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A DISCRETE MATHEMATICAL MODELLING AND OPTIMAL CONTROL OF MIGRATION DYNAMICS AMONG THE POLITICAL PARTIES IN MOROCCO

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Abstract. One of the phenomena that characterize any political process in any democratic country is the switch of political orientation for the politicians. This paper aims to study the dynamic of the change in political affiliation via a difference-equations model. The model deals with three types of population: The susceptible to switch their political party. The influencers are members of the political party. These are people working for the party to attract people to join the political party. Finally, the followers of the political party who stick to their support their political party in the elections. Our goal is to find the optimal control strategy that helps to reduce the switching of the political party. The characterization of the sought optimal control is derived based on Pontryagin's maximum principle. Finally, numerical examples are given to illustrate the obtained results.

1. Introduction

Party Switching is one phenomenon that been seen in many democratise in the world [11, 19, 20], yet little studies had focused on it [10]. Switching the political orientation means changing the political party that person supports. This phenomenon represents a topic of debate in the population as it affects the country's political image and the integrity of the electoral process. Such behavior is viewed as contempt, particularly since the switchers benefited from the recommendation of the political parties and received the privilege of representing the citizens of one of the governing branches of the country.

These switchings happened even after the election, which affected the formation of the political alliances and dramatically influenced the formation of teams at the parliament level. As a result, the political parties were in great embarrassment because switching from one group to another without clear criteria, which perpetuated a kind of instability and ambiguity in the work of this constitutional institution.

Many studies had investigated mathematical models of the legislative elections see [1], [5], [6], [12], [13] and [21]. Balatif in [4], discussed the problem of an electoral behavior model. The model described citizens' electoral behavior with regards to the electoral process in general and with regards to a political party in particular. It included three controls representing strategies that can help to increase the participation rate at elections and boost a political party's chances of getting more votes. In [2, 3] the author proposed a mathematical model that investigated the dynamics of registered voters and the negative impacts of abstainers on the potential electors. However, there not study that tried to mathematically model the party switchers and to find the best approach to limit the switching.

This paper aims to investigate a mathematical model that defines the political party's interconnection and the switching of individuals from one political party to another. The goal is to find an optimal control strategy that stop the switching. The first control strategy is by having legislation that limits this phenomenon. A political party can submit an offer as soon as a member changes the party to freeze his membership and tasks all the responsibility. The second control is by having influencers that mobilize other members to keep their party. Because of their conviction in their party, they cannot change their political affiliation. Our optimal control problem is formulated to be subject to an optimization criterion represented by the minimization of an objective function.

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In recent years, many attempts have been made to develop the control strategy for different systems [7], [15] and [17]. There are a number of different methods for calculating the optimal control for a specific mathematical model. For example, Pontryagin's maximum principle allows the calculation of the optimal control for an ordinary equation model system with a given constraint. The optimality system is solved based on an [14] and [22] iterative discrete schema that converges following an appropriate test similar to the one related to the Forward-Backward Sweep Method (FBSM).

The paper is organized as follows. In Section 2, the discrete-time model of the dynamics of an interconnected political party is described. In Section 3, we give some results concerning the existence of the optimal control. We use Pontryagin's maximum principle to analyze control strategies and determine the necessary condition for optimal control. Numerical simulations are given in Section 4. Finally, we conclude the paper in Section 5.

2. The description of the Model SIA

We consider a discrete-time model model SIA of the dynamic of political parties. We assume that there are M political parties in a country. Let N_i^k be the total population of the political parties k in time i.

Our model classifies the political parties, as a population in society, into three compartments: S_i^p represents the number of susceptible to leave the political party, I_i^p is the number of the influencers with the political party and who work to attract more people to the political party, and A_i^p is the number of non-influencers with the political party. The following system describes a difference equation model of the dynamics of the political parties.

$$S_{i+1}^p = S_i^p + \delta_p A_i^p - \sum_{k\in\{0,1,\dots,p\}} \alpha_{kp} \frac{S_i^p I_i^k}{N_i^p + N_i^k} - \mu_p S_i^p \tag{2.1}$$

$$I_{i+1}^p = I_i^p + \sum_{k\in\{0,1,\dots,p\}} \beta_{pk} \frac{S_i^k I_i^p}{N_i^p + N_i^k} + \nu_p A_i^p - (\mu_p + \sigma_p) I_i^p \tag{2.2}$$

$$A_{i+1}^p = \Lambda_p + A_i^p + \sigma_p I_i^p - (\nu_p + \delta_p + \mu_p) A_i^p + \sum_{k\in\{0,1,\dots,p\}} \gamma_{pk} \frac{S_i^k I_i^p}{N_i^p + N_i^k} \tag{2.3}$$

where S_0^p, A_0^p and I_0^p are the given initial state in the political party p. In equations (2.1)-(2.3), all parameters are non-negative and defined in Table 1. Following the contact between the spreader of party p and the susceptible to leave party k with α_{pk} rate, a part of the contact $\alpha_{pk} I_i^p S_i^k$ will be added to the number of individuals with the political party p with a number of $\gamma_{pk} I_i^p S_i^k$ and the other part will join the spreader of p with a $\beta_{pk} I_i^p S_i^k$ number with condition that $\alpha_{pk} = \beta_{pk} + \gamma_{pk}$.

In model (2.1)-(2.3) it has been assumed that individuals who have political party p and who desires the spread it, can not directly be susceptible to leave the political party, they must imperatively pass through the compartment of individuals endowed with political party and who do not wish to spread it, before being susceptible to leave it. And it has been supposed that natalities in political party p are directly added to the number of individuals who are endowed with political party p and who do not wish to spread it.

$N_i^p = S_i^p + I_i^p + A_i^p$ is the population size corresponding to political party p at time i. It is clear that the population size remains constant if $\mu_p = \frac{\Lambda_p}{N_i^p}$, in fact

$$\begin{aligned}
\sum_{j=1}^{p} N_{i+1}^j &= \sum_{j=1}^{p} (S_{i+1}^j + I_{i+1}^j + A_{i+1}^j) \\
&= \sum_{j=1}^{p} (S_i^j + I_i^j + A_i^p) + \sum_{j=1}^{p} \Lambda_j - \sum_{j=1}^{p} \mu_j \left(S_i^j + I_i^j + A_i^j\right) \\
&= \sum_{j=1}^{p} N_i^j + \sum_{j=1}^{p} \Lambda_j - \sum_{j=1}^{p} \mu_j N_i^j = \sum_{j=1}^{p} N_i^j.
\end{aligned}$$

TABLE 1. The description of parameters used for the definition of discrete time systems (2.1)-(2.3).

Parameter	Description
δ_p	Rate of individuals who wish to leave the political party p.
α_{kp}	Contact rate of a spreader of k with a susceptible to leave the political party p.
μ_p	Natural mortality rate in the political party k
β_{pk}	Rate of individuals who join the spreader of party p after leaving the political party k.
ν_p	Rate of individuals with the political party p who wants to spread it.
σ_p	Rate of individuals with the political party p who no longer want to spread it.
Λ_p	Birth rate in party p.
γ_{pk}	Rate of individuals who join the political party p after leaving the political party k.

3. An optimal control problem

Optimal control approach have been applied to models (2.1)-(2.3) to reduce the number of susceptible to leave the political party S_i^p and increase the number of individuals A_i^p, I_i^p endowed with target political party p along the control strategy period. For this we introduce a control variable u_i which represents the awareness to not leave the political party and the control (v_i^p) to show the effectiveness rate of the travel approach which boost contacts between the influencers with the political party and who wishes to spread the targeted the political party p and susceptible people to leave the political party k. Thus, the mathematical model with controls is described based on the following differential system

$$S_{i+1}^p = S_i^p + \delta_p A_i^p - \sum_{k\in\{0,1,\dots,p\}} \alpha_{kp} \frac{S_i^p I_i^k}{N_i^p} - \alpha_{pp} \frac{S_i^p I_i^p}{2N_i^p} v_i^p - \mu_p S_i^p - u_i S_i^p \tag{3.1}$$

$$I_{i+1}^p = I_i^p + \sum_{k\in\{0,1,\dots,p\}} \beta_{pk} \frac{S_i^k I_i^p}{N_i^p + N_i^p}(1+v_i^p) + \nu_p A_i^p - (\mu_p + \sigma_p) I_i^p \tag{3.2}$$

$$A_{i+1}^p = \Lambda_p + A_i^p + \sigma_p I_i^p - (\nu_p + \delta_p + \mu_p) A_i^p + \sum_{k\in\{0,1,\dots,p\}} \gamma_{pk} \frac{S_i^k I_i^p}{N_i^p + N_i^k}(1+v_i^k) + u_i S_i^p. \tag{3.3}$$

We are interested in controlling the population of the political party p. Then, the problem is to minimize the objective functional given by

$$J_p(u,v) = \sum_{i=1}^{N} (\psi_1 S_i^p - \psi_2 A_i^p) + \sum_{i=1}^{N-1} \left(\frac{\tau_1}{2}(u_i)^2 + \sum_{k\in\{0,1,2,\dots,p\}} \frac{\tau_2}{2}(v_i^k)^2 \right) \tag{3.4}$$

subject to system (3.1)-(3.3). Here ψ_1 and ψ_2 are positive constants to keep a balance in the size of S_i^p and A_i^p respectively. In the objective functional, τ_1 and τ_2 are the positive weight parameters which are associated with the controls u_i and (v_i^p).
Our goal is to minimize the susceptible group, minimize the systemic costs attempting to increase the number of individuals who have the political party and the spreader individuals in p. In other words, we are seeking an optimal control u_i^* and (v_i^{p*}) such that

$$J(u_i^*, (v_i^{p*})) = \min\{J(u,v) \mid (u,v) \in \mathcal{U}_{ad}\}, \tag{3.5}$$

where $\mathcal{U}_{ad}$ is the set of admissible controls defined by

$$\begin{aligned} \mathcal{U}_{ad} = \{ & (u,v) \mid u = (u_i)_i,\ v = ((v_i^{p*}))_i,\ u^{min} \le u_i \le u^{max}, \\ & v^{min} \le v_i^p \le v^{max},\ \ i \in \{0.\dots, N-1\} \}, \end{aligned}$$

where $(u^{min}, u^{max}) \in]0,1[^2$ and $(v^{min}, v^{max}) \in]0, M[^2$ with $M \ge 1$.

The sufficient condition for existence of an optimal control for the problem follows from the following theorem

Theorem 3.1. *(Sufficient conditions)*
For the optimal control problem given by (3.5) along with the state equations (3.1)-(3.3), there exists a controls $(u_i{}^*, (v_i^{p*})) \in \mathcal{U}_{ad}$ *such that*

$$J_p(u_i{}^*, (v_i^{p*})) = min\{J(u,v)/(u,v) \in \mathcal{U}_{ad}\}.$$

Proof. See Dabbs, K [[8], Theorem 1]. □

At the same time by using Pontryagin's Maximum Principle [14], we derive necessary conditions for our optimal control. For this purpose we define the Hamiltonian as

$$\begin{aligned}
\mathcal{H}(\Omega) &= \left(\psi_1 S_i^p - \psi_2 A_i^p + \frac{\tau_1}{2}(u_i)^2 + \sum_{k\in\{0,1,\ldots,p\}} \frac{\tau_2}{2}(v_i^p)^2\right) \\
&+ \zeta_{1,i+1}\left[S_i^p + \delta_p A_i^p - \sum_{k\in\{0,1,\ldots,p\}} \alpha_{kp}\frac{S_i^p I_i^p}{N_i^p + N_i^p} - \mu_p S_i^p - u_i S_i^p \right. \\
&\left. -\alpha_{pp}\frac{S_i^p I_i^p}{2N_i^p} v_i^p\right] \\
&+ \zeta_{2,i+1}\left[I_i^p + \sum_{k\in\{0,1,\ldots,p\}} \beta_{pk}\frac{S_i^p I_i^p}{N_i^p + N_i^p}\left(1 + v_i^p\right)\right. \\
&\left. + \nu_p A_i^p - (\mu_p + \sigma_p) I_i^p\right] \\
&+ \zeta_{3,i+1}\left[\Lambda_p + A_i^p + \sigma_p I_i^p - (\nu_p + \delta_p + \mu_p) A_i^p\right. \\
&\left. + \sum_{k\in\{0,1,\ldots,p\}} \gamma_{pk}\frac{S_i^p I_i^p}{N_i^p + N_i^p}\left(1 + v_i^p\right) + u_i S_i^p\right].
\end{aligned}$$

Theorem 3.2. *(Necessary Conditions)*
Given an optimal controls $(u_i^*, (v_i^{p*}))$ *and solutions* S^{p*}, I^{p*} *and* A^{p*}*, there exists* $\zeta_{k,i}$, $i = 1...N$, $k = 1,2,3$*, the adjoint variables satisfying the following equations*

$$\begin{aligned}
\zeta_{1,i} &= \psi_1 + \zeta_{1,i+1}\left(2 - \sum_{k\in\{0,1,\ldots,p\}} \alpha_{kp}\frac{I_i^k}{N_i^p + N_i^k} - \mu_p - u_i - \alpha_{pp}\frac{I_i^p}{2N_i^p} v_i^p\right) + \\
&\quad \zeta_{2,i+1}\beta_{pp}\frac{I_i^p}{2N_i^p}(1 + v_i^p) + \zeta_{3,i+1}\left(\gamma_{pp}\frac{I_i^p}{2N_i^p}(1 + v_i^p) + u_i\right) \\
\zeta_{2,i} &= -\zeta_{1,i+1}\alpha_{pp}\frac{S_i^p}{2N_i^p}(1 + v_i^p) + \zeta_{2,i+1}\left(2 + \sum_{k\in\{0,1,\ldots,p\}} \beta_{pk}\frac{S_i^k}{N_i^p + N_i^k}(1+\right. \\
&\quad \left. v_i^k) - (\mu_p + \sigma_p)\right) + \zeta_{3,i+1}\left(\sigma_p + \sum_{k\in\{0,1,\ldots,p\}} \gamma_{pk}\frac{S_i^k}{N_i^p + N_i^k}(1 + v_i^k)\right) \\
\zeta_{3,i} &= -\psi_2 + \zeta_{1,i+1}\delta_p + \zeta_{2,i+1}\nu_p + \zeta_{3,i+1}(2 - \nu_p - \delta_p - \mu_p)
\end{aligned}$$

with transversality conditions $\zeta_{1,N} = -\psi_1$, $\zeta_{2,N} = 0$ *and* $\zeta_{3,N} = \psi_2$.
Furthermore, the optimal control $(u_i{}^*, v_i^{k*})$ *is given for* $i = 1, ..., n$ *by*

$$u_i^* = min\{max\{u^{min}, \frac{(\zeta_{1,i+1} - \zeta_{3,i+1})S_i^p}{\tau_1}\}, u^{max}\},$$

$$v_i^{p^*} = \quad min\{max\{v^{min}, \frac{S_i^k I_i^p}{2\tau_2 N_i^p}(\alpha_{pp}\zeta_{1,i+1} - \beta_{pp}\zeta_{2,i+1} - \gamma_{pp}\zeta_{3,i+1})\}, v^{max}\},$$

and

$$\begin{aligned} v_i^{k^*} &= min\{max\{v^{min}, -\frac{S_i^k I_i^p}{\tau_2(N_i^p + N_i^k)}(\beta_{pk}\zeta_{2,i+1} + \gamma_{pk}\zeta_{3,i+1})\}, v^{max}\} \\ & k \neq p. \end{aligned}$$

Proof. Using Pontryagin's Maximum Principle [14], and setting $S_i^p = S_i^{p^*}$, $I_i^p = I_i^{p^*}$, $A_i^p = A_i^{p^*}$ and $u_i = u_i^*, v_i^k = v_i^{k^*}$ we obtain the following adjoint equations

$$\begin{aligned} \Delta\zeta_{1,i} &= -\frac{\partial\mathcal{H}}{\partial S_i^p} \\ &= -\left[\psi_1 + \zeta_{1,i+1}\left(1 - \sum_{k\in\{0,1,\ldots,p\}} \alpha_{kp}\frac{I_i^k}{N_i^p+N_i^k} - \mu_p - u_i - \alpha_{pp}\frac{I_i^p}{2N_i^p}v_i^p\right) + \right. \\ &\left. \zeta_{2,i+1}\beta_{pp}\frac{I_i^p}{2N_i^p}(1+v_i^p) + \zeta_{3,i+1}\left(\gamma_{pp}\frac{I_i^p}{2N_i^p}(1+v_i^p) + u_i\right)\right] \\ \Delta\zeta_{2,i} &= -\frac{\partial\mathcal{H}}{\partial I_i^p} \\ &= -\left[-\zeta_{1,i+1}\alpha_{pp}\frac{S_i^p}{2N_i^p}(1+v_i^p) + \zeta_{2,i+1}\left(1 + \sum_{k\in\{0,1,\ldots,p\}} \beta_{pk}\frac{S_i^k}{N_i^p+N_i^k}(1+ \right.\right. \\ &\left.\left. v_i^k) - (\mu_p+\sigma_p)\right) + \zeta_{3,i+1}\left(\sigma_p + \sum_{k\in\{0,1,\ldots,p\}} \gamma_{pk}\frac{S_i^k}{N_i^p+N_i^k}(1+v_i^k)\right)\right] \\ \Delta\zeta_{3,i}^p &= -\frac{\partial\mathcal{H}}{\partial A_i^p} \\ &= -[-\psi_2 + \zeta_{1,i+1}\delta_p + \zeta_{2,i+1}\nu_p + \zeta_{3,i+1}(2 - \nu_p - \delta_p - \mu_p)] \end{aligned}$$

then

$$\begin{aligned} \zeta_{1,i} &= \psi_1 + \zeta_{1,i+1}\left(2 - \sum_{k\in\{0,1,\ldots,p\}} \alpha_{kp}\frac{I_i^k}{N_i^p+N_i^k} - \mu_p - u_i - \alpha_{pp}\frac{I_i^p}{2N_i^p}v_i^p\right) + \\ & \zeta_{2,i+1}\beta_{pp}\frac{I_i^p}{2N_i^p}(1+v_i^p) + \zeta_{3,i+1}\left(\gamma_{pp}\frac{I_i^p}{2N_i^p}(1+v_i^p) + u_i\right) \\ \zeta_{2,i} &= -\zeta_{1,i+1}\alpha_{pp}\frac{S_i^p}{2N_i^p}(1+v_i^p) + \zeta_{2,i+1}\left(2 + \sum_{k\in\{0,1,\ldots,p\}} \beta_{pk}\frac{S_i^k}{N_i^p+N_i^k}(1+ \right. \\ &\left. v_i^k) - (\mu_p+\sigma_p)\right) + \zeta_{3,i+1}\left(\sigma_p + \sum_{k\in\{0,1,\ldots,p\}} \gamma_{pk}\frac{S_i^k}{N_i^p+N_i^p}(1+v_i^p)\right) \\ \zeta_{3,i} &= -\psi_2 + \zeta_{1,i+1}\delta_p + \zeta_{2,i+1}\nu_p + \zeta_{3,i+1}(2 - \nu_p - \delta_p - \mu_p). \end{aligned}$$

with transversality conditions $\zeta_{1,N} = -\psi_1$, $\zeta_{2,N} = 0$ and $\zeta_{3,N} = \psi_2$.
To obtain the optimality conditions we take the variation with respect to control u_i, v_i^k and set it equal to zero

$$\frac{\partial\mathcal{H}}{\partial u_i} = \tau_1 u_i - \zeta_{1,i+1}S_i^p + \zeta_{3,i+1}S_i^p = 0,$$

$$\frac{\partial\mathcal{H}}{\partial v_i^p} = \left(\tau_2 v_i^p - \zeta_{1,i+1}\alpha_{pp}\frac{S_i^p I_i^p}{2N_i^p} + \zeta_{2,i+1}\beta_{pp}\frac{S_i^p I_i^p}{2N_i^p} + \zeta_{3,i+1}\gamma_{pk}\frac{S_i^p I_i^p}{2N_i^p}\right) = 0$$

and for $k \neq p$

$$\frac{\partial\mathcal{H}}{\partial v_i^k} \quad = \quad \left(\tau_2 v_i^k - \zeta_{1,i+1}\alpha_{pp}\frac{S_i^k I_i^p}{N_i^p+N_i^k} + \zeta_{2,i+1}\beta_{pp}\frac{S_i^k I_i^p}{N_i^p+N_i^k} + \zeta_{3,i+1}\gamma_{pk}\frac{S_i^k I_i^p}{N_i^p+N_i^k}\right) = 0.$$

TABLE 2. The description of parameters used for the definition of discrete time systems (3.1)-(3.3).

	S_0	I_0	A_0	β_{jk}	α_{kj}	γ_{jk}	Λ_j	σ_j	ν_j	μ_j	δ_j
$p_1 = cell\,(5,5)$	80	120	400	0.001	0.01	0.011	7	0.02	0.02	0.009	0.03
$p_2 = cell\,(5,2)$	50	20	100	0.025	0.02	0.045	3	0.03	0.04	0.001	0.02
$p_3 = cell\,(2,2)$	150	50	50	0.02	0.01	0.03	4	0.02	0.04	0.001	0.02
$p_4 = cell\,(2,5)$	150	100	150	0.04	0.03	0.07	2	0.03	0.04	0.001	0.01

Then we obtain the optimal controls

$$u_i^* = \frac{(\zeta_{1,i+1} - \zeta_{3,i+1})S_i^p}{\tau_1}, \quad i = 1, ..., n$$

$$v_i^{p*} = \frac{S_i^k I_i^p}{2\tau_2 N_i^p}(\alpha_{pp}\zeta_{1,i+1} - \beta_{pp}\zeta_{2,i+1} - \gamma_{pp}\zeta_{3,i+1}), \quad i = 1, ..., n$$

and

$$v_i^{k*} = -\frac{S_i^k I_i^p}{\tau_2(N_i^p + N_i^k)}(\beta_{pk}\zeta_{2,i+1} + \gamma_{pk}\zeta_{3,i+1}), \quad i = 1, ..., n, \text{ and } k \neq p.$$

By the bounds in $\mathcal{U}_{ad}$ of the control, it is easy to obtain, $u_i^*, v_i^{k^*}$ for $i = 1, ..., n$ in the following form

$$u_i^* = min\{max\{u^{min}, \frac{(\zeta_{1,i+1} - \zeta_{3,i+1})S_i^p}{\tau_1}\}, u^{max}\},$$

$$v_i^{p*} = min\{max\{v^{min}, \frac{S_i^k I_i^p}{2\tau_2 N_i^p}(\alpha_{pp}\zeta_{1,i+1} - \beta_{pp}\zeta_{2,i+1} - \gamma_{pp}\zeta_{3,i+1})\}, v^{max}\},$$

and

$$v_i^{k^*} = min\{max\{v^{min}, -\frac{S_i^k I_i^p}{\tau_2(N_i^p + N_i^k)}(\beta_{pk}\zeta_{2,i+1} + \gamma_{pk}\zeta_{3,i+1})\}, v^{max}\} \quad k \neq p.$$

□

4. Numerical simulation

In this section, we carry out numerical simulations to demonstrate our theoretical results . We wrote a code in MATLAB based on an iterative discrete scheme that converges following an appropriate test similar the one related to the forward–backward sweep method (FBSM), and simulated our results using data cited in table 2. The optimality systems are solved using an iterative method. Where the state system with an initial guess is solved forward in time and then the adjoint system is solved backward in time because of the transversality conditions. Afterwards, we updated the optimal controls values using the values of state and costate variables obtained in the previous steps. Finally, we execute the previous steps till a tolerance criterion is reached. In order to show the importance of our work, and without loss of generality.

Considering the critical level of control, we give an optimal control sufficient to reduce the number of susceptible to leave the political party S_i^p and increase the number of individuals A_i^p , I_i^p endowed with target political party p.

In the following, we can see that the optimal control function has a very desirable effect upon the members of the political party and who wishes to spread it, while the members likely to leave the political party are decreasing for almost the entire duration of the process. The graphs below allow us to compare changes in the number of S^p, I^p and A^p before and after the introduction of control.

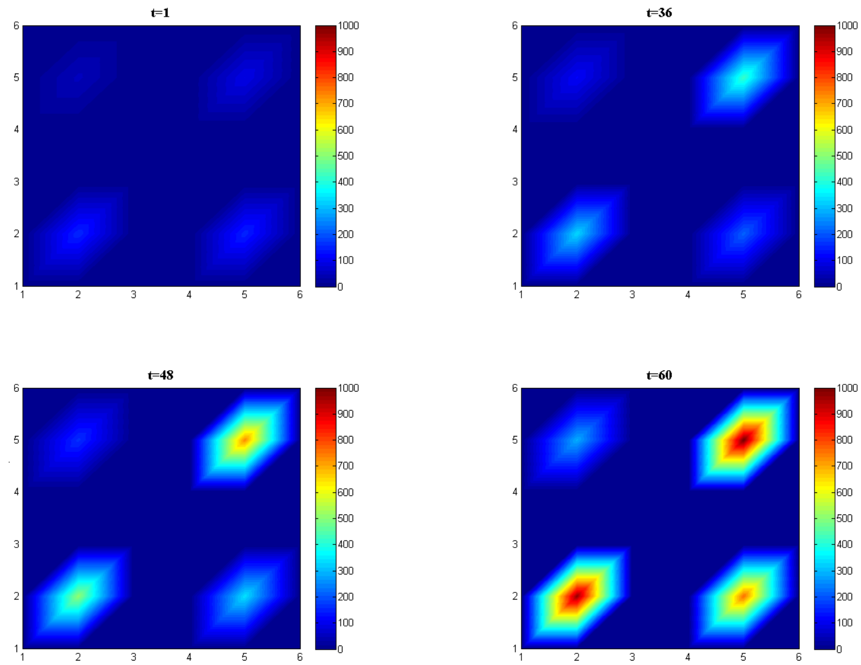

FIGURE 1. Time evolution of the individual members S^p without controls.

In Figures. 1, 2 and 3 we give the numerical results in absence of controls. For time t = 1 to t = 60 we can see that spreads in four the political parties characterized by different parameters, the number of susceptible that can change there party from $p_1 = 80$, $p_2 = 50$, $p_3 = 150$ and $p_4 = 150$ as initial conditions to 729, 163, 512 and 340, in the four political parties respectively.

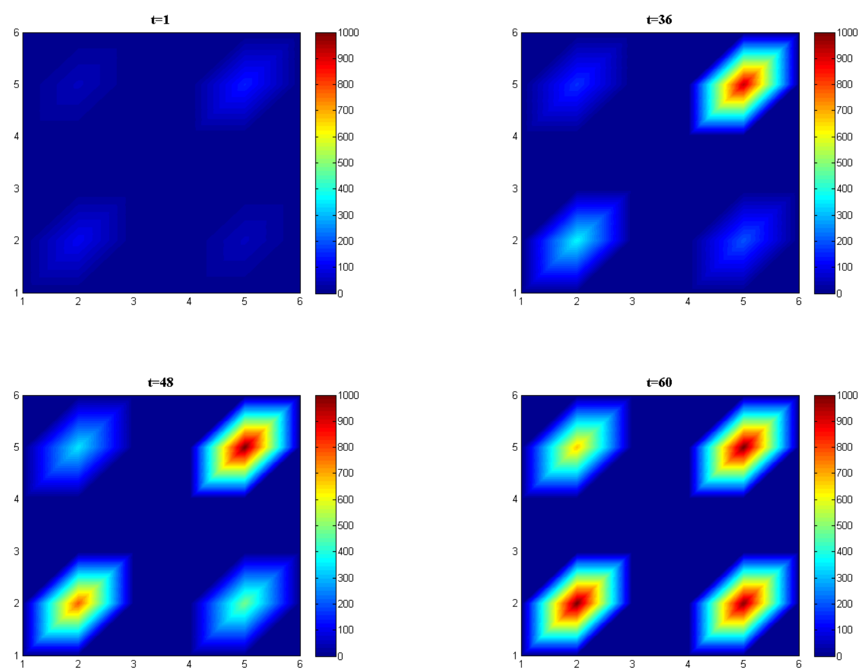

FIGURE 2. Time evolution of the non-influencers members I^p without controls.

Once a control is introduced in the system (4)–(6), particularly in the equations that describes dynamics of S^{p_1}, I^{p_1} and A^{p_1} functions associated to the first political party, we can deduce its effect on the number of susceptible members that can leave the party in Fig. 4, from 729 when there was yet no control strategy, to 181 when there is the control u^{p_1}. One of the major benefits of that control, is to increase the number of the influencers, and this can be observed in the of Fig. 6, where the number of the influencers political party the members becomes approximately equal to 975.

Figure 5 is added here, to show the advantage of the control v^k, in increasing the number of the influencers with political party in p_1 and who wishes to spread by hosting people coming from p_2, p_3 and p_4. In one hand, the number of those people rises from 286, 971 and 141 respectively when there were no controls yet in Fig 2, but in the other hand it decreases from 125, 182 and 13 in the three political parties respectively, when the control v^k is introduced. So we participate to an decrease of the I^{p_2}, I^{p_3} and I^{p_4} function that can obviously be proved by this control taking, and which lead to the effectiveness of the control strategy in the first political party with a rate that varies from Fig 2 to Fig 5, from a value equal to 358 towards a value equal to 691.

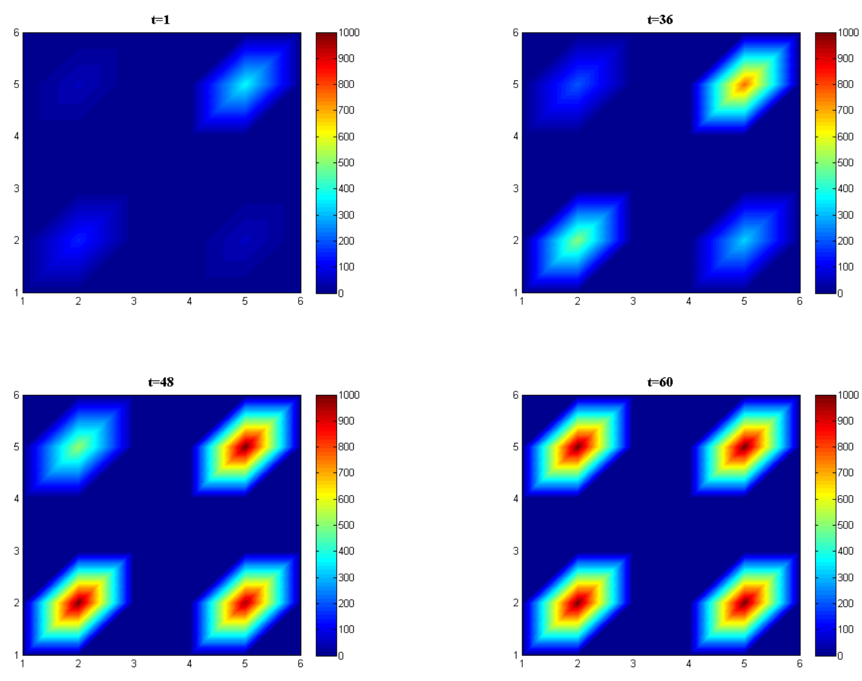

FIGURE 3. Time evolution of the individual members A^p without controls.

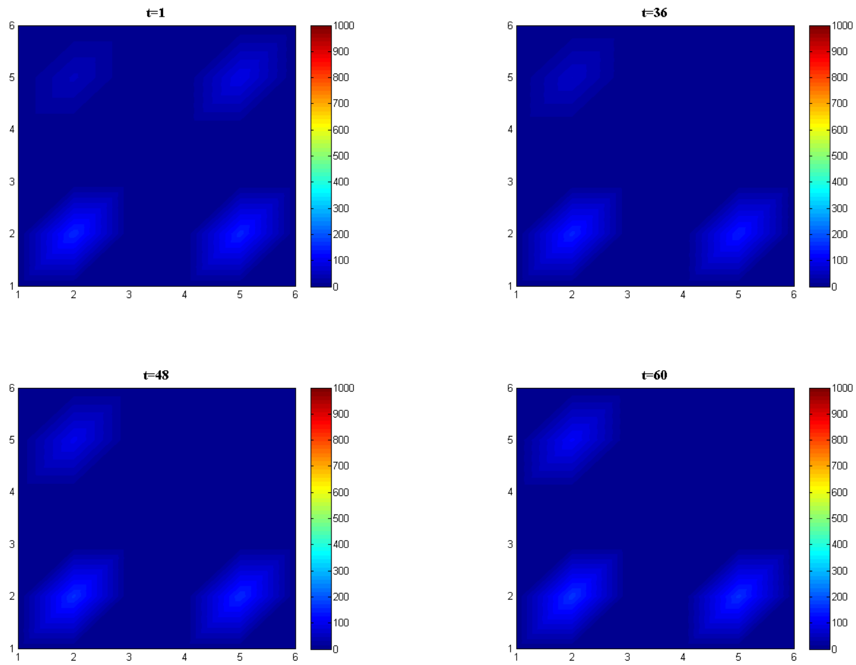

FIGURE 4. Time evolution of the individual members S^p with controls.

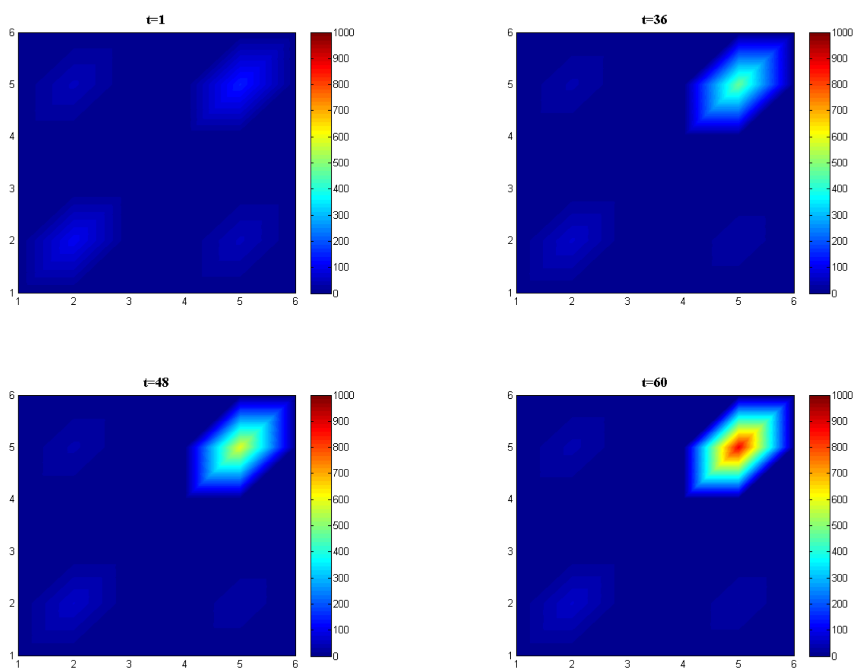

FIGURE 5. Time evolution of the individual members I^p with controls.

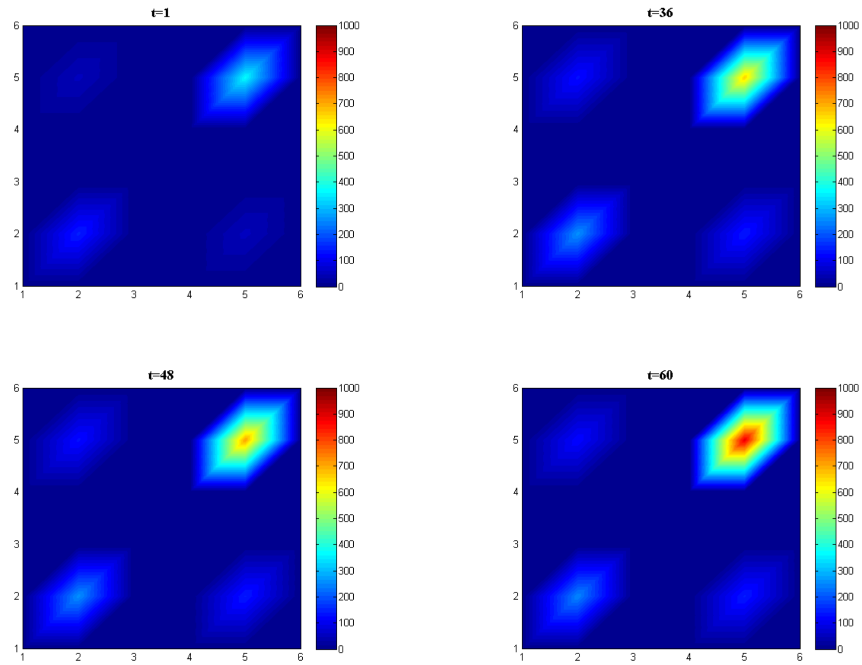

FIGURE 6. Time evolution of the individual members A^p with controls.

5. Conclusion

Political party switchers is a phenomenon that characterizes any democracies in the world and creates significant confusion for voters and mistrust in the political processes. This paper considers a difference equation model of the dynamics of the political party switchers. This work aims to determine an optimal control strategy that helps political parties prevent members from switching their political part, which might help maintain their political power. The model classifies the members of every political party dynamics into three compartments: (1) the susceptible to switch the political party, (2) the influencers in the political party, i.e., the party's active members that try to influence politicians to switch their party, and (3) the non-influencers with the political party. These are the party members that are not targeting the switchers. The optimal control aimed to minimize the switching between the parties. Using a discrete version of Pontryagin's maximum principle, we determine control strategies and the necessary conditions for optimal control. Finally, the numerical simulation illustrated the obtained results numerical simulation by showing that the optimal control was able to reduce the number of susceptible to leave the political party and increase the number of influencers and non-influencers. This finding shows that to eradicate the political party switchers, the number of non-influencers should have increased. This means if influencers face a high number of non -influencers, then the political parties will be safe from losing their members to other parties.

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