Journal of Mathematical Control Science & Applications
Vol. 11 No. 1 (January-June, 2025)
ISSN: 0974-0570

THE EXISTENCE AND CONTROLLABILITY OF NONAUTONOMOUS SYSTEMS INFLUENCED BY IMPULSES ON BOTH STATE AND CONTROL

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Abstract. This paper investigates the approximate controllability for non-autonomous impulsive integro-differential equations in Hilbert spaces, emphasizing their importance. First, we consider a linear problem and established the approximate controllability results by finding a feedback control. We then establish the existence of a mild solution using the fixed-point approach. Conditions for approximate controllability are derived with the aid of linear evolution ssytems, impulsive resolvent operators and the adjoint problem. An illustrative example is provided to demonstrate the applicability of the proposed results.

1. Introduction

Controllability is a fundamental concept in mathematical control theory, essential for addressing various control challenges, such as stabilizing unstable systems via feedback control [1], ensuring the irreducibility of transition semigroups [2] and solving optimal control problems [3]. Several types of controllability have been developed, including exact, null, approximate, interior, boundary, and finite-approximate controllability [4]. Exact controllability ensures that a system can be guided to any desired final state, whereas approximate controllability guarantees that the system can be brought arbitrarily close to a given final state. Studies on infinite-dimensional control systems indicate that exact controllability is rarely achieved (cf. [5]). In contrast, approximate controllability is more prevalent and, in many applications, is entirely sufficient (see, for example, [6, 7, 8]). Consequently, investigating this weaker yet practically significant notion of controllability is both important and necessary, particularly for nonlinear systems.

The theory of impulsive differential equations provides a broader framework than traditional differential equations by capturing abrupt changes in system dynamics. Such systems exhibit unique behaviors like oscillations, solution merging, and discontinuous trajectories, making them applicable in diverse fields.

2000 *Mathematics Subject Classification.* 34K45; 37L05; 93C25; 93B05; 93C10.
Key words and phrases. Non-autonomous differential equations, impulsive systems, approximate Controllability, Krasnoselskii's fixed point theorem.

Recently, Mahmudov [9] investigated the approximate controllability of an impulsive system in a Hilbert space $\mathbb{H}$, given by

$$\begin{aligned} x'(t) &= \mathrm{A}x(t) + \mathrm{B}u(t), \quad t \in J = [0,b] \setminus \{t_1, \ldots, t_m\}, \\ \Delta x(t_k) &= \mathrm{D}_k x(t_k) + \mathrm{E}_k v_k, \quad k = 1, \ldots, m, \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

Where $x(.) \in \mathbb{H}$, $u(.) \in L^2([0,b], \mathbb{U})$, and $v_k \in \mathbb{U}$. If $v_k = u(t_k)$, the jump $\Delta x(t_k)$ depends on both the control and state at t_k.

Previous studies primarily focused on finite-dimensional cases [10, 11, 12, 13, 14], whereas Mahmudov's work represents the first extension of this impulse structure to infinite-dimensional spaces. Using semigroup theory and impulsive operators, he established necessary and sufficient conditions for approximate controllability. Traditional impulsive models use

$$\Delta x(t_k) = I_k\left(x\left(t_k^-\right)\right),$$

where I_k is a predefined impulse function. While suitable for predictable systems, this approach may not capture complex or unpredictable impulses. The alternative formulation

$$\Delta x(t_{k+1}) = \mathrm{D}_{k+1} x(t_{k+1}) + \mathrm{E}_{k+1} v_{k+1},$$

offers greater flexibility for modeling frequent or intricate abrupt shifts. Further advancements in this area have been made by Asadzade et al. [15, 16], who explored the existence and optimal control of impulsive stochastic evolution systems and the approximate controllability of semilinear systems.

Impulsive integro-differential equations arise naturally in various real-world systems where abrupt changes or discontinuities occur at specific moments in time. In many practical situations, the presence of memory effects and hereditary properties, which are inherently captured by integro-differential equations, makes their study crucial [17, 18].

Recent research has extended controllability concepts to impulsive integro-differential systems, considering the combined effects of memory terms and sudden changes in system dynamics. Approximate controllability results for such systems have been established using techniques based on semigroup theory, fixed point theorems, and operator theory [19, 20]. The study of approximate controllability for impulsive integro-differential systems is essential for the development of efficient control strategies in fields such as medicine, economics, and environmental sciences.

The study of approximate controllability in non-autonomous systems is essential for managing time-dependent dynamics in real-world applications. It allows systems to adapt and achieve desired behaviors despite uncertainties, using flexible and adaptive control strategies. This has significant implications in fields such as robotics (following time-varying paths), climate science (modeling seasonal changes), economics (managing market fluctuations), and biomedical engineering (targeted therapies). For a comprehensive exploration of non-autonomous systems, we refer readers to the book by Kloeden et al. [21]. Noteworthy contributions in this area include the articles by Arora et al. [22] and Ravikumar et al., which provide valuable insights into the approximate controllability of non-autonomous systems. The above literature review clearly indicates that the approximate controllability of non-autonomous semilinear integro differential systems with the specified impulse structure, as in system (1.1), has not yet been explored. This gap has motivated us to investigate non-autonomous integro-impulsive systems within a separable Hilbert space, as described below:

$$\begin{aligned} x'(t) &= \mathrm{A}(t)x(t) + \mathrm{B}u(t) + f\left(t, x(t)\right) + \int_0^t q(t-s)\xi(s, x(s))ds, \\ &\qquad t \in J = [0, b] \backslash \left\{t_1, \ldots, t_m\right\}, \\ \Delta x\left(t_k\right) &= \mathrm{D}_k x\left(t_k\right) + \mathrm{E}_k v_k, \quad k = 1, \ldots, m, \\ x(0) &= x_0, \end{aligned} \tag{1.2}$$

where $\{\mathrm{A}(t) : t \in J\}$ is a family of linear operators (not necessarily bounded) on $\mathbb{H}$. Control $u : J \rightarrow \mathbb{U}$, where $\mathbb{U}$ is Hilbert space identified with its own dual is given in $L^2([0, b], \mathbb{U})$, $v_k \in \mathbb{U}, k = 1, \ldots, m$. $\mathrm{B} : \mathbb{U} \rightarrow \mathbb{H}$, $\mathrm{D}_k : \mathbb{H} \rightarrow \mathbb{H}$, $\mathrm{E}_k : \mathbb{U} \rightarrow \mathbb{H}$ are bounded linear operators and $\|\mathrm{B}\|_{\mathcal{L}} = M_B$. The functions $f, \xi : J \times \mathbb{H} \rightarrow \mathbb{H}$ are satisfying some suitable assumptions. $q : [0, b] \rightarrow \mathbb{H}$ is continuous and $q \in L^1\left([0, b], \mathbb{R}^+\right)$.

At the points of discontinuance t_k (where $k = 1, \ldots, m$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{n+1} = b$), the state variable's abrupt change is determined by $\Delta x\left(t_k\right) = x(t_k^+) - x(t_k^-)$, with $x(t_k^{\pm}) = \lim_{h \rightarrow 0^{\pm}} x(t_k + h)$ and the supposition that $x(t_k^-) = x(t_k)$. $\prod_{j=1}^{k} A_j$ denotes the operator composition $A_1 A_2 \ldots A_k$. For $j = k+1$ to k, $\prod_{j=k}^{k+1} A_j = 1$. In the same way, $\prod_{j=k}^{1} A_j$ represents the composition $A_k A_{k-1} \ldots A_1$ and $\prod_{j=k+1}^{k} A_j = 1$.

2. Preliminaries and Assumptions

This section contains some essential definitions and specified assumptions which are required to derive the sufficient conditions for ensuring the approximate controllability of system (1.2). Also we include the list of important symbols which are frequently used throughtout in this paper

$\mathbb{H} \rightarrow$ Hilbert space
$\mathbb{N} \rightarrow$ set of natural numbers
$\mathbb{R} \rightarrow$ set of real numbers
$\mathbb{X} \rightarrow$ Banach space
$\mathcal{L}(\mathbb{X}) \rightarrow$ the set of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$
$\mathbb{U} \rightarrow$ Hilbert sace
$J = [0, b] \rightarrow$ time interval
$x \rightarrow$ state variable
$\mathrm{A} \rightarrow$ linear (not necessarily bounded) operator
B, D_k, $\mathrm{E}_k \rightarrow$ linear bounded operators for $k = 1 \ldots m$
$\mathrm{I} \rightarrow$ identity operator
$\Omega \rightarrow$ bounded linear operator
$\mathrm{U}(t, s) \rightarrow$ evolution family for $(t, s) \in J \times J$, $t \geq s$

$\mathcal{N}, \mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2$, N, C_i, $\mathscr{L} \to$ are non-negative real numbers
Let us define the function space

$$\mathcal{PC}(J;\mathbb{H}) := \{\psi : J \to \mathbb{H} : \psi(\cdot) \text{ is piecewise continuous with jump discontinuity at } t_k \text{ satisfying } x(t_k^-) = x(t_k)\}.$$

For $x \in \mathcal{PC}(J;\mathbb{H})$, we define $\|x\|_{\mathcal{PC}} = \sup_{t\in J} \|x(t)\|$.

2.1. Evolution family. An evolution family is an essential concept in the study of nonautonomous systems, particularly when dealing with time-dependent differential equations. Here is a formal definition:

Definition 2.1 ([23])**.** Let $\mathbb{X}$ be a Banach space, and let $J = [0, b]$, be an interval of the real line. An evolution family $\{\mathrm{U}(t, s)\}_{(t,s)\in J\times J,\, t\geq s}$ is a two-parameter family of bounded linear operators on $\mathbb{X}$ with the following properties:

(1) Initial Condition:

$$\mathrm{U}(s, s) = \mathrm{I} \quad \text{for all } s \in J,$$

where I is the identity operator on $\mathbb{X}$.

(2) Semigroup Property (also called the cocycle condition):

$$\mathrm{U}(t, s) = \mathrm{U}(t, r)\mathrm{U}(r, s) \quad \text{for all } s \leq r \leq t \text{ in } J.$$

(3) Strong Continuity: The mapping $(t, s) \mapsto \mathrm{U}(t, s)x$ is continuous for each fixed $x \in \mathbb{X}$.

Definition 2.2. If A is a linear, not necessarily bounded, operator in $\mathbb{X}$, the resolvent set $\rho(\mathrm{A})$ of A is the set of all comlex numbers λ for which $\lambda\mathrm{I} - \mathrm{A}$ is invertible, i.e., $(\lambda\mathrm{I}-\mathrm{A})^{-1}$ is a bounded linear operator in $\mathbb{X}$. The family $\mathrm{R}(\lambda, \mathrm{A}) = (\lambda\mathrm{I} - \mathrm{A})^{-1}, \quad \lambda \in \rho(\mathrm{A})$ of bounded linear operators is called the resolvent of A.

To construct an evolution family, let us impose the following assumptions on the family of linear operators $\{\mathrm{A}(t) : t \in J\}$ (see, chapter 5, [23]).

(R1) The linear operator $\mathrm{A}(t)$ is closed for each $t \in J$ and the domain $\mathcal{D}(\mathrm{A}(t)) = \mathcal{D}(\mathrm{A})$ is dense in $\mathbb{X}$ and independent of t.

(R2) The resolvent operator $\mathrm{R}(\lambda, \mathrm{A}(t))$ for $t \in J$ exists for all λ with $Re\lambda \leq 0$ and there exists $K > 0$ such that

$$\|\mathrm{R}(\lambda, \mathrm{A}(t))\|_{\mathcal{L}(\mathbb{X})} \leq \frac{K}{|\lambda| + 1}.$$

(R3) There exist constants $P > 0$ and $0 < \delta \leq 1$ such that

$$\left\|(\mathrm{A}(t) - \mathrm{A}(s))\mathrm{A}^{-1}(\tau)\right\|_{\mathcal{L}(\mathbb{X})} \leq P|t - s|^{\delta}, \text{for all } t, s, \tau \in J.$$

(R4) The operator $\mathrm{R}(\lambda, \mathrm{A}(t)), t \in J$ is compact for some $\lambda \in \rho(\mathrm{A}(t))$, where $\rho(\mathrm{A}(t))$ is the resolvent set of $\mathrm{A}(t)$.

Lemma 2.3 (Theorem 6.1, Chapter 5, [23])**.** *Suppose that (R1)-(R3) hold true. Then there esists a unique evolution family* $\mathrm{U}(t, s)$ *on* $0 \leq s \leq t \leq b$ *satisfying the following:*

(1) *For* $0 \leq s \leq t \leq b$*, we have* $\|\mathrm{U}(t, s)\|_{\mathcal{L}(\mathbb{X})} \leq M$.

(2) *The operator* $\mathrm{U}(t,s):\mathbb{X}\mapsto\mathcal{D}(\mathrm{A})$ *for* $0\le s\le t\le b$ *and the mapping* $t\mapsto\mathrm{U}(t,s)$ *is strongly differentiable in* $\mathbb{X}$*. The derivative* $\frac{\partial}{\partial t}\mathrm{U}(t,s)\in\mathcal{L}(\mathbb{X})$ *and it is strongly continuous on* $0\le s\le t\le b$*. Moreover,*

$$\frac{\partial}{\partial t}\mathrm{U}(t,s)-\mathrm{A}(t)\mathrm{U}(t,s)=0,\ \text{for } 0\le s\le t\le b,$$

$$\left\|\frac{\partial}{\partial t}\mathrm{U}(t,s)\right\|_{\mathcal{L}(\mathbb{X})}=\|\mathrm{A}(t)\mathrm{U}(t,s)\|_{\mathcal{L}(\mathbb{X})}\le\frac{M}{t-s},$$

and

$$\left\|\mathrm{A}(t)\mathrm{U}(t,s)\mathrm{A}(s)^{-1}\right\|_{\mathcal{L}(\mathbb{X})}\le M,\ \text{for } 0\le s\le t\le b.$$

(3) *For each* $t\in J$ *and every* $v\in\mathcal{D}(\mathrm{A})$*,* $\mathrm{U}(t,s)v$ *is differentiable with respect to* s *on* $0\le s\le t\le b$ *and*

$$\frac{\partial}{\partial t}\mathrm{U}(t,s)v=-\mathrm{U}(t,s)\mathrm{A}(s)v.$$

Lemma 2.4 (Proposition 2.1,[24])**.** *Supose* $\{\mathrm{A}(t):t\in J\}$ *satisfies the assumptions (R1)-(R4). Let* $\{\mathrm{U}(t,s):0\le s\le t\le b\}$ *be the linear evolution family generated by* $\{\mathrm{A}(t):t\in J\}$*, then* $\{\mathrm{U}(t,s):0\le s\le t\le b\}$ *is a compact operator, whenever* $t-s>0$.

Definition 2.5. A mild solution $x:J\to\mathbb{H}$ of the system (1.2) satisfying $x(0)=x_0$ and $\Delta x(t_k)=\mathrm{D}_k x(t_k)+\mathrm{E}_k v_k,\ k=1,\dots,m$ on the intervals $t_{k-1}<t\le t_k$ is continuous, which is given by

$$x(t)=\begin{cases}\mathrm{U}(t,0)x(0)+\int_0^t\mathrm{U}(t,s)[\mathrm{B}u(s)\\ \qquad +f(s,x(s))+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau]ds,0\le t\le t_1,\\ \mathrm{U}(t,t_k)x(t_k^+)\\ +\int_{t_k}^t\mathrm{U}(t,s)[\mathrm{B}u(s)+f(s,x(s))+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau]ds,\\ \quad t_k<t\le t_{k+1},k=1,\dots,m,\end{cases}\tag{2.1}$$

with

$$\begin{aligned}x(t_k^+)=&\prod_{j=k}^{1}(\mathrm{I}+\mathrm{D}_j)\mathrm{U}(t_j,t_{j-1})x_0+\sum_{i=1}^{k}\prod_{j=k}^{i+1}(\mathrm{I}+\mathrm{D}_j)\mathrm{U}(t_j,t_{j-1})(\mathrm{I}+\mathrm{D}_i)\\ &\times\left(\int_{t_{i-1}}^{t_i}\mathrm{U}(t_i,s)[\mathrm{B}u(s)+f(s,x(s))+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau]ds\right)\\ &+\sum_{i=2}^{k}\prod_{j=k}^{i}(\mathrm{I}+\mathrm{D}_j)\mathrm{U}(t_j,t_{j-1})\mathrm{E}_{i-1}v_{i-1}+\mathrm{E}_k v_k.\end{aligned}$$

Definition 2.6. [8] The system (1.2) is considered approximately controllable on the interval J if the closure of the reachable set equals the entire space $\mathbb{H}$. The reachable set is defined by

$$\mathfrak{R}_t=\{x\in\mathbb{H}\mid x=x(t,0,u),\ u(\cdot)\in\mathrm{L}^2(J;\mathbb{U})\}.$$

Lemma 2.7 (Theorem 1, [25])**.** *(Krasnoselskii's Fixed Point Theorem) Let $\mathcal{E}$ be a closed, bounded and convex subset of a Banach space $\mathbb{X}$ and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two mappings of $\mathcal{E}$ into $\mathbb{X}$ such that $\mathcal{G}_1(w)+\mathcal{G}_2(x)\in\mathcal{E}$, whenever $w,x\in\mathcal{E}$. If $\mathcal{G}_1$ is continuous and $\mathcal{G}_1(\mathcal{E})$ is relatively compact subset of $\mathcal{E}$. Also $\mathcal{G}_2$ is a contraction map. Then there exists $z\in\mathcal{E}$ such that $z=\mathcal{G}_1(z)+\mathcal{G}_2(z)$.*

2.2. Linear nonautonomous system. The linear nonautonomous impulsive system corresponding to system (1.2) in $\mathbb{H}$ is given by:

$$\begin{aligned} x'(t) &= \mathrm{A}(t)x(t)+\mathrm{B}u(t), \quad t\in J=[0,b]\setminus\{t_1,\dots,t_m\},\\ \Delta x(t_k) &= \mathrm{D}_k x(t_k)+\mathrm{E}_k v_k, \quad k=1,\dots,m,\\ x(0) &= x_0. \end{aligned} \tag{2.2}$$

The mild solution of the above linear system is given by the following expression

$$x(t)=\begin{cases}\mathrm{U}(t,0)x(0)+\int_0^t \mathrm{U}(t,s)\mathrm{B}u(s)ds, \quad 0\le t\le t_1\\ \mathrm{U}(t,t_k)\,x\left(t_k^+\right)+\int_{t_k}^t \mathrm{U}(t,s)\mathrm{B}u(s)ds,\\ \quad t_k<t\le t_{k+1}, k=1,\dots,m,\end{cases} \tag{2.3}$$

with

$$\begin{aligned} x\left(t_k^+\right) =& \prod_{j=k}^{1}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)x_0+\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\left(\mathrm{I}+\mathrm{D}_i\right)\\ &\times\int_{t_{i-1}}^{t_i}\mathrm{U}\left(t_i,s\right)\mathrm{B}u(s)+\sum_{i=2}^{k}\prod_{j=k}^{i}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\mathrm{E}_{i-1}v_{i-1}+\mathrm{E}_k v_k. \end{aligned}$$

To demonstrate the approximate controllability of the linear system mentioned above, we introduce a bounded linear operator $\Omega: L^2(J,\mathbb{U})\times\mathbb{U}^m\to\mathbb{H}$ as follows:

$$\begin{aligned} &\Omega\left(u(\cdot),\{v_k\}_{k=1}^m\right)\\ &=\mathrm{U}\left(b,t_m\right)\sum_{i=1}^{m}\prod_{j=m}^{i+1}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\left(\mathrm{I}+\mathrm{D}_i\right)\int_{t_{i-1}}^{t_i}\mathrm{U}\left(t_i,s\right)\mathrm{B}u(s)ds\\ &\quad+\int_{t_m}^{b}\mathrm{U}(b,s)\mathrm{B}u(s)ds+\mathrm{U}\left(b,t_m\right)\sum_{i=2}^{m}\prod_{j=m}^{i}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\mathrm{E}_{i-1}v_{i-1}\\ &\quad+\mathrm{U}\left(b,t_m\right)\mathrm{E}_m v_m. \end{aligned}$$

Remark 2.8. We can verify Lemma 7 and Lemma 9 in [9] for the linear system (2.2) in similar way.

The operator Ω^* is the adjoint of Ω and has the following form (it can be verified in the similar way as in Lemma 9,[9])

$$\begin{aligned} &\Omega^*\varphi=\left(\mathrm{B}^*\psi(\cdot),\left\{\mathrm{D}_k^*\psi\left(t_k^+\right)\right\}_{k=1}^m\right),\\ &\mathrm{B}^*\psi(t)=\begin{cases}\mathrm{B}^*\mathrm{U}^*(b,t)\varphi,\ t_m<t\le b,\\ \mathrm{B}^*\mathrm{U}^*\left(t_k,t\right)\left(\mathrm{I}+\mathrm{D}_k^*\right)\prod_{i=k+1}^{m}\mathrm{U}^*\left(t_i,t_{i-1}\right)\left(\mathrm{I}+\mathrm{D}_i^*\right)\mathrm{U}^*\left(b,t_m\right)\varphi,\\ t_{k-1}<t\le t_k,\end{cases} \end{aligned}$$

$$\mathrm{E}_k^*\psi\left(t_k^+\right)=\begin{cases}\mathrm{E}_m^*\mathrm{U}^*\left(b,t_m\right)\varphi, \quad k=m,\\ \mathrm{E}_k^*\prod_{i=k+1}^m \mathrm{U}^*\left(t_i,t_{i-1}\right)\left(\mathrm{I}+\mathrm{D}_i^*\right)\mathrm{U}^*\left(b,t_m\right)\varphi, \quad k=m-1,\dots,1,\end{cases}$$

where the operators U^*, B^*, D_k^*, E_m^* are the adjoint operators of U, B, D_k and E_k respectively and $\psi(.)$ is the solution of the adjoint problem associated with system (2.2). The operator $\Omega\Omega^*:\mathbb{H}\to\mathbb{H}$ has the following form:

$$\Omega\Omega^*=\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b,$$

where $\Gamma_{t_m}^b,\widetilde{\Gamma}_{t_m}^b,\Theta_0^{t_m},\widetilde{\Theta}_0^{t_m}:\mathbb{H}\to\mathbb{H}$ are non-negative operators and defined as follows:

$$\Gamma_{t_m}^b:=\int_{t_m}^b \mathrm{U}(b,s)\mathrm{B}\mathrm{B}^*\mathrm{U}(b,s)ds, \quad \widetilde{\Gamma}_{t_m}^b:=\mathrm{U}\left(b,t_m\right)\mathrm{E}_m\mathrm{E}_m^*\mathrm{U}^*\left(b,t_m\right),$$

$$\begin{aligned}\Theta_0^{t_m}:=&\mathrm{U}\left(b,t_m\right)\\ &\times\sum_{i=1}^m\prod_{j=m}^{i+1}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\left(\mathrm{I}+\mathrm{D}_i\right)\int_{t_{i-1}}^{t_i}\mathrm{U}\left(t_i,s\right)\mathrm{B}\mathrm{B}^*\mathrm{U}^*\left(t_k,s\right)ds\\ &\times\left(\mathrm{I}+\mathrm{D}_i^*\right)\prod_{k=i+1}^m\mathrm{U}^*\left(t_k,t_{k-1}\right)\left(\mathrm{I}+\mathrm{D}_k^*\right)\mathrm{U}^*\left(b,t_m\right),\\ \widetilde{\Theta}_0^{t_m}:=&\mathrm{U}\left(b,t_m\right)\sum_{i=2}^m\prod_{j=m}^{i}\left(I+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\mathrm{E}_{i-1}\mathrm{E}_{i-1}^*\\ &\times\prod_{k=i}^m\mathrm{U}^*\left(t_k,t_{k-1}\right)\left(I+\mathrm{D}_k^*\right)\mathrm{U}^*\left(b,t_m\right).\end{aligned}$$

Remark 2.9. The linear system 2.2 is said to be approximately controllable on $[0,b]$ if $\overline{\operatorname{Im}\Omega}=\mathbb{H}$.

Now we will prove the approximate controllability of linear non-autonomous system (2.2).

Theorem 2.10. *Under the assumptions (R1)-(R4), for the system (2.2), the following statements are equivalent:*

(a) *System (2.2) is approximately controllable on* $[0,b]$*.*
(b) $\Omega^*\varphi=0$ *implies that* $\varphi=0$*.*
(c) $\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b$ *is strictly positive.*
(d) $\lambda\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b\right)^{-1}$ *converges to zero operator as* $\lambda\to0^+$ *in strong operator topology.*
(e) $\lambda\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b\right)^{-1}$ *converges to zero operator as* $\lambda\to0^+$ *in weak operator topology.*

Proof. The proof of the equivalence $(a)\Longleftrightarrow(b)$ is standard. Approximately controllability of system (2.2) on $[0,b]$ is equivalent to $\operatorname{Im}\Omega$ is dense in $\mathbb{H}$. That means, the kernel of Ω^* is trivial in $\mathbb{H}$. Equivalently,

$$\Omega^*\varphi=\left(\mathrm{B}^*\psi(\cdot),\left\{\mathrm{E}_k^*\psi\left(t_k^+\right)\right\}_{k=1}^m\right)=0,$$

implies that $\varphi = 0$. For the equivalence $(a) \iff (c)$ is clear from [9]. The equivalence $(d) \iff (e)$ is a consequence of positivity of

$$\lambda\left(\lambda \mathrm{I} + \Theta_0^{t_m} + \Gamma_{t_m}^b + \widetilde{\Theta}_0^{t_m} + \widetilde{\Gamma}_{t_m}^b\right)^{-1}.$$

We prove only $(a) \iff (d)$. To do so, consider the functional

$$J_\lambda(\varphi) = \frac{1}{2}\left\|\Omega^*\varphi\right\|^2 + \frac{\lambda}{2}\|\varphi\|^2 - \left\langle \varphi, h - \mathrm{U}\left(b, t_m\right)\prod_{j=m}^{1}\left(\mathrm{I} + \mathrm{D}_j\right)\mathrm{U}\left(t_j, t_{j-1}\right)x_0 \right\rangle.$$

The map $\varphi \to J_\lambda(\varphi)$ is continuous and strictly convex. The functional $J_\lambda(\cdot)$ admits a unique minimum $\widehat{\varphi}_\lambda$ that defines a map $\Phi : \mathbb{H} \to \mathbb{H}$. Since $J_\lambda(\varphi)$ is Frechet differentiable at $\widehat{\varphi}_\lambda$, by the optimality of $\widehat{\varphi}_\lambda$, we must have

$$\begin{aligned}\frac{d}{d\varphi}J_\lambda(\varphi) =& \Theta_0^{t_m}\widehat{\varphi}_\lambda + \Gamma_{t_m}^b\widehat{\varphi}_\lambda + \widetilde{\Theta}_0^{t_m}\widehat{\varphi}_\lambda + \widetilde{\Gamma}_{t_m}^b\widehat{\varphi}_\lambda + \lambda\widehat{\varphi}_\lambda - h \\ &+ \mathrm{U}\left(b, t_m\right)\prod_{j=m}^{1}\left(\mathrm{I} + \mathrm{D}_j\right)\mathrm{U}\left(t_j, t_{j-1}\right)x_0 = 0. \qquad (2.4)\end{aligned}$$

By solving above equation (2.4) for $\widehat{\varphi}_\lambda$, we get

$$\begin{aligned}\widehat{\varphi}_\lambda =& \left(\lambda \mathrm{I} + \Theta_0^{t_m} + \Gamma_{t_m}^b + \widetilde{\Theta}_0^{t_m} + \widetilde{\Gamma}_{t_m}^b\right)^{-1} \\ &\times \left(h - \mathrm{U}\left(b, t_m\right)\prod_{j=m}^{1}\left(\mathrm{I} + \mathrm{D}_j\right)\mathrm{U}\left(t_j, t_{j-1}\right)x_0\right). \qquad (2.5)\end{aligned}$$

Now we define control $u(s)$ as following

$$\begin{aligned}u(s) = \Bigg(&\sum_{k=1}^{m}\mathrm{B}^*\mathrm{U}(t_k, s)^*\prod_{i=k+1}^{m}\mathrm{U}(t_i, t_{i-1})^*\mathrm{U}(b, t_m)^*\chi(t_{k-1}, t_k) \\ &+\mathrm{B}^*\mathrm{U}(b, s)^*\chi(t_m, b)\Bigg)\widehat{\varphi}_\lambda, \qquad (2.6)\end{aligned}$$

$$v_m = \mathrm{E}_m^*\mathrm{U}(b, t_m)^*\widehat{\varphi}_\lambda, \quad v_k = \mathrm{E}_k^*\prod_{i=k}^{m}\mathrm{U}(t_i, t_{i-1})^*(\mathrm{I} + \mathrm{D}_i^*)\mathrm{U}(b, t_m)^*\widehat{\varphi}_\lambda.$$

Let $x_\lambda(b)$ be the solution at the final point b corresponding to the above defined control, can be expressed as:

$$\begin{aligned}x_\lambda(b) =& \mathrm{U}(b, t_m)\prod_{j=p}^{1}\left(\mathrm{I} + \mathrm{D}_j\right)\left(\mathrm{U}(t_j - t_{j-1})\right)x_0 \\ &+ \Theta_0^{t_m}\widehat{\varphi}_\lambda + \Gamma_{t_p}^b\widehat{\varphi}_\lambda + \widetilde{\Theta}_0^{t_p}\widehat{\varphi}_\lambda + \widetilde{\Gamma}_{t_p}^b\widehat{\varphi}_\lambda + \lambda\widehat{\varphi}_\lambda.\end{aligned}$$

. Now from (2.3), (2.5) and (2.6) we get

$$x_\lambda(b) - h = -\lambda\left(\lambda I + \Theta_0^{t_m} + \Gamma_{t_m}^b + \widetilde{\Theta}_0^{t_m} + \widetilde{\Gamma}_{t_m}^b\right)^{-1}$$

$$\times\left(h-\mathrm{U}\left(b,t_m\right)\prod_{j=m}^{1}\left(I+D_j\right)T\left(t_j-t_{j-1}\right)x_0\right). \tag{2.7}$$

The above expression shows that the linear system (2.2) is approximately controllable iff $\lambda\left(\lambda I+\Theta_0^{t_m}+\Gamma_{t_m}^{b}+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^{b}\right)^{-1}$ converges to zero operator as $\lambda\mapsto 0^+$ in strong operator topology. Therefore, $(a)\iff(d)$. □

In order to establish the existence results for the system (1.2), we require the following assumptions:

(A1) For every $x\in\mathbb{H}$, $\lambda\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^{b}+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^{b}\right)^{-1}$ converges to zero operator as $\lambda\to 0^+$ in strong operator topology.

(A2) (i) The function $f:[0,b]\times\mathbb{H}\to\mathbb{H}$ is continuous and there is a constant L_f such that for every $t\in[0,b]$ and $x,y\in\mathbb{H}$,

$$\|f(t,x)-f(t,y)\|\le L_f\|x-y\|,$$

(ii) there exists C_f such that $\|f\left(t,x(t)\right)\|\le C_f$ for $t\in[0,b]$.

(A3) (i) The function $\xi:[0,b]\times\mathbb{H}\to\mathbb{H}$ is continuous and there is a constant $\tilde{L}_\xi$ such that for every $t\in[0,b]$ and $x,y\in\mathbb{H}$,

$$\|\xi(t,x)-\xi(t,y)\|\le L_\xi\|x-y\|,$$

(ii) there exists C_ξ such that $\|\xi\left(t,x(t)\right)\|\le C_\xi$ for $t\in[0,b]$.

3. Existence and Approximate Controllability of Semilinear System

The primary goal of this section is to identify sufficient conditions for the solvability of system (1.2). To achieve this, we will first demonstrate that, for each λ and a fixed $h\in\mathbb{H}$, system (1.2) possesses at least one mild solution. We prove the existence of a mild solution of the system (1.2) with the control

$$u(s)=\left(\sum_{k=1}^{m}\mathrm{B}^*\mathrm{U}(t_k,s)^*\prod_{i=k+1}^{m}\mathrm{U}(t_i,t_{i-1})^*\mathrm{U}(b,t_m)^*\chi(t_{k-1},t_k)\right.$$
$$\left.+\mathrm{B}^*\mathrm{U}(b,s)^*\chi(t_m,b)\right)\widehat{\varphi}_\lambda, \tag{3.1}$$

$$v_m=\mathrm{E}_m^*\mathrm{U}(b,t_m)^*\widehat{\varphi}_\lambda,\quad v_k=\mathrm{E}_k^*\prod_{i=k}^{m}\mathrm{U}(t_i,t_{i-1})^*(\mathrm{I}+\mathrm{D}_i^*)\mathrm{U}(b,t_m)^*\widehat{\varphi}_\lambda,$$

with

$$\widehat{\varphi}_\lambda=\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^{b}+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^{b}\right)^{-1}\times g\left(x(\cdot)\right),$$

where

$$g\left(x(\cdot)\right)=\left(h-\mathrm{U}\left(b,t_m\right)\prod_{j=m}^{1}(\mathrm{I}+\mathrm{D}_j)\mathrm{U}\left(t_j,t_{j-1}\right)x_0\right.$$
$$-\int_{t_m}^{b}\mathrm{U}(b,s)\left(f\left(s,x(s)\right)+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds$$

$$- \mathrm{U}(b, t_m) \sum_{i=1}^{m} \prod_{j=m}^{i+1} (\mathrm{I}+\mathrm{D}_j)\, \mathrm{U}(t_j, t_{j-1})\, (\mathrm{I}+\mathrm{D}_i)$$
$$\times \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s) \left(f(s, x(s)) + \int_0^s q(s-\tau)\xi(\tau, x(\tau)) d\tau \right) ds \Bigg).$$

With these assumptions established, we are now ready to prove the existence and uniqueness of the mild solution for (1.2) using the fixed point theorem 2.7.

Theorem 3.1. *If the assumptions (R1)-(R4) and (A2)-(A3) are satisfied. Then for every $\lambda > 0$ and for fixed $h \in \mathbb{H}$, the system* (1.2) *has at least one mild solution in $\mathcal{PC}([0,b], \mathbb{H})$ provided that*

$$\max\{\mathcal{N}, \mathcal{K}_1\} < 1, \tag{3.2}$$

and

$$\max\{M; \mathscr{L}\} < 1, \tag{3.3}$$

where $\mathcal{N}$ and $\mathcal{K}_1$ are given by:

$$\begin{cases} \mathcal{N} = M + \frac{M^3 M_{\mathrm{B}}^2 b}{\lambda}, \\ \mathcal{K}_1 = M^{k+1} \prod_{j=i}^{k} (1+\|\mathrm{D}_j\|) \Bigg(1 + \frac{M^2 M_B^2 b}{\lambda} \left(m M^{m-k}+1\right)(MN+1) + \mathcal{K}_0 \\ \qquad + \frac{M^2}{\lambda} \|\mathrm{E}_k\| \, \|\mathrm{E}_k^* \prod_{i=k}^{m} \|\mathrm{U}^*(t_i, t_{i-1})\| \, (\mathrm{I}+\mathrm{D}_i^*)\| \Bigg), \\ \mathcal{K}_0 = \frac{M^2}{\lambda} \sum_{i=2}^{k} \prod_{j=i}^{k} (1+\|\mathrm{D}_j\|) \, \|\mathrm{U}(t_j, t_{j-1})\| \\ \qquad \times \|\mathrm{E}_{i-1}\| \left\| \mathrm{E}_{i-1}^* \prod_{l=i-1}^{m} \|\mathrm{U}(t_l, t_{l-1})^*\| \, (\mathrm{I}+\mathrm{D}_l^*) \right\|, \\ C_i = \prod_{j=k}^{i+1} (1+\|D_j\|) \, \|T(t_j - t_{j-1})\| \, (1+\|D_i\|), \quad N = \sum_{i=1}^{k} C_i, \\ \mathscr{L} = M^{k+1} \prod_{j=1}^{k} (1+\|\mathrm{D}_j\|) + M^2 N b \left(L_f + q^* L_\xi\right) \ \text{and}\ q^* = \int_0^t |q(t-s)| ds. \end{cases}$$

Proof. For each constant $r_0 > 0$, let

$$\mathcal{B}_{r_0} = \{x \in \mathcal{PC}([0,b], \mathbb{H}) : \quad \|x\|_{\mathcal{PC}} \le r_0\}.$$

It is easy to see that $\mathcal{B}_{r_0}$ is a bounded closed convex set. Define operators F_1 and F_2 on $\mathcal{B}_{r_0}$ as follows:

$(F_1 x)(t)=$

$$\begin{cases} \mathrm{U}(t,0)x_0, \quad \text{for} \quad t_0 < t \le t_1, \\ \mathrm{U}(t, t_k) \prod_{j=k}^{1} (\mathrm{I}+\mathrm{D}_j)\, \mathrm{U}(t_j, t_{j-1})\, x_0 \\ + \mathrm{U}(t, t_k) \sum_{i=1}^{k} \prod_{j=k}^{i+1} (\mathrm{I}+\mathrm{D}_j)\, \mathrm{U}(t_j, t_{j-1})\, (\mathrm{I}+\mathrm{D}_i) \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s)\, \mathrm{B}u(s) ds \\ + \mathrm{U}(t, t_k) \sum_{i=1}^{k} \prod_{j=k}^{i+1} (\mathrm{I}+\mathrm{D}_j)\, \mathrm{U}(t_j, t_{j-1})\, (\mathrm{I}+\mathrm{D}_i) \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s) \big[f(s, x(s)) \\ + \int_0^s q(s-\tau)\xi(\tau, x(\tau)) d\tau \big] ds + \mathrm{U}(t, t_k) \sum_{i=2}^{k} \prod_{j=k}^{i} (\mathrm{I}+\mathrm{D}_j)\, \mathrm{U}(t_j, t_{j-1})\, \mathrm{E}_{i-1} v_{i-1} \\ \qquad + \mathrm{U}(t, t_k)\, \mathrm{E}_k v_k, \ \text{for}\ t_k < t \le t_{k+1}, k \ge 1, \end{cases}$$

and

$$(F_2 x)(t) = \begin{cases} \int_0^t \mathrm{U}(t,s) \big[\mathrm{B}u(s) + f(s, x(s)) \\ \qquad + \int_0^s q(s-\tau)\xi(\tau, x(\tau)) d\tau \big] ds, \ \text{for}\ t_0 < t \le t_1 \\ \int_{t_k}^t \mathrm{U}(t,s) \big[\mathrm{B}u(s) + f(s, x(s)) \\ \qquad + \int_0^s q(s-\tau)\xi(\tau, x(\tau)) d\tau \big] ds, \ \text{for}\ t_k < t \le t_{k+1}, k \ge 1. \end{cases}$$

Clearly, x is a mild solution of (2) if and only if the operator equation $x = F_1x + F_2x$ has a solution. To establish this, we will demonstrate that the operator $F_1 + F_2$ has a fixed point by applying theorem 2.7 . For this, we proceed in several steps. **Step 1:**To prove that there exists a positive number r_0 such that $F_1x + F_2y \in \mathcal{B}_{r_0}$ whenever $x, y \in \mathcal{B}_{r_0}$, we choose

$r_0 \geq$

$$\max\left(\frac{\left(\frac{M^2M_{\mathrm{B}}^2b}{\lambda}\|h\| + \frac{M^2M_{\mathrm{B}}^2b}{\lambda}\left(MbC_f + MbC_\xi q^*\right) + MbC_f + MbC_\xi q^*\right)}{1-\mathcal{N}},\ \frac{\mathcal{K}_2}{1-\mathcal{K}_1}\right),$$

$$\begin{aligned}\text{where}\quad \mathcal{K}_2 = \Bigg(&\frac{M^2M_{\mathrm{B}}^2b}{\lambda}\left(MN+1\right)\left(mM^{m-k}+1\right)\\ &+\mathcal{K}_0\frac{M^2}{\lambda}\|\mathrm{E}_k\|\left\|\mathrm{E}_k^*\prod_{i=k}^{m}\|\mathrm{U}(t_i,t_{i-1})^*\|\left(\mathrm{I}+\mathrm{D}_i^*\right)\right\|\Bigg)\\ &\times\left(\|h\| + M\left(C_f+q^*C_\xi\right)b + M^2NCb\right) + Mb\left(C_f+q^*C_\xi\right)\left(MN+1\right).\end{aligned}$$

First, we calculate for $t_0 < t \leq t_1$ and $s \in [0, b]$,

$$\begin{aligned}u(s) = \mathrm{B}^*\mathrm{U}x(t_1,s)^*\left(\lambda I + \Gamma_0^{t_1}\right)^{-1}\Bigg[&h - \mathrm{U}(t,0)x_0\\ &-\int_0^{t_1}\mathrm{U}(t_1,s)\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds\Bigg].\end{aligned}$$

Using the triangle inequality, Lipschitz conditions, and the boundedness of the evolution family $\mathrm{U}(t,s)$, the norm $\|u(s)\|_{\mathbb{U}}$ can be calculated as:

$$\begin{aligned}\|u(s)\|_{\mathbb{U}} = \Bigg\|&\mathrm{B}^*\mathrm{U}(t_1,s)^*\left(\lambda I + \Gamma_0^{t_1}\right)^{-1}\Bigg[h - \mathrm{U}(t,0)x_0\\ &-\int_0^{t_1}\mathrm{U}(t_1,s)\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds\Bigg]\Bigg\|\\ \leq{}& \|\mathrm{B}^*\|_{\mathcal{L}}\|\mathrm{U}(t_1-s)^*\|_{\mathbb{H}}\left\|\left(\lambda I + \Gamma_0^{t_1}\right)^{-1}\right\|\Bigg\|h - \mathrm{U}(t,0)x_0\\ &-\int_0^{t_1}\mathrm{U}(t_1,s)\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds\Bigg\|\\ \leq{}& \frac{MM_B}{\lambda}\Bigg(\|h\| + \|\mathrm{U}(t_1,0)\|_{\mathbb{H}}\|x_0\|\\ &+\|\mathrm{U}(t_1,s)\|_{\mathbb{H}}\int_0^{t_1}\left\|\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)\right\|ds\Bigg)\\ \leq{}& \frac{MM_B}{\lambda}\left(\|h\| + Mr_0 + MC_fb + MC_\xi q^*\right).\end{aligned}$$

To calculate the norm of u for $t_k < t \leq t_{k+1}$, $k \geq 1$ and $s \in [0, b]$, first we find the norm of $\widehat{\varphi}_\lambda$ as follows:

$$\|\widehat{\varphi}_\lambda\| \leq \left\|\left(\lambda\mathrm{I} + \Theta_0^{t_m} + \Gamma_{t_m}^b + \widetilde{\Theta}_0^{t_m} + \widetilde{\Gamma}_{t_m}^b\right)^{-1}\right\|$$

$$
\begin{aligned}
&\times \Bigg(\|h\| + \left\|\mathrm{U}\left(b,t_m\right) \prod_{j=m}^{1} (\mathrm{I}+\mathrm{D}_j)\mathrm{U}\left(t_j,t_{j-1}\right)x_0\right\| \\
&\quad + \left\|\int_{t_m}^{b} \mathrm{U}(b,s)\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds\right\| \\
&\quad + \Bigg\|\mathrm{U}\left(b,t_m\right) \sum_{i=1}^{m} \prod_{j=m}^{i+1} (\mathrm{I}+\mathrm{D}_j)\,\mathrm{U}\left(t_j-t_{j-1}\right)(\mathrm{I}+\mathrm{D}_i) \\
&\qquad \times \int_{t_{i-1}}^{t_i} \mathrm{U}\left(t_i-s\right)\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds\Bigg\|\Bigg) \\
&\le \frac{1}{\lambda}\Bigg(\|h\| + M^{k+1}\prod_{j=1}^{k}\left(1+\|\mathrm{D}_j\|\right)\|x_0\| + Mb(C_f+q^*C_\xi) \\
&\quad + M^2\sum_{i=1}^{m} C_i \int_{t_{i-1}}^{t_i}\left\|\left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)\right\|ds\Bigg) \\
&\le \frac{1}{\lambda}\Bigg(\|h\| + M^{k+1}\prod_{j=1}^{k}\left(1+\|\mathrm{D}_j\|\right)\|x_0\| + Mb(C_f+q^*C_\xi) \\
&\quad + M^2Nb(C_f+q^*C_\xi)\Bigg).
\end{aligned}
$$

With the above help we can find the norm of u as follows:

$$
\begin{aligned}
\|u(s)\|_{\mathbb{U}} &\le \Bigg\|\Bigg(\sum_{k=1}^{m} \mathrm{B}^*\mathrm{U}(t_k,s)^* \prod_{i=k+1}^{m} \mathrm{U}^*(t_i,t_{i-1})\mathrm{U}^*(b,t_m)\chi(t_{k-1},t_k) \\
&\qquad + \mathrm{B}^*\mathrm{U}(b,s)^*\chi(t_m,b)\Bigg)\Bigg\|\,\|\widehat{\varphi}_\lambda\| \\
&\le \frac{1}{\lambda}\left(mM_{\mathrm{B}}M^{m+1-k} + M_{\mathrm{B}}M\right) \times \Bigg(\|h\| \\
&\quad + M^{k+1}\prod_{j=1}^{k}\left(1+\|\mathrm{D}_j\|\right)r_0 + Mb(C_f+q^*C_\xi) + M^2N(C_f+q^*C_\xi)\Bigg).
\end{aligned}
$$

Now, for $0 \le t \le t_1$, we have

$$
\begin{aligned}
\left\|\left(F_1x\right)(t) + \left(F_2x\right)(t)\right\| &\le \|\mathrm{U}(t,0)x(0)\|_{\mathbb{H}} + \left\|\int_0^t \mathrm{U}(t,s)\mathrm{B}u(s)ds\right\| + \Bigg\|\int_0^t \mathrm{U}(t,s) \\
&\qquad \times \left(f\left(s,x(s)\right) + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)ds\Bigg\| \\
&\le M\|x_0\| + M\|\mathrm{B}\|_{\mathcal{L}}\int_0^t u(s)ds + MbC_f + MbC_\xi q^* \\
&\le Mr_0 + \frac{M^2M_{\mathrm{B}}^2 b}{\lambda}\left(\|h\| + Mr_0 + MbC_f + MbC_\xi q^*\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + MbC_f + MbC_\xi q^* \\
&= \mathcal{N} r_0 + \Bigg(\frac{M^2 M_{\mathrm{B}}^2 b}{\lambda} \|h\| \\
&\qquad + \left(\frac{M^2 M_{\mathrm{B}}^2 b}{\lambda} + 1 \right) \left(MbC_f + MbC_\xi q^* \right) \Bigg) \\
&\leq r_0.
\end{aligned}
$$

For $t_k < t \leq t_{k+1}$ for $k \geq 1$, we have,

$$
\begin{aligned}
&\left\| \left(F_1 x\right)(t) + \left(F_2 x\right)(t) \right\| \\
&\leq \left\| \mathrm{U}\left(t, t_k\right) \prod_{j=k}^{1} \left(\mathrm{I} + \mathrm{D}_j\right) T\left(t_j - t_{j-1}\right) x_0 \right\| \\
&\quad + \left\| \mathrm{U}\left(t, t_k\right) \sum_{i=1}^{k} \prod_{j=k}^{i+1} \left(\mathrm{I} + \mathrm{D}_j\right) \mathrm{U}\left(t_j, t_{j-1}\right) \left(\mathrm{I} + \mathrm{D}_i\right) \int_{t_{i-1}}^{t_i} \mathrm{U}\left(t_i, s\right) \mathrm{B} u(s) ds \right\| \\
&\quad + \Bigg\| \mathrm{U}\left(t, t_k\right) \sum_{i=1}^{k} \prod_{j=k}^{i+1} \left(\mathrm{I} + \mathrm{D}_j\right) \mathrm{U}\left(t_j, t_{j-1}\right) \left(\mathrm{I} + \mathrm{D}_i\right) \\
&\qquad \times \int_{t_{i-1}}^{t_i} \mathrm{U}\left(t_i, s\right) \left(f\left(s, x(s)\right) + \int_0^s q(s-\tau) \xi(\tau, x(\tau)) d\tau \right) ds \Bigg\| \\
&\quad + \left\| \mathrm{U}\left(t, t_k\right) \sum_{i=2}^{k} \prod_{j=k}^{i} \left(\mathrm{I} + \mathrm{D}_j\right) \mathrm{U}\left(t_j, t_{j-1}\right) \mathrm{E}_{i-1} v_{i-1} \right\| \\
&\quad + \left\| \mathrm{U}\left(t, t_k\right) \mathrm{E}_k v_k \right\| + \left\| \int_{t_k}^{t} \mathrm{U}(t, s) \mathrm{B} u(s) ds \right\| \\
&\quad + \left\| \int_{t_k}^{t} \mathrm{U}(t, s) \left(f\left(s, x(s)\right) + \int_0^s q(s-\tau) \xi(\tau, x(\tau)) d\tau \right) ds \right\| \\
&\leq M^{k+1} \prod_{j=1}^{k} \left(1 + \|\mathrm{D}_j\|\right) r_0 + M^2 \|\mathrm{B}\|_{\mathcal{L}} \sum_{i=1}^{k} C_i \int_{t_{i-1}}^{t_i} \|u(s)\| \;\; ds \\
&\quad + M^2 \sum_{i=1}^{k} C_i \int_{t_{i-1}}^{t_i} \| \left(f\left(s, x(s)\right) + \int_0^s q(s-\tau) \xi(\tau, x(\tau)) d\tau \right) \| ds \\
&\quad + M \sum_{i=2}^{k} \prod_{j=i}^{k} \left(1 + \|\mathrm{D}_j\|\right) \|\mathrm{U}(t_j, t_{j-1})\| \, \|\mathrm{E}_{i-1}\| \, \|v_{i-1}\| + M \, \|\mathrm{E}_k\| \, \|v_k\| \\
&\quad + M \|\mathrm{B}\|_{\mathcal{L}} \int_{t_k}^{t} u(s) ds + M \int_{t_k}^{t} \left\| \left(f\left(s, x(s)\right) + \int_0^s q(s-\tau) \xi(\tau, x(\tau)) d\tau \right) \right\| ds \\
&\leq \mathcal{K}_1 r_0 + \mathcal{K}_2 \\
&\leq r_0.
\end{aligned}
$$

Consequently, $F_1 + F_2$ maps $\mathcal{B}_{r_0}$ to $\mathcal{B}_{r_0}$.

Step 2: The next step is to prove that F_1 is a contraction.

To demonstrate that F_1 is a contraction mapping on the set $\mathcal{B}_r$, it is necessary to show that there exists a constant $0 < \mathscr{L} < 1$ such that for all $x, y \in \mathcal{B}_r$,

$$\|F_1 x - F_1 y\|_{\mathcal{PC}} \leq \mathscr{L} \|x - y\|_{\mathcal{PC}}.$$

Let $x, y \in \mathcal{B}_r$. We will estimate $\|F_1 x - F_1 y\|_{\mathcal{PC}}$ for $t_0 < t \leq t_1$ and $t_k < t \leq t_{k+1}$. For $t_0 < t \leq t_1$:

$$\|(F_1 x)(t) - (F_1 y)(t)\| = \|\mathrm{U}(t,0)(x(0) - y(0))\|_{\mathbb{H}}.$$

Using the properties of the evolution operator $\mathrm{U}(t,s)$:

$$\|\mathrm{U}(t,0)(x(0) - y(0))\| \leq M\|x(0) - y(0)\|,$$

for $M < 1$, F_1 is a contraction map.

For $t_k < t \leq t_{k+1}, k \geq 1$:

$$\begin{aligned}
\|(F_1 x)(t) - (F_1 y)(t)\| \leq & \left\| \mathrm{U}(t, t_k) \prod_{j=k}^{1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1})(x_0 - y_0) \right\| \\
+ & \Bigg\| \mathrm{U}(t, t_k) \sum_{i=1}^{k} \prod_{j=k}^{i+1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1})(\mathrm{I} + \mathrm{D}_i) \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s) \\
& \times \left[(f(s, x(s)) - f(s, y(s))) + \int_0^s q(s-\tau)(\xi(\tau, x(\tau)) - \xi(\tau, y(\tau)))\, d\tau \right] ds \Bigg\|.
\end{aligned}$$

Using the properties of $\mathrm{U}(t,s)$, the boundedness of operators D_j, and assumptions on f :

$$\left\| \mathrm{U}(t, t_k) \prod_{j=k}^{1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1})(x_0 - y_0) \right\| \leq M^{k+1} \prod_{j=1}^{k} (1 + \|\mathrm{D}_j\|) \|x_0 - y_0\|.$$

Since $x, y \in \mathcal{B}_r$ this gives $\|x_0 - y_0\| \leq \|x - y\|_{\mathcal{PC}}$.

Thus,

$$\left\| \mathrm{U}(t, t_k) \prod_{j=k}^{1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1})(x_0 - y_0) \right\| \leq M^{k+1} \prod_{j=1}^{k} (1 + \|\mathrm{D}_j\|) \|x - y\|_{\mathcal{PC}}.$$

For the second term, using the properties of $\mathrm{U}(t,s)$ and D_j, and assumption (A2) of f :

$$\begin{aligned}
& \Bigg\| \mathrm{U}(t, t_k) \sum_{i=1}^{k} \prod_{j=k}^{i+1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1})(\mathrm{I} + \mathrm{D}_i) \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s) \\
& \quad \times \left[(f(s, x(s)) - f(s, y(s))) + \int_0^s q(s-\tau)(\xi(\tau, x(\tau)) - \xi(\tau, y(\tau)))\, d\tau \right] ds \Bigg\|
\end{aligned}$$

$$\leq M^2\Bigg(\sum_{i=1}^{k}\prod_{j=i+1}^{k}\left(1+\|\mathrm{D}_j\|\right)\|\mathrm{U}\left(t_j,t_{j-1}\right)\|_{\mathbb{H}}\left(1+\|\mathrm{D}_i\|\right)$$
$$\times\int_{t_{i-1}}^{t_i}\Big\|\left(f\left(s,x(s)\right)-f\left(s,y(s)\right)\right)$$
$$+\int_0^s q(s-\tau)\left(\xi(\tau,x(\tau))-\xi(\tau,y(\tau))\right)d\tau\Big\|ds\Bigg)$$
$$\leq M^2\Bigg(\sum_{i=1}^{k}C_i\int_{t_{i-1}}^{t_i}\Big[\|\left(f\left(s,x(s)\right)-f\left(s,y(s)\right)\right)\|$$
$$+\int_0^s\|q(s-\tau)\|\,\|\xi(\tau,x(\tau))-\xi(\tau,y(\tau))\|\,d\tau\Big]ds\Bigg)$$
$$\leq M^2Nb\left(L_f+q^*L_\xi\right)\|x-y\|_{\mathcal{PC}}.$$

Combining all terms, we get:

$$\|\left(F_1x\right)(t)-\left(F_1y\right)(t)\|$$
$$\leq\left(M^{k+1}\prod_{j=1}^{k}\left(1+\|\mathrm{D}_j\|\right)+M^2Nb\left(L_f+q^*L_\xi\right)\right)\|x-y\|_{\mathcal{PC}}.$$

To show that F_1 is a contraction, we need the right-hand side to be less than $\|x-y\|_{\mathcal{PC}}$. Hence, we need

$$M^{k+1}\prod_{j=1}^{k}\left(1+\|\mathrm{D}_j\|\right)+M^2Nb\left(L_f+q^*L_\xi\right)<1.$$

Therefore, there exists a constant $\mathscr{L}\in(0,1)$ such that:

$$\|F_1x-F_1y\|_{\mathcal{PC}}\leq\mathscr{L}\|x-y\|_{\mathcal{PC}}.$$

This shows that on $\mathcal{B}_{r_0}$, F_1 is a contraction map .

Step 3: Now we will show that F_2 is continuous and $F_2(\mathcal{B}_{r_0})$ is relatively compact subset of $\mathcal{B}_{r_0}$.

First, we need to prove that the mapping F_2 is continuous on $\mathcal{B}_{r_0}$. To do this, let $x_n\to x$ in $\mathcal{B}_{r_0}$. Then, we have:

$$f\left(t,x_n(t)\right)\to f\left(t,x(t)\right)\ \text{ and }\ \xi\left(t,x_n(t)\right)\to\xi\left(t,x(t)\right)\ \text{ as } n\to\infty.$$

Moreover, for $t_0\leq t\leq t_1$ by Lebesgue dominated convergence theorem, we get

$$\Big\|\int_0^t\mathrm{U}(t,s)\Big[f\left(t,x_n(t)\right)+\int_0^s q(s-\tau)\xi(\tau,x_n(\tau))d\tau-f\left(t,x(t)\right)$$
$$-\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\Big]ds\Big\|$$
$$\leq M\int_0^t\Big[\|f\left(t,x_n(t)\right)-f\left(t,x(t)\right)\|$$

$$+\int_0^s \|q(s-\tau)\| \, \|\xi(\tau,x_n(\tau))-\xi(\tau,x(\tau))\| \Big] ds \to 0 \text{ as } n\to\infty.$$

$$\begin{aligned}\|F_2(x_n)-F_2(x)\| \leq &\Big\| \int_0^t \mathrm{U}(t,s)\big[f(t,x_n(t))+\int_0^s q(s-\tau)\xi(\tau,x_n(\tau))d\tau \\ &\qquad - f(t,x(t))-\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\big]ds\Big\| \\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

For $t_k < t \leq t_{k+1}$ with $k \geq 1$, the argument is similar to that for $t_0 < t \leq t_1$. Hence, it follows that F_2 is continuous on $\mathcal{B}_{r_0}$.

Next, we demonstrate that for any $t \in [0,b]$, the set $\mathscr{V}(t)=\{F_2(x)(t) \mid x \in \mathcal{B}_{r_0}\}$ is relatively compact in $\mathbb{H}$. To establish this, we will utilize the extended version of the Ascoli-Arzelà theorem (Theorem 2.1, [26]). For $t=0$, it is evident that $\mathscr{V}(0)$ is relatively compact in $\mathbb{H}$. Now, for $0<t\leq b$, let $\epsilon \in (0,t)$. By applying Lemma 2.4, we find that the operator $\mathrm{U}(t,t-\epsilon)$ is compact. We define an operator F^ϵ on $\mathcal{B}_{r_0}$ by:

$$(F^\epsilon x)(t)=\begin{cases} \int_0^{t-\epsilon}\mathrm{U}(t,s)\left[\mathrm{B}u(s)+f(s,x(s))+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right]ds \\ =\mathrm{U}(t,t-\epsilon)\int_0^{t-\epsilon}\mathrm{U}(t-\epsilon,s)\Big[\mathrm{B}u(s)+f(s,x(s)) \\ \qquad +\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\Big]ds \text{ if } t_0<t\leq t_1, \\ \int_{t_k}^{t-\epsilon}\mathrm{U}(t,s)\left[\mathrm{B}u(s)+f(s,x(s))+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right]ds \\ =\mathrm{U}(t,t-\epsilon)\int_{t_k}^{t-\epsilon}\mathrm{U}(t-\epsilon,s)\Big[\mathrm{B}u(s)+f(s,x(s)) \\ \qquad +\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\Big]ds \text{ if } t_k<t\leq t_{k+1}, k\geq 1. \end{cases}$$

Then the set $\{(F^\epsilon)(t) : x \in \mathcal{B}_{r_0}\}$ is relatively compact in $\mathbb{H}$ because $\mathrm{U}(t,t-\epsilon)$ is compact. This compactness helps us establish the desired continuity properties. Now, let's consider the case for $t_0<t\leq t_1$:

$$\begin{aligned}\|(F_2x)(t)-(F^\epsilon x)(t)\| \leq &\left\|\int_{t-\epsilon}^t \mathrm{U}(t,s)\mathrm{B}u(s)ds\right\| + \Big\|\int_{t-\epsilon}^t \mathrm{U}(t,s)\Big[f(s,x(s)) \\ &\qquad + \int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\Big]ds\Big\|.\end{aligned}$$

To estimate the component involving $\mathrm{B}u^\lambda(s)$, we apply the triangle inequality followed by the Cauchy-Schwarz inequality. This yields:

$$\left\|\int_{t-\epsilon}^t \mathrm{U}(t,s)\mathrm{B}u(s)ds\right\| \leq MM_B\epsilon^{\frac{1}{2}}\left(\int_{t-\epsilon}^t \|u(s)\|^2 ds\right)^{\frac{1}{2}}$$

Using assumptions (A2) and (A3), we have

$$\left\|\int_{t-\epsilon}^t \mathrm{U}(t,s)\left[f(s,x(s))+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right]ds\right\|$$

$$\begin{aligned}
&\le \left(\int_{t-\epsilon}^{t} \left\| \mathrm{U}(t,s)\left[f\left(s,x(s)\right)+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right]\right\| ds\right)\\
&\le M\int_{t-\epsilon}^{t} \left\| \left(f\left(s,x(s)\right)+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)\right\| ds\\
&\le M\left(C_f-q^*C_\xi\right)\epsilon.
\end{aligned}$$

Combining all terms, we get:

$\|(F_2x)(t)-(F^\epsilon x)(t)\| \le M\left(C_f-q^*C_\xi\right)\epsilon + MM_B\epsilon^{\frac{1}{2}}\left(\int_{t-\epsilon}^{t}\|u(s)\|^2 ds\right)^{\frac{1}{2}}.$

As $\epsilon \to 0$:

$$\|(F_2x)(t)-(F^\epsilon x)(t)\| \to 0.$$

For $t_k < t \le t_{k+1}$, with $k \ge 1$, the definitions of F_2 and F^ϵ allow us to derive similar results as previously discussed.

Therefore, since F_2x can be approximated arbitrarily closely by $F^\epsilon x$, and $F^\epsilon x$ is relatively compact in $\mathbb{H}$, it follows that $\mathscr{V}(t)=\{F_2(x)(t) \mid x\in\mathcal{B}_{r_0}\}$ is relatively compact in $\mathbb{H}$.

Finally, we show that $F_2(\mathcal{B}_{r_0})$ is equicontinuous on $[0,b]$. Let $0\le s_1\le s_2\le t_1$ for any $x\in\mathcal{B}_{r_0}$, we consider the following estimate

$$\begin{aligned}
&\left\|F_2x(s_2)-F_2x(s_1)\right\|\\
&\quad\le \left\|\int_0^{s_1}\left[\mathrm{U}(s_2,s)-\mathrm{U}(s_1,s)\right]\left[Bu(s)+f\left(s,x(s)\right)\right.\right.\\
&\qquad\qquad\left.\left.+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right]ds\right\|\\
&\qquad+\left\|\int_{s_1}^{s_2}\mathrm{U}(s_2,s)\left[Bu(s)+f\left(s,x(s)\right)+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right]ds\right\|\\
&\quad\le \int_{s_1}^{s_2}\|\mathrm{U}(s_2,s)\|_{\mathcal{L}(\mathbb{H})}\|\mathrm{B}\|_{\mathcal{L}}\|u(s)\|_{\mathbb{U}}ds\\
&\qquad+\int_{s_1}^{s_2}\|\mathrm{U}(s_2,s)\|_{\mathcal{L}(\mathbb{H})}\left\|\left(f\left(s,x(s)\right)+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)\right\|\\
&\qquad+\int_0^{s_1}\|\mathrm{U}(s_2,s)-\mathrm{U}(s_1,s)\|_{\mathcal{L}(\mathbb{H})}\|\mathrm{B}\|_{\mathcal{L}}\|u(s)\|_{\mathbb{U}}ds\\
&\qquad+\int_0^{s_1}\|\mathrm{U}(s_2,s)-\mathrm{U}(s_1,s)\|_{\mathcal{L}(\mathbb{H})}\left\|\left(f\left(s,x(s)\right)\right.\right.\\
&\qquad\qquad\qquad\left.\left.+\int_0^s q(s-\tau)\xi(\tau,x(\tau))d\tau\right)\right\|ds\\
&\quad\le MM_{\mathrm{B}}\|u(t)\|_{L^2(J;\mathbb{U})}(s_2-s_1)^{\frac{1}{2}}\\
&\qquad+M_{\mathrm{B}}\|u(t)\|_{L^2(J;\mathbb{U})}\int_0^{s_1}\|\mathrm{U}(s_2,s)-\mathrm{U}(s_1,s)\|_{\mathcal{L}(\mathbb{H})}ds\\
&\qquad+M\left(C_f+q^*C_\xi\right)(s_2-s_1)\\
&\qquad+\left(C_f+q^*C_\xi\right)\int_0^{s_1}\|\mathrm{U}(s_2,s)-\mathrm{U}(s_1,s)\|_{\mathcal{L}(\mathbb{H})}ds. \qquad (3.4)
\end{aligned}$$

The right hand side of the inequality ((3.4)) converges to zero uniformly for $x \in \mathcal{B}_{r_0}$ as $|s_2 - s_1| \to 0$, since the operator $\mathrm{U}(t,s)$ is continuous in operator topology for $t \geq 0$. For $t_k < t \leq t_{k+1}$, $k \geq 1$, we can show the equicontinuity of F_2 for any $x \in \mathcal{B}_r$in the same way as above. Therefore, the image of $\mathcal{B}_{r_0}$ under F_2 is equicontinuous. This suggests that $F_2\left(B_{r_0}\right)$ is equicontinuous.As a result, by applying the extended version of the Arzelà-Ascoli theorem, we conclude that, $F_2\left(B_{r_0}\right)$ is relatively compact set. Hence, by Lemma2.7, the operator $F_1 + F_2$ possesses at least one fixed point $x \in \mathcal{B}_{r_0}$, which coincides with the mild solution of system (1.2). □

Remark 3.2. We can also show the uniqueness of the mild solution by using the contraction mapping principle with the constant $k = \max\{k_1, k_2\} < 1$, where k_1 and k_2 are defined as

$$k_1 = Mb\left(L_f + q^* L_\xi\right), \quad k_2 = \left(M^2 Nb + Mb\right)\left(L_f + q^* L_\xi\right).$$

.

Our next target is to prove the approximate controllability of semilinear system(1.2).

Theorem 3.3. *Let the assumptions (R1)-(R4), (A1)-(A3) and the conditions of theorem 3.1 are true. Then, the system* (1.2) *is approximately controllable.*

Proof. From theorem3.1, we know that for every $\lambda > 0$ and $h \in \mathbb{H}$, there exists a mild solution $x_\lambda \in \mathcal{PC}([0,b], \mathbb{H})$ such that

$$x_\lambda(t) = \begin{cases} \mathrm{U}(t,0)x(0) + \int_0^t \mathrm{U}(t,s)\big[\mathrm{B}u(s) + f(s, x_\lambda(s)) \\ \qquad\qquad + \int_0^s q(s-\tau)\xi(\tau, x_\lambda(\tau))d\tau\big]ds, \quad 0 \leq t \leq t_1 \\ \mathrm{U}\left(t, t_k\right) x\left(t_k^+\right) + \int_{t_k}^t \mathrm{U}(t,s)\big[\mathrm{B}u(s) + f(s, x_\lambda(s)) \\ \qquad\qquad + \int_0^s q(s-\tau)\xi(\tau, x_\lambda(\tau))d\tau\big]ds, t_k < t \leq t_{k+1}, k = 1, \ldots, m, \end{cases} \tag{3.5}$$

where

$$\begin{aligned} x\left(t_k^+\right) = &\prod_{j=k}^{1}\left(\mathrm{I} + \mathrm{D}_j\right)\mathrm{U}\left(t_j, t_{j-1}\right)x_0 + \sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(\mathrm{I} + \mathrm{D}_j\right)\mathrm{U}\left(t_j, t_{j-1}\right)\left(\mathrm{I} + \mathrm{D}_i\right) \\ &\times \int_{t_{i-1}}^{t_i}\mathrm{U}\left(t_i, s\right)\left[\mathrm{B}u(s) + f(s, x_\lambda(s)) + \int_0^s q(s-\tau)\xi(\tau, x_\lambda(\tau))d\tau\right]ds \\ &+ \sum_{i=2}^{k}\prod_{j=k}^{i}\left(\mathrm{I} + \mathrm{D}_j\right)\mathrm{U}\left(t_j, t_{j-1}\right)\mathrm{E}_{i-1}v_{i-1} + \mathrm{E}_k v_k. \end{aligned}$$

The control $u(s)$ is defined as

$$u(s) = \Bigg(\sum_{k=1}^{m} \mathrm{B}^* \mathrm{U}^*(t_k, s) \prod_{i=k+1}^{m} \mathrm{U}(t_i, t_{i-1})^* \mathrm{U}(b, t_m)^* \chi(t_{k-1}, t_k) + \mathrm{B}^* \mathrm{U}(b, s)^* \chi(t_m, b) \Bigg) \widehat{\varphi}_\lambda, \tag{3.6}$$

$$v_m = \mathrm{E}_m^* \mathrm{U}(b, t_m)^* \widehat{\varphi}_\lambda, \quad v_k = \mathrm{E}_k^* \prod_{i=k}^{m} \mathrm{U}(t_i, t_{i-1})^* (\mathrm{I} + \mathrm{D}_i^*) \mathrm{U}(b, t_m)^* \widehat{\varphi}_\lambda,$$

$$\text{with} \quad \widehat{\varphi}_\lambda = \Big(\lambda \mathrm{I} + \Theta_0^{t_m} + \Gamma_{t_m}^b + \widetilde{\Theta}_0^{t_m} + \widetilde{\Gamma}_{t_m}^b \Big)^{-1} g(x_\lambda(.)),$$

$$\begin{aligned} \text{and} \quad g(x_\lambda(.)) = \Bigg(& h - \mathrm{U}(b, t_m) \prod_{j=m}^{1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1}) x_0 \\ & - \int_{t_m}^{b} \mathrm{U}(b, s) \big[f(s, x_\lambda(s)) + \int_0^s q(s-\tau) \xi(\tau, x_\lambda(\tau)) d\tau \big] ds \\ & - \mathrm{U}(b, t_m) \sum_{i=1}^{m} \prod_{j=m}^{i+1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1}) (\mathrm{I} + \mathrm{D}_i) \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s) \\ & \big[f(s, x_\lambda(s)) + \int_0^s q(s-\tau) \xi(\tau, x_\lambda(\tau)) d\tau \big] ds \Bigg). \end{aligned}$$

∎

Using (3.5) and (3.6) we can easily obtain that

$$x_\lambda(b) - h = \lambda \widehat{\varphi}_\lambda = \lambda \Big(\lambda \mathrm{I} + \Theta_0^{t_m} + \Gamma_{t_m}^b + \widetilde{\Theta}_0^{t_m} + \widetilde{\Gamma}_{t_m}^b \Big)^{-1} g(x_\lambda(.)).$$

Now, by using assumptions (A2) , we find

$$\int_0^b \| f(s, x_\lambda(s)) \|_{\mathbb{H}}^2 \, ds \le C_f^2 b, \text{ and}$$

and the boundedness of the sequence $\{f(., x_\lambda(.)) : \lambda > 0\}$ in $\mathrm{Ł}^2([0, b]; \mathbb{H})$. Then there is a subsequence still denoted by $\{f(., x_\lambda(.))\}$ that weakly converges to, say $f(.)$ in $\mathrm{Ł}^2([0, b]; \mathbb{H})$. Similarly by using (A3), we obtain the weak convergence of $\{\xi(., x_\lambda(.))\}$ that weakly converges to, say $\xi(.)$ in $\mathrm{Ł}^2([0, b]; \mathbb{H})$. Then by Corollary 3.3 (chapter 3) [27], we obtain

$$\begin{aligned} \| g(x_\lambda(.)) - \omega \| \le \Bigg\| & \int_{t_m}^{b} \mathrm{U}(b, s) \big[\left(f(s, x_\lambda(s)) - f(s) \right) \\ & + \int_0^s q(s-\tau) \left(\xi(\tau, x_\lambda(\tau)) - \xi(\tau) \right) d\tau \big] ds \\ & - \mathrm{U}(b, t_m) \sum_{i=1}^{m} \prod_{j=m}^{i+1} (\mathrm{I} + \mathrm{D}_j) \mathrm{U}(t_j, t_{j-1}) (\mathrm{I} + \mathrm{D}_i) \int_{t_{i-1}}^{t_i} \mathrm{U}(t_i, s) \\ & \times \big[\left(f(s, x_\lambda(s)) - f(s) \right) \end{aligned} \tag{3.7}$$

$$+\int_0^s q(s-\tau)\left(\xi(\tau,x_\lambda(\tau))-\xi(\tau)\right)d\tau\Big]ds\Bigg\|$$
$$\to 0, \tag{3.8}$$

where $$\omega = h - \mathrm{U}\left(b,t_m\right)\prod_{j=m}^{1}(\mathrm{I}+\mathrm{D}_j)\mathrm{U}\left(t_j,t_{j-1}\right)x_0$$
$$-\int_{t_m}^{b}\mathrm{U}(b,s)\left[f(s)-\int_0^s q(s-\tau)\xi(\tau)d\tau\right]ds$$
$$-\mathrm{U}\left(b,t_m\right)\sum_{i=1}^{m}\prod_{j=m}^{i+1}\left(\mathrm{I}+\mathrm{D}_j\right)\mathrm{U}\left(t_j,t_{j-1}\right)\left(\mathrm{I}+\mathrm{D}_i\right)$$
$$\times\int_{t_{i-1}}^{t_i}\mathrm{U}\left(t_i,s\right)\left[f(s)-\int_0^s q(s-\tau)\xi(\tau)d\tau\right]ds,$$

as $\lambda \to 0^+$. The first term in the right hand side of the expression 3.7 goes to zero because of the compactness of the operator $(\mathrm{Q}f)(.) = \int_0^b \mathrm{U}(.,s)f(s)ds : \mathrm{Ł}^2([0,b];\mathbb{H}) \to \mathcal{PC}([0,b],\mathbb{H})$(see Lemma 4.1 and theorem 4.2 in [28]) and the second term tends to zero by using the compactness of the operator $\mathrm{U}(t,s)$, for $t \geq 0$. Finally we compute $\|x_\lambda(b)-h\|_{\mathbb{H}}$ as

$$\|x_\lambda(b)-h\| = \left\|\lambda\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b\right)^{-1}g(x_\lambda(.))\right\|,$$
$$\leq \left\|\lambda\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b\right)^{-1}\omega\right\|$$
$$+\left\|\lambda\left(\lambda\mathrm{I}+\Theta_0^{t_m}+\Gamma_{t_m}^b+\widetilde{\Theta}_0^{t_m}+\widetilde{\Gamma}_{t_m}^b\right)^{-1}\left(g\left(x_\lambda(.)\right)-\omega\right)\right\|.$$

By estimate (3.7) and assumption (A1), we obtain

$$\|x_\lambda(b)-h\|_{\mathbb{H}} \to 0 \text{ as } \lambda \to 0^+.$$

which guarantee that the system (1.2) is approximately controllable in $\mathbb{H}$. □

4. Application

We consider the following impulsive semilinear functional heat problem on $\mathbb{H} = \mathbb{U} = L^2([0,\pi];\mathbb{R})$:

$$\begin{cases} \frac{\partial}{\partial t}z(t,\zeta) = a(t)\frac{\partial^2}{\partial \zeta^2}z(t,\zeta) + \mu(t,\zeta) + \frac{e^{-t}z\left(t,\zeta\right)}{(9+e^t)(1+z(t,\zeta))} \\ \qquad\qquad + \int_0^t e^{t-s}\frac{e^s z\left(s,\zeta\right)}{5+z\left(s,\zeta\right)}ds, \quad \zeta\in[0,\pi], \;\; t\in[0,1], \;\; t\neq\left\{\frac{1}{2}\right\}, \\ z(t,0) = 0 = z(t,\pi), \quad t\in[0,1], \\ \Delta z\left(\frac{1}{2},\zeta\right) = \mathrm{D}_1 z\left(\frac{1}{2},\zeta\right) + \mathrm{E}v_1, \\ \Delta z\left(1,\zeta\right) = \mathrm{D}_2 z\left(1,\zeta\right) + \mathrm{E}_2 v_2, \\ z\left(0,\zeta\right) = \phi(\zeta). \end{cases} \tag{4.1}$$

Where $a:[0,1]\mapsto\mathbb{R}^+$, is Holder continuous function of order $0< \ \leq 1$, that is there exists a positive constant C_a such that

$$|a(t)-a(s)|\leq C_a|t-s|, \text{for all} t,s\in[0,1].$$

For $\mathbb{H}=L^2\left([0,\pi];\mathbb{R}\right)$, the operator $\mathrm{A}(t)g(\zeta)=a(t)g''(\zeta)$, with the domain $\mathcal{D}(\mathrm{A}(t)) = \mathcal{D}(\mathrm{A}) = \mathrm{H}^2\left([0,\pi];\mathbb{R}\right)\cap \mathrm{W}_0^{1,2}([0,\pi];\mathbb{R})$. We define the operator $\mathrm{A}(t)$ as $\mathrm{A}g(\zeta)= g''$, $\zeta\in[0,\pi]$, with the domain $\mathcal{D}(\mathrm{A})$. Moreover, for $t\in[0,1]$ and $g\in\mathcal{D}(\mathrm{A})$, the operator $\mathrm{A}(t)$ can be expressed as

$$\mathrm{A}(t)g=\sum_{n=1}^{\infty}(-n^2a(t))\langle g,w_n\rangle w_n, \quad g\in\mathcal{D}(\mathrm{A}), \text{ for } \quad \langle g,w_n\rangle=\int_0^{\pi}g(\zeta)w_n 9\zeta d\zeta,$$

where, $-n^2(n\in\mathbb{N})$ and $w_n(\zeta)=\sqrt{\frac{2}{\pi}}\sin(n\zeta)$, are the eigenvalues and the corresponding normalizes eigenfunctions of the operator A respectively. The operator $\mathrm{A}(t)$ satisfies all the conditions (R1)-(R4) (see application section of [28]). Then by applying Lemma 2.3, we obtain the existence of a unique evolution system $\{\mathrm{U}(t,s):0\leq s\leq t\leq 1\}$. From Lemma 2.4, we observe that the evolution system $\{\mathrm{U}(t,s):0\leq s\leq t\leq 1\}$ is compact for $t-s>0$. The evolution system $\mathrm{U}(t,s)$ can be explicitly as

$$\mathrm{U}(t,s)g=\sum_{n=1}^{\infty}e^{-n^2\int_s^t a(\tau)d\tau}\langle g,w_n\rangle w_n, \quad \text{for each} \quad g\in\mathbb{H}.$$

We also have

$$\mathrm{U}(t,s)^*g^*=\sum_{n=1}^{\infty}e^{-n^2\int_s^t a(\tau)d\tau}\langle g^*,w_n\rangle w_n, \quad \text{for each} \quad g^*\in\mathbb{H}.$$

Next, we define operator $\mathrm{B}:L^2\left([0,\pi];\mathbb{R}\right)\to\mathbb{H}$ such that

$$\mathrm{B}\left(u(t)\right)(\zeta)=u(t)\left(\zeta\right)=\mu\left(t,\zeta\right), \quad t\in[0,1], \quad \zeta\in[0,\pi].$$

We can see, the operator B defined as above is a linear bounded operator. We also define $\mathrm{D}_k=\mathrm{E}_k=\mathrm{I}$, for $k=1,2$.

Let the function $x:J\to\mathbb{H}$ be given by

$$x(t)\left(\zeta\right)=z\left(t,\zeta\right), \quad \zeta\in[0,\pi].$$

The nonlinear functions $f, \xi : [0,1] \times D \to \mathbb{H}$ is defined as

$$f\left(t, x(t)\right)(\zeta) = \frac{e^{-t} z\left(t, \zeta\right)}{(9+e^{t})(1+z(t,\zeta))} \text{ and } \xi\left(t, x(t)\right)(\zeta) = \frac{e^{t} z\left(t, \zeta\right)}{5 + z\left(t, \zeta\right)}, \quad \zeta \in [0, \pi].$$

We can check that for f and ξ, assumptions (A2) and (A3) are satisfied with $L_f = \frac{1}{10}$, $L_\xi = \frac{e}{25}$, $C_f = \frac{1}{10}$, $C_\xi = \frac{e}{5}$. We take $v_1 = \sin(\pi t)$, $v_2 = \cos(\pi t)$ and $q^* = e - 1$. By the above settings we can transform system(4.1) in the abstract form as system (1.2).

Since all the conditions are satisfied therefore, there exists a mild solution the system (4.1) and is approximately controllable.

5. Conclusion

In this study, we have investigated the existence and controllability of a class of non-autonomous impulsive integro-differential systems in a Hilbert space. Initially, we established the existence of mild solutions using Krasnoselskii's fixed point theorem. Furthermore, we proved the approximate controllability of the system and provided a detailed example to illustrate the theoretical results. This research enhances the understanding of control methods for impulsive nonlinear systems and can be extended to second-order systems in Banach space. The study of finite approximate controllability for second-order and stochastic systems remains an open problem and will be explored in future work.

6. Acknowledgment

We sincerely thank the anonymous reviewers for their insightful comments and suggestions, which significantly enhanced the quality of this paper.

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