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BROWNIAN SUPER-EXPONENTS

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Abstract. We introduce a transform on the class of stochastic exponentials for d-dimensional Brownian motions. Each stochastic exponential generates another stochastic exponential under the transform. The new exponential process is often merely a supermartingale even in cases where the original process is a martingale. We determine a necessary and sufficient condition for the transform to be a martingale process. The condition links expected values of the transformed stochastic exponential to the distribution function of certain time-integrals.

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1. Introduction

If $\mathbf{X}(t)$ is a d-dimensional progressively measurable process and $\mathbf{W}$ is a Brownian motion under a measure P, the *stochastic exponential* determined by $\mathbf{X}$ is the process

$$Z_{\mathbf{X}}(t) = \exp\left\{\int_0^t \mathbf{X}(u)\cdot d\mathbf{W}(u) - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 du\right\}.$$

The problem of checking whether $Z_{\mathbf{X}}(t)$ is a true martingale is important for the use of Girsanov's theorem. Two well-known sufficient conditions are due to Novikov and to Kazamaki; see for example Revuz and Yor [7]. Examples where the process $Z_{\mathbf{X}}(t)$ is strictly a supermartingale appear in Goodman and Kim [3], Levental and Skorohod [6], and Wong and Heyde [9].

In their recent paper, Wong and Heyde [9] present a necessary and sufficient condition for any stochastic exponential to form a martingale process. Their condition is formulated in terms of an explosion time. We consider a class of stochastic exponentials for which their condition becomes more explicit. We begin with any stochastic exponential and we describe a modification, or transform, of it which generates another stochastic exponential.

The transform involves a *time-integral* of the form

$$\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u) du.$$

We derive a necessary and sufficient condition for the transform to be a martingale. Our condition is formulated in terms of the distribution of time integrals, and we use the relation to obtain bounds on the tail behavior of these distributions.

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Definition 1.1. Suppose that $\mathbf{X}(t)$ is a progressively measurable process such that for some $T > 0$,

$$P\left\{\int_0^T ||\mathbf{X}(u)||^2 du < \infty\right\} = 1. \tag{1.1}$$

If $Z_{\mathbf{X}}(t)$ is the stochastic exponential generated by $\mathbf{X}(t)$ and $y > 0$, the associated *super-exponent process* $Y_{\mathbf{X}}(t)$, defined for $t \leq T$, is

$$Y_{\mathbf{X},y}(t) = \frac{Z_{\mathbf{X}}(t)}{y^{-1} + \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du}. \tag{1.2}$$

Notice from Equation (1.2) that $Y_{\mathbf{X},y}(0) = y$. In addition, $Y_{\mathbf{X},y}(t)$ is positive so that the random variable

$$\exp(Y_{\mathbf{X},y}(t))$$

is greater than one. We show that this random variable has a finite expected value which is less than or equal to e^y. This result is surprising since $Y_{\mathbf{X},y}(t)$ is used as an exponent here. According to Definition 1.1, $Y_{\mathbf{X},y}(t)$ itself contains an exponential factor $Z_{\mathbf{X}}(t)$. For this reason, we say that the process $Y_{\mathbf{X},y}(t)$ is a *Brownian super-exponent.*

2. Transform Properties

Proposition 2.1. *Suppose that a progressively measurable process* $\mathbf{X}$ *satisfies condition (1.1). Let* $Y_{\mathbf{X},y}(t)$ *denote the super-exponent process in Definition 1.1. Then for each* $t \leq T$,

$$Y_{\mathbf{X},y}(t) = y + \int_0^t Y_{\mathbf{X},y}(u)\mathbf{X}(u)\cdot d\mathbf{W}(u) - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 Y^2_{\mathbf{X},y}(u)du. \tag{2.1}$$

Moreover, the process

$$\exp\left(Y_{\mathbf{X},y}(t)\right) \tag{2.2}$$

is a positive supermartingale on the interval $0 \leq t \leq T$. *In addition, the process*

$$\tilde{Z}(t) = \exp\left(Y_{\mathbf{X},y}(t) - y\right) \tag{2.3}$$

is a stochastic exponential for $\mathbf{W}$. *This stochastic exponential is generated by the d-dimensional process*

$$Y_{\mathbf{X},y}(t)\mathbf{X}(t). \tag{2.4}$$

Proof. It follows from the definition of $Z_{\mathbf{X}}(t)$ that

$$dZ_{\mathbf{X}} = Z_{\mathbf{X}}\mathbf{X}\cdot d\mathbf{W}, \quad d\int_0^t ||\mathbf{X}||^2 Z_{\mathbf{X}} du = ||\mathbf{X}||^2 Z_{\mathbf{X}} dt.$$

Direct calculation shows that

$$\begin{aligned} dY_{\mathbf{X},y} &= \frac{dZ_{\mathbf{X}}}{y^{-1} + \frac{1}{2}\int_0^t ||\mathbf{X}||^2 Z_{\mathbf{X}} du} + Z_{\mathbf{X}}\ d(y^{-1} + \frac{1}{2}\int_0^t ||\mathbf{X}||^2 Z_{\mathbf{X}} du)^{-1} \\ &= Y_{\mathbf{X},y}\mathbf{X}\cdot d\mathbf{W} - \frac{1}{2}||\mathbf{X}||^2 Y^2_{\mathbf{X},y} dt. \end{aligned} \tag{2.5}$$

From this equation we see that $Y_{\mathbf{X},y} - y$ is the sum of the Itô integral of $Y_{\mathbf{X},y}\mathbf{X}$ and the elementary integral of $-\frac{1}{2}||Y_{\mathbf{X},y}\mathbf{X}||^2$. This establishes Equation (2.1). It follows immediately from Equation (2.1) that $Y_{\mathbf{X},y} - y$ is the exponent of a stochastic exponential. Therefore, the process

$$\exp\left(Y_{\mathbf{X},y}(t) - y\right)$$

is a positive local martingale. It is well known that a positive local martingale is a supermartingale; see, for instance, Karatzas and Shreve [4]. In addition, Equation (2.3) is a direct consequence of Equation (2.1) and the definition of stochastic exponential processes. □

Theorem 2.2. *Suppose that* $\mathbf{X}(t)$ *is a deterministic function such that for some* $T > 0$

$$\int_0^T ||\mathbf{X}(u)||^2 du < \infty.$$

Let $Z_{\mathbf{X}}(t)$ *and* $Y_{\mathbf{X},y}$ *denote the stochastic exponential and super-exponent process generated by* $\mathbf{X}(t)$*. Then for each non-negative measurable function* $G(u)$*,* $u > 0$*, and* $t < T$*,*

$$E[G(Y_{\mathbf{X},y}(t))\exp(Y(t) - y)]$$
$$= E\left[G\Big(\frac{Z_{\mathbf{X}}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du}\Big) \; ; \; \int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du < \frac{2}{y} \right]. \tag{2.6}$$

Proof. For $N = 1, 2, \ldots$ let τ_N be the stopping time defined by

$$\tau_N = \inf\{t \le T \;:\; Y_{\mathbf{X},y}(t) \ge N\}.$$

It follows from Equation (2.1) that

$$Y_{\mathbf{X},y}(t\wedge\tau_N) - y = \int_0^{t\wedge\tau_N} Y_{\mathbf{X},y}(u)\mathbf{X}(u)\cdot d\mathbf{W}(u) - \frac{1}{2}\int_0^{t\wedge\tau_N} ||\mathbf{X}(u)||^2 Y^2_{\mathbf{X},y}(u)du.$$

From this equation we see that $\exp(Y_{\mathbf{X},y}(t\wedge\tau_N) - y)$ is another stochastic exponential which is generated by

$$Y_{\mathbf{X},y}(u)1_{\{u<\tau_N\}}\mathbf{X}(u).$$

Since this process is uniformly bounded in $L^2[0,T]$, it satisfies Novikov's condition. It is well known (see Karatzas and Shreve [4]) that the associated stochastic exponential is a martingale. We apply Girsanov's Theorem to change measure using the Radon-Nykodym derivative

$$\Lambda(T) = \exp(Y_{\mathbf{X},y}(T\wedge\tau_N) - y).$$

The probability measure Q_N is given by

$$\frac{dQ_N}{dP} = \Lambda(T).$$

Then with respect to Q_N the process

$$\tilde{\mathbf{W}}(t) = \mathbf{W}(t) - \int_0^{t\wedge\tau_N} Y_{\mathbf{X},y}(u)\mathbf{X}(u)du$$

is a Brownian motion for $t \le T$. Since $Y_{\mathbf{X},y}(t)$ is a strong solution to equation (2.1), we may consider its SDE with respect to the Brownian motion $\tilde{\mathbf{W}}$:

For $t < \tau_N$

$$\begin{aligned} dY_{\mathbf{X},y} &= Y_{\mathbf{X},y}\mathbf{X}\cdot d\mathbf{W} - \frac{1}{2}||\mathbf{X}||^2 Y_{\mathbf{X},y}^2 dt \\ &= Y_{\mathbf{X},y}\mathbf{X}\cdot \{\, d\tilde{\mathbf{W}} + Y_{\mathbf{X},y}\mathbf{X}dt\} - \frac{1}{2}||\mathbf{X}||^2 Y_{\mathbf{X},y}^2 dt \\ &= Y_{\mathbf{X},y}\mathbf{X}\cdot d\tilde{\mathbf{W}} + \frac{1}{2}||\mathbf{X}||^2 Y_{\mathbf{X},y}^2 dt. \end{aligned} \tag{2.7}$$

Now we have an explicit solution to the SDE in equation (2.7):

$$Y_{\mathbf{X},y}(t) = \frac{\tilde{Z}_{\mathbf{X}}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 \tilde{Z}_{\mathbf{X}}(u)du}. \tag{2.8}$$

In this equation, $\tilde{Z}_{\mathbf{X}}(t)$ denotes the stochastic exponential (generated by $\mathbf{X}$) with respect to the Brownian motion $\tilde{\mathbf{W}}$. Now we consider

$$\begin{aligned} &E[G(Y_{\mathbf{X},y}(t))\exp(Y_{\mathbf{X},y}(t) - y)1_{\{t<\tau_N\}}] \\ &= E[G(Y_{\mathbf{X},y}(t))\Lambda(T)1_{\{t<\tau_N\}}] \\ &= E_{Q_N}[G(Y_{\mathbf{X},y}(t))1_{\{t<\tau_N\}}] \\ &= E_{Q_N}\left[G\left(\frac{\tilde{Z}_{\mathbf{X}}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 \tilde{Z}_{\mathbf{X}}(u)du}\right)1_{\{t<\tau_N\}}\right]. \end{aligned} \tag{2.9}$$

Here we used the identity for $Y_{\mathbf{X},y}$ in Equation (2.8).

Moreover, from Equation (2.8) we also have

$$t < \tau_N \quad \text{if and only if} \quad \max_{s\le t}\left(\frac{\tilde{Z}_{\mathbf{X}}(s)}{y^{-1} - \frac{1}{2}\int_0^s ||\mathbf{X}(u)||^2 \tilde{Z}_{\mathbf{X}}(u)du}\right) < N.$$

This allows us to write the last expected value in Equation (2.9) as

$$E\left[G\left(\frac{Z_{\mathbf{X}}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2\ Z_{\mathbf{X}}(u)du}\right); \max_{s\le t}\left(\frac{Z_{\mathbf{X}}(s)}{y^{-1} - \frac{1}{2}\int_0^s ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du}\right) < N\right]$$

since the integrand involves only the distribution of a Brownian motion for each choice of N. The limit of this expected value as $N \to \infty$ is

$$E\left[G\left(\frac{Z_{\mathbf{X}}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2\ Z_{\mathbf{X}}(u)du}\right)\ ;\ \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du) < y^{-1}\right].$$

Since the limit of the first expected value in Equation (2.9) is

$$E[G(Y_{\mathbf{X},y}(t))\exp(Y_{\mathbf{X},y}(t) - y)],$$

the theorem is proved. □

3. Examples Using the Transform

Proposition 3.1. *Suppose that* $\mathbf{X}(t)$ *is a deterministic function such that*

$$\int_0^t ||\mathbf{X}(u)||^2 du$$

is strictly increasing and finite for $t \leq T < \infty$. *Let* $Z_{\mathbf{X}}(t)$ *and* $Y_{\mathbf{X},y}$ *denote the stochastic exponential and super-exponent process generated by* $\mathbf{X}(t)$. *Then the process*

$$\exp(Y_{\mathbf{X},y}(t))$$

is a strict supermartingale for $t \leq T$. *Moreover,*

$$E[\exp(Y_{\mathbf{X},y}(t))] = e^y Pr\left\{\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du < \frac{2}{y}\right\}. \tag{3.1}$$

Proof. We apply Theorem 2.2 using the choice $G(u) \equiv 1$. Equation (2.6) becomes

$$E[\exp(Y_{\mathbf{X},y}(t) - y)] = Pr\left\{\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du < \frac{2}{y}\right\},$$

and Equation (3.1) follows. Now since each $Z_{\mathbf{X}}(u)$ is a log normal random variable, the process

$$\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du \tag{3.2}$$

has strictly increasing sample paths. It follows that the right hand expression in Equation (3.1) is strictly decreasing. Therefore, $\exp(Y_{\mathbf{X},y}(t))$ is a strict supermartingale. □

Remark 3.2. Equation (3.1) provides a useful tool for investigating the distribution of a time integral given by Equation (3.2). Since each super-exponent

$$Y_{\mathbf{X},y}(t) = \frac{Z_{\mathbf{X}}(t)}{y^{-1} + \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du}$$

is point-wise increasing as a function of y, it follows from the identity

$$Pr\left\{\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du < a\right\} = \exp(-\frac{2}{a})E[\exp(Y_{\mathbf{X},2/a}(t))]$$

that the distribution function is the product of a decreasing function of a and the explicit factor $\exp(-2/a)$.

It is not known whether $\exp(Y_{\mathbf{X},\infty}(t))$ has finite expectation. A finite expected value would produce sharp estimates for the lower tail probability of (3.2). We conjecture that

$$E\left[\frac{2Z_{\mathbf{X}}(t)}{\int_0^t ||\mathbf{X}(u)||^2 Z_{\mathbf{X}}(u)du}\right] = \infty.$$

Example 3.3. In the case of $d = 1$ the choice $X(t) \equiv \sigma$ specializes the time integral in (3.2) to a time integral of *geometric Brownian motion:*

$$\int_0^t \exp(\sigma W(u) - \sigma^2 u/2) du. \tag{3.3}$$

Expected values involving related time integrals appear in computational problems of financial mathematics. Consequently, distribution properties of these time integrals have been studied by many authors; see Dufresne [1], Geman and Yor [2], Rogers and Shi [8], and Goodman and Kim [3].

Although most works have used analytic techniques to express the distribution in various integral forms, in Goodman and Kim [3] martingales techniques are used exclusively. A special case of Equation (3.1) appears in [3], Theorem 4.1:

$$\begin{aligned} &Pr\left\{ \int_0^t \exp(W(u) - u/2) du \le a \right\} \\ &= \exp\left(-\frac{2}{a}\right) E\left[\exp\left(\frac{2\exp(W(t) - t/2)}{a + \int_0^t \exp(W(u) - u/2) du}\right)\right]. \end{aligned}$$

The right hand expression for the distribution can be differentiated with respect to a. Consequently, it is shown in [3] that the density function multiplied by $a^2/2$ equals the difference between two distribution functions of time integrals of slightly different geometric Brownian motions.

Example 3.4. In contrast to deterministic choices for $\mathbf{X}(t)$, where the stochastic exponential

$$\exp(Y_{\mathbf{X},y}(t))$$

is never a martingale, stochastic choices for $\mathbf{X}$ may produce martingales. Of course, the introduction of a stopping time, as we have seen in the proof of Theorem 2.2, may produce a martingale. In other cases, stopping times are not required.

Consider the example of $X(t) = \cos(W(t))$, again in the case $d = 1$. Then

$$\begin{aligned} Z_{\mathbf{X}}(t) &= \exp\left(\int_0^t \cos(W(u)) dW(u) - \frac{1}{2}\int_0^t \cos^2(W(u)) du\right) \\ &= \exp\left(\sin(W(t)) + \frac{1}{2}\int_0^t [\sin(W(u)) - \cos^2(W(u))] du\right) \end{aligned}$$

is a bounded random variable. Therefore, its super-exponent, $Y_{\cos(W),y}(t)$ is also bounded. Then since the local martingale

$$\exp(Y_{\cos(W),y}(t))$$

is also bounded, it is a martingale. It is of interest then to know when a super-exponent generates a martingale process.

4. The Martingale Condition

Theorem 1 of Wong and Heyde [9] identifies a necessary and sufficient condition for a progressively measurable process $\tilde{\mathbf{X}}$ to generate a martingale stochastic exponential process. For completeness, we state their result here.

Proposition 4.1. ([9], Proposition 1) *Consider a d-dimensional progressively measurable process* $\tilde{\mathbf{X}}(t) = \xi(\mathbf{W}(\cdot), t)$. *Then there will also exist a d-dimensional progressively measurable process*

$$\tilde{\mathbf{R}}(t) = \xi(\mathbf{W}(\cdot) + \int_0^{\cdot} \tilde{\mathbf{R}}(u)du, t)$$

defined possibly up to an explosion time $\tau^{M_{\mathbf{R}}}$, *where*

$$\tau^{M_R} = \inf \left\{ t \in \mathbb{R}^+ : M_{\mathbf{R}}(t) = \int_0^t ||\tilde{\mathbf{R}}(u)||^2 du = \infty \right\}.$$

Theorem 4.2. ([9], Theorem 1) *Consider* $\tilde{\mathbf{X}}(t)$ *and* $\tilde{\mathbf{R}}(t)$ *as defined in Proposition 4.1. The stochastic exponential* $Z_{\tilde{\mathbf{X}}}(T)$ *satisfies*

$$P(\tau^{M_{\mathbf{R}}} > T) = E_P[Z_{\tilde{\mathbf{X}}}(T)]$$

and hence is a martingale if and only if $P(\tau^{M_{\mathbf{R}}} > T) = 1$.

We apply Theorem 1 of [9] using $\tilde{\mathbf{X}}(t) = Y_{\mathbf{X},y}(t)\mathbf{X}(t)$. That is, our generating process is the one in Proposition 2.1 where the stochastic exponential process is

$$\exp(Y_{\mathbf{X},y}(t) - y).$$

We first show that each generating process $\mathbf{X}$ implicitly defines another process $\mathbf{X}'$. This allows us to identify the process $\tilde{\mathbf{R}}(t)$.

Proposition 4.3. *Suppose that a d-dimensional progressively measurable process* $\mathbf{X}(t)$ *satisfies*

$$Pr\left(\int_0^T ||\mathbf{X}(u)||^2 du < \infty\right) = 1$$

for some $T > 0$. *Then there exists another progressively measurable process* $\mathbf{X}'(t)$, *so that if* $\tilde{\mathbf{X}}(t) := Y_{\mathbf{X},y}(t)\mathbf{X}(t)1_{\{t \le T\}}$ *in Proposition 4.1, then the process* $\tilde{\mathbf{R}}(t)$ *of the proposition satisfies*

$$\tilde{\mathbf{R}}(t) = \frac{Z_{\mathbf{X}'}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}'(u)||^2 Z_{\mathbf{X}'}(u)du}\mathbf{X}'(t)$$

for all $t < \tau^{M_{\mathbf{R}}}$. *Moreover,*

$$\tau^{M_R} = \inf \left\{ t \in \mathbb{R}^+ : \int_0^{t \wedge T} ||\mathbf{X}'(u)||^2 Z_{\mathbf{X}'}(u)du = 2/y \right\}.$$

Proof. We follow the proof of Proposition 4.1. Let

$$\tilde{\mathbf{X}}(t) := Y_{\mathbf{X},y}(t)\mathbf{X}(t)1_{\{t \le T\}}.$$

For each $N = 1, 2, \dots$ we define a sequence of stopping times by

$$\tau_N = \inf \left\{ t \in \mathbb{R}^+ : \int_0^t Y^2_{\mathbf{X},y}(u)||\mathbf{X}(u)||^2 1_{\{u \le T\}} du \ge N \right\}.$$

It follows from Equation (2.1) that

$$Z_{\tilde{\mathbf{X}}}(t \wedge \tau_N) = \exp(Y_{\mathbf{X},y}(t \wedge \tau_N) - y)$$

forms a martingale. As in the proof of Theorem 2.2, we apply Girsanov's theorem using the Radon-Nikodym derivative

$$\Lambda(T) = \exp(Y_{\mathbf{X},y}(T \wedge \tau_N) - y)$$

to obtain the probability measure Q_N where

$$dQ_N = \Lambda(T)dP.$$

With respect to the measure Q_N, the process

$$\tilde{\mathbf{W}}(t) = \mathbf{W}(t) - \int_0^{t\wedge\tau_N} Y_{\mathbf{X},y}(u)\mathbf{X}(u)du$$

is a Brownian motion. Hence, on the set $\{t \leq \tau_N \wedge T\}$ we have

$$\tilde{\mathbf{X}}(t) = \xi(\tilde{\mathbf{W}}(\cdot) + \int_0^{\cdot} Y_{\mathbf{X},y}(u)\mathbf{X}(u)du, t),$$

that is,

$$Y_{\mathbf{X},y}(t)\mathbf{X}(t) = \xi(\tilde{\mathbf{W}}(\cdot) + \int_0^{\cdot} Y_{\mathbf{X},y}(u)\mathbf{X}(u)du, t). \tag{4.1}$$

Now the process $Y_{\mathbf{X},y}(t)$ can also be described in terms of the Brownian motion $\tilde{\mathbf{W}}$. The calculations in Equation (2.7) also apply to the stochastic case. Equation (2.8) gives an explicit formula for $Y_{\mathbf{X},y}$:

$$Y_{\mathbf{X},y}(t) = \frac{\tilde{Z}_{\mathbf{X}}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}(u)||^2 \tilde{Z}_{\mathbf{X}}(u)du}. \tag{4.2}$$

We see that each term of Equation (4.1) is a functional of $\mathbf{X}$ and the Brownian motion $\tilde{\mathbf{W}}$. This demonstrates the existence of a process $\mathbf{X}$ so that (4.1) and (4.2) hold up to a time τ_N defined by the integral of $Y_{\mathbf{X},y}(u)\mathbf{X}(u)$, using the Brownian motion $\tilde{\mathbf{W}}$.

Therefore, using the identical distribution of $\mathbf{W}$ and the (original) measure P, we see that there exists a progressively measurable process $\mathbf{X}'(t)$ so that

$$\frac{Z_{\mathbf{X}'}(t)}{y^{-1} - \frac{1}{2}\int_0^t ||\mathbf{X}'(u)||^2 Z_{\mathbf{X}'}(u)du}\mathbf{X}'(t) = \xi(\mathbf{W}(\cdot) + \int_0^{\cdot} Y_{\mathbf{X}',y}(u)\mathbf{X}'(u)du, t).$$

Here, we have abbreviated the complete expression on the right hand side using (4.2) to provide the notation. That is, $Y_{\mathbf{X}',y}$ denotes the expression in Equation (4.2) but *in the original Brownian motion* and $\mathbf{X}$ is replaced by the process $\mathbf{X}'$.

As $N \to \infty$ the stopping time τ_N increases to the stopping time

$$\tau = \inf\left\{t \leq T : \int_0^t Y^2_{\mathbf{X}',y}(u)||\mathbf{X}'(u)||^2 du = \infty\right\}.$$

By construction, the new process $\mathbf{X}'$ satisfies

$$\int_0^T ||\mathbf{X}'(u)||^2 du < \infty \quad \text{a. s.} \quad \text{and} \quad \mathbf{X}'(u) = 0 \text{ for } u > T.$$

Therefore, the process $Y_{\mathbf{X}',y}$ (again, defined as in (4.2)) is bounded along each sample path up to the time where its denominator first hits zero. This defines the stopping time τ^{M_R} of the Proposition. $\square$

Theorem 4.4. *Suppose that* $\mathbf{X}(t)$ *and* $\mathbf{X}'(t)$ *are d-dimensional processes as defined in Proposition 4.3. Then the super-exponent process* $Y_{\mathbf{X},y}(t)$ *satisfies*

$$E[\exp(Y_{\mathbf{X},y}(t)-y)] = Pr\left\{\int_0^t ||\mathbf{X}'(u)||^2 Z_{\mathbf{X}}(u)du < 2/y\right\} \tag{4.3}$$

for $t \leq T$.

Proof. From Theorem 4.2 and Proposition 4.3 we have

$$\begin{aligned}
&E[\exp(Y_{\mathbf{X},y}(t)-y)] = Pr\left\{\tau^{M_R} > t\right\} \\
&= Pr\left\{\int_0^t Y^2_{\mathbf{X}',y}(u)||\mathbf{X}'(u)||^2 du < \infty\right\} \\
&= Pr\left\{\int_0^t \frac{Z_{\mathbf{X}'}(u)}{y^{-1}-\frac{1}{2}\int_0^u ||\mathbf{X}'(r)||^2 Z_{\mathbf{X}'}(r)dr}||\mathbf{X}'(u)||^2 du < \infty\right\} \\
&= Pr\left\{\int_0^t ||\mathbf{X}'(u)||^2 Z_{\mathbf{X}}(u)du < 2/y\right\}.
\end{aligned}$$

□

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