_1}, B_{T_s^\alpha}^{H_1}) \sim \frac{\alpha^{\frac{1}{2}} s t^{\alpha H_1 - 1}}{(\Gamma(\alpha+1))^{H_1} \sqrt{E(B_{T_s^\alpha}^{H_1})^2}}, \quad as\ t \to \infty.$$

Hence we obtain the following result

Corollary 4.3. *The fBm time changed by inverse α-stable subordinator with index $\alpha \in (0,1)$ has the LRD property for all Hurst index $H \in (0,1)$.*

In the case $n = 2$ and $H_1 = \frac{1}{2}$ we obtain the following result proved in [1].

Corollary 4.4. *The mfBm time changed by inverse α-stable subordinator has the LRD property for every $H \in (0,1)$.*

The case $n = 2$ we obtain the following result proved in [15].

Corollary 4.5. *The fmfBm motion time changed by inverse α-stable subordinator exibits LRD for every $H_1 < H_2$.*

As an application to the original process we obtain the following. .

The case of mfBm when $n = 2$ and $H_1 = 1/2$. Assume $H = H_2$. Let $H > 1/2$. When $\alpha \uparrow 1$, in Eqs. (3.3) and (3.4) we have

$$lim_{\alpha\to 1} Corr(L^{H,a}_{T^\alpha_t}, L^{H,a}_{T^\alpha_s}) = c_1(s)t^{-H} + c_2(s)t^{H-1}, \quad as \;\; t \to \infty.$$

Hence using Remark 3.2 and Corollary 4 we can see that

Corollary 4.6. *The mfBm of parameters a_1, a_2 and H has the LRD property for all $H > 1/2$.*

5. Conclusion

The time-changed generalized mixed fractional Brownian motion is defined by taking linear combinations of a finite number of independent fractional Brownian motions with different Hurst indices, and then time-changing the process by an inverse α-stable subordinator with index $\alpha \in (0, 1)$. In this paper, it is shown that the time-changed generalized mixed fractional Brownian motion process exhibits a long-range dependence property under some conditions on the Hurst indices. It has a number of advantages over other processes, including its flexibility, ease of simulation, and theoretical properties. We deduce that the mixed fractional Brownian motion has long-range dependence for each $H > 1/2$. These studies demonstrate the potential of the time-changed stochastic processes for modeling a wide range of natural phenomena that exhibit long-range dependence. Other challenging problem in financial mathematics is the evaluation of geometric Asian power options under time changed generalized mixed fractional Brownian motion which can be used to model the dynamics of asset prices with long-range dependence and anomalous diffusion. The details of this investigation we postpone for a forthcoming paper [16].

Conflicts of Interest:
The authors declare no conflict of interest.

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