

**THE LARGE SCALE BEHAVIOR OF SUPER-BROWNIAN  
MOTION IN THREE DIMENSIONS WITH  
A SINGLE POINT SOURCE \***

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ABSTRACT. In a recent work, Fleischmann and Mueller (2004) showed the existence of a super-Brownian motion in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with extra birth at the origin. Their construction made use of an analytical approach based on the fundamental solution of the heat equation with a one point potential worked out by Albeverio et al. (1995). The present note addresses two properties of this measure-valued process in the three-dimensional case, namely the scaling of the process and the large scale behavior of its mean.

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**1. Introduction and result**

A super-Brownian motion in  $\mathbb{R}$  with a single point source  $\delta_0$  was constructed in Engländer & Fleischmann [3]. It was shown that its expected mass grows exponentially in time, and is in the mass-rescaled limit distributed in space as  $x \mapsto e^{-|x|}$ . In Engländer & Turaev [4] it is even proved that the random measures themselves grow in law exponentially as time increases, and are otherwise in the mass-rescaled limit spatially situated with the same shape except an overall random factor. The probabilistic effect behind the non-trivial existence of the model is the fact that a Brownian particle in  $\mathbb{R}$  hits the origin with certainty and that it has there a non-degenerate local time, serving as an additional birth rate for the random creation of mass.

In higher dimensions, a Brownian particle fails to hit the origin, and a local time would degenerate. Nevertheless, Fleischmann & Mueller [6] succeeded in constructing a super-Brownian motion in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a single point source. They heavily used well-known analytical facts from mathematical physics concerning Laplace operators with one-point-potentials. Heuristically, some additional rescaling enters the regularization of the delta function (serving as single point source). Properties of this new super-Brownian motion are not known so far. The *purpose* of the present note is to get some progress by studying its scaling and the large scale behavior of its expectation in the three-dimensional case.

**1.1. The heat equation with one-point-potential.** The Schrödinger equation with a one-point-potential is studied in quantum theory to describe singular

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electromagnetic effects on quantum particles, see e.g. the monograph Albeverio et al. [2, Part I]. By analytic continuation, solutions to the Schrödinger equation can be (at least formally) obtained via solutions of the heat equation.

Formally, the heat equation with a one-point-potential is given by

$$\partial_t u = \Delta u + \delta_0^{(\alpha)} u =: \Delta^{(\alpha)} u, \quad (1.1)$$

where  $\partial_t$  denotes the derivative with respect to time,  $\Delta$  is the  $d$ -dimensional Laplacian, and  $u : (0, \infty) \times \mathring{\mathbb{R}}^d \rightarrow \mathbb{R}_+$  is a time-space field, where  $\mathring{\mathbb{R}}^d := \mathbb{R}^d \setminus \{0\}$  with the Euclidean metric is locally compact. If we denote by  $B_\varepsilon(y)$  an open ball around  $y \in \mathbb{R}^d$  of radius  $\varepsilon > 0$ , then having in mind that  $\varepsilon^{-d} \mathbf{1}_{B_\varepsilon(0)} \approx \delta_0$ , the operator  $\Delta^{(\alpha)} := \Delta + \delta_0^{(\alpha)}$  is heuristically the limit as  $\varepsilon \downarrow 0$  of the operator

$$\Delta_\varepsilon^{(\alpha)} := \Delta + h(d, \alpha, \varepsilon) \varepsilon^{-d} \mathbf{1}_{B_\varepsilon(0)}, \quad (1.2)$$

where  $h(d, \alpha, \varepsilon)$  is some additional rescaling factor which depends on a parameter  $\alpha$  at least. Restricting to  $d = 3$ , the function  $h$  can be chosen as

$$h(3, \alpha, \varepsilon) := \frac{\pi^2}{4} \varepsilon - 8\pi^2 \alpha \varepsilon^2, \quad \alpha \in \mathbb{R}, \quad \varepsilon > 0, \quad (1.3)$$

(cf. [2, (H.74)]).

Physically,  $\alpha$  in the case  $\alpha < 0$  is related to the *scattering length*  $\text{sl}_\alpha := -(4\pi\alpha)^{-1}$  of the free Laplace operator  $\Delta$  with respect to the interaction Laplacian  $\Delta^{(\alpha)}$ . Roughly speaking, the scattering length describes the average distance a free particle manages to go before any interaction takes place. So, if  $\alpha \downarrow -\infty$  the scattering length  $\text{sl}_\alpha \downarrow 0$  becomes smaller and we expect more interaction. For  $\alpha \geq 0$  there is *no* proper physical interpretation of  $\text{sl}_\alpha$  as the point spectrum of  $\Delta^{(\alpha)}$  is empty (see [2, Theorem I.1.4]).

The *fundamental solution*  $p^\alpha$  to the equation

$$\partial_t u = \Delta^{(\alpha)} u \quad \text{on } (0, \infty) \times \mathring{\mathbb{R}}^d, \quad d = 2, 3, \quad (1.4)$$

which provides the basis for the analytical construction of the superprocess in [6], has been computed in Albeverio et al. [1]. In  $d = 3$  (the two-dimensional case is more delicate, which is the reason we restrict to  $d = 3$ ), the *one-point-interaction heat kernel*  $p^\alpha$  for  $\alpha \in \mathbb{R}$  is given by

$$\begin{aligned} p_t^\alpha(x, y) &= p_t(x, y) + \frac{2t}{|x||y|} p_t(|x| + |y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty du e^{-4\pi\alpha u} p_t(u + |x| + |y|), \end{aligned} \quad (1.5)$$

$t > 0$ ,  $x, y \in \mathring{\mathbb{R}}^3$ , where  $p$  is the usual *free* heat kernel defined by,

$$p_t(x, y) := (4\pi t)^{-d/2} \exp(-|y - x|^2/4t), \quad (1.6)$$

and with a slight abuse of notation,

$$p_t(r) := (4\pi t)^{-d/2} \exp(-r^2/4t), \quad t > 0, \quad r \geq 0. \quad (1.7)$$

Also recall the scaling of the free heat kernel, i.e. for all  $k, t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$p_t(x, y) = k^{d/2} p_{kt}(k^{1/2}x, k^{1/2}y). \quad (1.8)$$

Note, that the last term in (1.5) is always finite and disappears for  $\alpha = 0$ . Moreover,  $\alpha \mapsto p^\alpha$  is pointwise continuous and decreasing, and we have the (pointwise)

convergences  $p^\alpha \uparrow +\infty$  as  $\alpha \downarrow -\infty$  (i.e. the fundamental solution explodes which can be interpreted as *immediate interaction*), whereas  $p^\alpha \downarrow p$  as  $\alpha \uparrow +\infty$  leads the free case (i.e. the interaction disappears).

Rigorously, the family  $\{\Delta^{(\alpha)} : \alpha \in \mathbb{R}\}$  of operators are defined as *all* self-adjoint extensions on the Hilbert space  $\mathcal{L}^2(\dot{\mathbb{R}}^d, dx)$  of the Laplacian  $\Delta$  acting on  $\mathcal{C}_{\text{com}}^\infty(\dot{\mathbb{R}}^d)$ , the space of infinitely differentiable functions on  $\dot{\mathbb{R}}^d = \mathbb{R}^d \setminus \{0\}$  with compact support (see e.g. [2, Chapters I.1 and I.5]). Hence, although the  $p^\alpha$  from (1.5) differ from the free heat kernel  $p$ , they solve the heat equation

$$\partial_t p_t^\alpha(x, y) = \Delta p_t^\alpha(x, y) \quad \text{on } (0, \infty) \times \dot{\mathbb{R}}^3, \quad (1.9)$$

with the Laplacian  $\Delta$  acting either on the variable  $x$  or  $y$ . In particular,  $(t, x, y) \mapsto p_t^\alpha(x, y)$  is jointly continuous on  $(0, \infty) \times \dot{\mathbb{R}}^3 \times \dot{\mathbb{R}}^3$ . Let us denote by  $S^\alpha$  the semigroup associated with the kernel  $p^\alpha$ , i.e.

$$S_t^\alpha \varphi(x) := \int_{\dot{\mathbb{R}}^3} dy \varphi(y) p_t^\alpha(x, y). \quad (1.10)$$

Note that  $S^\alpha$  is *not* a contraction semigroup and so there is *no* stochastic process generated by this flow. The following lemma shows that in the present three-dimensional case the kernel  $p^\alpha$  has a similar scaling behavior as the free heat kernel  $p$ .

**Lemma 1.1 (Scaling of the  $p^\alpha$ ).** *We have, for all  $k, t > 0$ ,  $x, y \in \dot{\mathbb{R}}^3$ , and  $\alpha \in \mathbb{R}$ ,*

$$p_t^\alpha(x, y) = k^{3/2} p_{kt}^{k^{-1/2}\alpha}(k^{1/2}x, k^{1/2}y). \quad (1.11)$$

*Proof.* It follows immediately from the definition (1.5) of the  $p^\alpha$  and the scaling (1.8) of the free heat kernel  $p$ .  $\square$

## 1.2. The flow associated with the one-point-interaction heat kernel.

This section is devoted to introducing a space of functions  $\Phi$  on which the flow  $S^\alpha$  acts as a strongly continuous linear semigroup (see [6, Section 2] for details). Let  $\phi$  denote the *weight and reference function*

$$\phi(x) := |x|^{-1}, \quad x \in \dot{\mathbb{R}}^3 = \mathbb{R}^3 \setminus \{0\}. \quad (1.12)$$

For fixed  $\varrho \in (1, 2)$ , let  $\mathcal{H} = \mathcal{H}^\varrho$  denote the space of measurable functions  $\varphi$  on  $\dot{\mathbb{R}}^3$  for which

$$\|\varphi\|_{\mathcal{H}} := \left( \int_{\dot{\mathbb{R}}^3} dx \phi(x) |\varphi(x)|^\varrho \right)^{1/\varrho} < \infty. \quad (1.13)$$

Then  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  is a Banach space, where as usual we do not distinguish between equivalence classes and their representatives. Now, let  $\Phi = \Phi^\varrho$  denote the set of all *continuous* functions  $\varphi : \dot{\mathbb{R}}^3 \rightarrow \mathbb{R}$  such that  $\varphi \in \mathcal{H}$  and

$$0 \leq \varphi \leq C \phi \quad \text{for some constant } C = C_\varphi > 0. \quad (1.14)$$

We endow  $\Phi$  with the topology inherited from  $\mathcal{H}$ . Note that the set  $\mathcal{C}_{\text{com}}^+ = \mathcal{C}_{\text{com}}^+(\dot{\mathbb{R}}^3)$  of all non-negative, continuous functions on  $\dot{\mathbb{R}}^3$  with compact support is contained in  $\Phi$ . We remark that  $\varphi \in \Phi$  might have a singularity at  $x = 0$  of order  $|x|^{-\xi}$  with  $0 < \xi < 1$ . The linear semigroup  $S^\alpha$  introduced in (1.10) is strongly continuous on the cone  $\Phi = \Phi^\varrho$ , cf. Corollary 2.12 in [6].

**1.3. Super-Brownian motion with a single point source.** The superprocess  $X$  we want to study was constructed in [6] via the so-called *log-Laplace approach*. Roughly speaking, the Markov process  $X$  is uniquely determined by its log-Laplace transition functionals (s. (1.15) below), and they are described by a function  $v$ , the so-called log-Laplace function. The point is, that  $v$  solves uniquely a non-linear equation, the so-called log-Laplace equation (s. (1.16) below). The main work in [6] was, to verify that the Cauchy problem for this equation is well-posed. Here uniqueness followed from a contraction argument, existence was shown via a Picard iteration, and non-negativity of  $v$  followed using an approximating linearized equation. This then allows to construct  $X$  via a Trotter product approach to the related log-Laplace semigroup, more precisely, via an approximating log-Laplace equation related to separating critical continuous-state branching and mass flow according to  $S^\alpha$  on alternate small time intervals.

Now we come to a precise description of  $X$ . Denote by  $\mathcal{M} = \mathcal{M}(\dot{\mathbb{R}}^3)$  the set of all (Radon) measures  $\mu$  on  $\dot{\mathbb{R}}^3$  such that  $\langle \mu, \varphi \rangle := \int_{\dot{\mathbb{R}}^3} \mu(dx) \varphi(x) < \infty$  for all  $\varphi \in \Phi$ . Recalling that  $\mathcal{C}_{\text{com}}^+ \subset \Phi$ , endow  $\mathcal{M}$  with the vague topology.

Fix a constant  $\eta > 0$  (branching rate). Suppose  $0 < \beta < 1$  (branching index; the finite variance branching case  $\beta = 1$  has been excluded in [6] for  $d = 3$  for technical reasons). Then for each  $\alpha \in \mathbb{R}$ , there is a non-degenerate  $\mathcal{M}$ -valued (time-homogeneous) Markov process  $X = X^\alpha$  such that for (deterministic) starting measures  $\mu \in \mathcal{M}$  and for  $\varphi \in \Phi$ ,

$$-\log \mathbb{E}_\mu \exp \langle X_t^\alpha, -\varphi \rangle = \langle \mu, v(t, \cdot) \rangle, \quad t > 0, \quad (1.15)$$

where  $v = \{v(t, x) : t \geq 0, x \in \dot{\mathbb{R}}^3\}$  is the unique non-negative solution of the integral equation related to the  $\Phi$ -valued evolution equation

$$\begin{cases} \partial_t v = \Delta^{(\alpha)} v - \eta v^{1+\beta} & \text{on } (0, \infty), \\ v(0+, \cdot) = \varphi \end{cases} \quad (1.16)$$

(see [6, Theorem 4.4]). That is,

$$v(t, x) = \int_{\dot{\mathbb{R}}^3} dy p_t^\alpha(x, y) \varphi(y) - \eta \int_0^t ds \int_{\dot{\mathbb{R}}^3} dy p_{t-s}^\alpha(x, y) v^{1+\beta}(s, y), \quad (1.17)$$

$t > 0, x \in \dot{\mathbb{R}}^3$ . Clearly, the first moments of  $X^\alpha$  are determined by the  $S^\alpha$  flow to be

$$\mathbb{E}_\mu \langle X_t^\alpha, \varphi \rangle = \langle \mu, S_t^\alpha \varphi \rangle, \quad (1.18)$$

for all starting measures  $\mu \in \mathcal{M}$ ,  $t \geq 0$ , and  $\varphi \in \Phi$ .

**1.4. Large scale behavior of the mean.** Before we can state our result, we have to introduce some notation. The limiting measure will be expressed by means of the kernel

$$\vartheta_t^\alpha(x, y) := \frac{2t}{|x||y|} p_t(|y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty du e^{-4\pi\alpha u} p_t(u + |y|), \quad (1.19)$$

for  $\alpha \in \mathbb{R}$ ,  $t > 0$ , and  $x, y \in \dot{\mathbb{R}}^3$ . Note that the integral is always finite, hence for  $\alpha = 0$  the second term disappears. Moreover, the kernel  $\vartheta^\alpha$  is always non-negative.

This holds trivially whenever  $\alpha < 0$ , and to see this for  $\alpha > 0$ , use the estimate

$$p_t(u + |y|) \leq p_t(|y|). \quad (1.20)$$

We extend the definition of  $\vartheta^\alpha$  by setting

$$\vartheta_t^\alpha(x, y) := \begin{cases} 0, & \text{if } \alpha = +\infty, \\ +\infty, & \text{if } \alpha = -\infty. \end{cases} \quad (1.21)$$

The so defined kernels  $\vartheta^\alpha$  turn out to be pointwise continuous in  $\alpha \in [-\infty, +\infty]$  (which follows from the arguments of the proof of Theorem 1.2 in Section 2.2 below).

**Theorem 1.2 (Large scale behavior of the mean).** *For  $t > 0$ ,  $\alpha, \lambda_k \in \mathbb{R}$ , and all starting measures  $X_0^\alpha = \mu \in \mathcal{M}$  satisfying  $\langle \mu, \phi \rangle < \infty$ , we have the convergence in  $\mathcal{M}$ ,*

$$\lim_{k \uparrow \infty} k^{-1/2} \mathbb{E}_\mu [X_{kt}^{\lambda_k \alpha}(k^{1/2} dy)] = \langle \mu, \vartheta_t^{\alpha^*}(\cdot, y) \rangle dy, \quad (1.22)$$

provided that  $\alpha^* := \lim_{k \uparrow \infty} k^{1/2} \lambda_k \alpha \in [-\infty, +\infty]$ .

Note that in the special case  $\lambda_k = k^{-1/2}$  we have  $\alpha^* = \alpha \in \mathbb{R}$ , whereas the particular case  $\alpha = 0$  implies  $\alpha^* = 0$  giving the simplified  $\vartheta^0$  in the limit term (without the second term in (1.19)).

*Remark 1.3 (Large scale total mass).* Taking  $\lambda_k \equiv 1$  and insert formally  $\varphi = 1$  as test function into convergence statement (1.22) yields

$$\lim_{k \uparrow \infty} k^{-1/2} \mathbb{E}_\mu \langle X_{kt}^\alpha, 1 \rangle = \begin{cases} 0 & \text{if } \alpha > 0, \\ 2t \langle \mu, \phi \rangle \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) & \text{if } \alpha = 0, \\ \infty & \text{if } \alpha < 0. \end{cases} \quad (1.23)$$

A rigorous argument can be given along the same lines as the proof of Theorem 1.2 below.  $\diamond$

**1.5. Discussion and open problems.** Let us comment on the *three cases*  $\alpha^* = +\infty$ ,  $\alpha^* \in \mathbb{R}$ , and  $\alpha^* = -\infty$  in Theorem 1.2. In the first case, the limiting mass disappears, more precisely, the scaled expression  $\mathbb{E}_\mu [X_{kt}^{\lambda_k \alpha}(k^{1/2} dy)]$  is of order  $o(k^{1/2})$ . Roughly speaking, if  $\alpha^* = +\infty$ , then there are no interactions in the scaling limit (free case). In the second case,  $\alpha^* \in \mathbb{R}$ , the former expectation is about  $k^{1/2} \langle \mu, \vartheta_t^{\alpha^*}(\cdot, y) \rangle dy$ . Note that these measures are decreasing in  $\alpha^*$ . Finally, if  $\alpha^* = -\infty$ , we have immediate interaction in the large scale limit leading to the explosion of the expected mass.

Clearly, to describe only the large scale behavior of the *expected processes* is unsatisfactory. It is desirable to get insight into the processes themselves. Recall that in the one-dimensional case the large time behavior in law of the process itself is known from [4] (for a sharpening of some results from [4], see Engländer & Winter [5]). However, we stress the fact, that the process in three dimensions is expected to have quite different features. For instance, if  $\alpha = 0$ , then according to Remark 1.3 the total mass grows with a power order, whereas in one dimension the growth is exponential. Moreover, in the three-dimensional case one needs additionally to contract the normalized measures to get a limit. For the measures

themselves, scaled as in Theorem 1.2, *there might be extinction in law* despite convergence of their expectations as in (1.22).

Another *open problem* is the large scale behavior of  $\mathbb{E}X^\alpha$  in the two-dimensional case, in which the fundamental solutions  $p^\alpha$  from [1] are analytically more delicate, see e.g. [6, formula (2.30)]. In particular, a scaling property as in Lemma 1.1 is not available.

*Remark 1.4 (Discrete version).* In physics literature, Redner & Kang [7], the following somehow related model occurs: Discrete time random walkers produce a fixed number of additional particles if they hit the origin in  $\mathbb{Z}^d$ . Dimension effects occur.  $\diamond$

## 2. Proofs

**2.1. A scaling property.** Recall that we are dealing with the three-dimensional case.

**Proposition 2.1 (A scaling property).** *Let  $t, k > 0$ ,  $\mu \in \mathcal{M}$ , and  $\alpha, \lambda_k \in \mathbb{R}$ . Then*

$$\begin{aligned} & \left\{ k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} \cdot) \mid X_0^{\lambda_k \alpha} = k^{1/\beta} \mu(k^{-1/2} \cdot) \right\} \\ & \stackrel{\mathcal{L}}{=} \left\{ X_t^{k^{1/2} \lambda_k \alpha} \mid X_0^{k^{1/2} \lambda_k \alpha} = \mu \right\}. \end{aligned} \quad (2.1)$$

Note that besides  $\lambda_k \equiv 1$ , the cases  $\lambda_k = k^{-1/2}$  or even  $\alpha = 0$  are particularly nice, since here the right hand side in (2.1) is independent of  $k$  (a kind of self-similarity).

*Proof.* For  $\varphi \in \Phi$  fixed,

$$\langle k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} dy), \varphi \rangle = \langle X_{kt}^{\lambda_k \alpha}, k^{-1/\beta} \varphi(k^{-1/2} \cdot) \rangle, \quad (2.2)$$

hence, by (1.15) and (1.17),

$$\begin{aligned} & -\log \mathbb{E}_{k^{1/\beta} \mu(k^{-1/2} \cdot)} \exp \langle k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} dy), -\varphi \rangle \\ & = -\log \mathbb{E}_{k^{1/\beta} \mu(k^{-1/2} \cdot)} \exp \langle X_{kt}^{\lambda_k \alpha}, -k^{-1/\beta} \varphi(k^{-1/2} \cdot) \rangle \\ & = \langle k^{1/\beta} \mu(k^{-1/2} \cdot), v(kt, \cdot) \rangle = \langle \mu, k^{1/\beta} v(kt, k^{1/2} \cdot) \rangle, \end{aligned} \quad (2.3)$$

where  $\{v(t', x') : t' \geq 0, x' \in \dot{\mathbb{R}}^3\}$  is the non-negative solution of the integral equation related to the function-valued evolution equation

$$\begin{cases} \partial_t v = \Delta^{(\lambda_k \alpha)} v - \eta v^{1+\beta} & \text{on } (0, \infty), \\ v(0+, \cdot) = k^{-1/\beta} \varphi(k^{-1/2} \cdot). \end{cases} \quad (2.4)$$

More precisely,

$$\begin{aligned} k^{1/\beta} v(kt, k^{1/2} x) & = k^{1/\beta} \int_{\dot{\mathbb{R}}^3} dy p_{kt}^{\lambda_k \alpha}(k^{1/2} x, y) k^{-1/\beta} \varphi(k^{-1/2} y) \\ & \quad - k^{1/\beta} \eta \int_0^{kt} ds \int_{\dot{\mathbb{R}}^3} dy p_{kt-s}^{\lambda_k \alpha}(k^{1/2} x, y) v^{1+\beta}(s, y). \end{aligned} \quad (2.5)$$

By a change of variable,

$$\begin{aligned} k^{1/\beta} v(kt, k^{1/2}x) &= \int_{\mathbb{R}^3} dy k^{3/2} p_{kt}^{\lambda_k \alpha}(k^{1/2}x, k^{1/2}y) \varphi(y) \\ &\quad - k^{1/\beta} \eta \int_0^t ds k \int_{\mathbb{R}^3} dy k^{3/2} p_{kt-ks}^{\lambda_k \alpha}(k^{1/2}x, k^{1/2}y) v^{1+\beta}(ks, k^{1/2}y). \end{aligned} \quad (2.6)$$

Hence, by Lemma 1.1,

$$\begin{aligned} k^{1/\beta} v(kt, k^{1/2}x) & \\ &= \int_{\mathbb{R}^3} dy p_t^{k^{1/2} \lambda_k \alpha}(x, y) \varphi(y) - k^{1/\beta} \eta \int_0^t ds k \int_{\mathbb{R}^3} dy p_{t-s}^{k^{1/2} \lambda_k \alpha}(x, y) v^{1+\beta}(ks, k^{1/2}y). \end{aligned} \quad (2.7)$$

Since  $1/\beta + 1 - (1/\beta)(1 + \beta) = 0$  we see that  $k^{1/\beta} v(kt, k^{1/2}x) =: w_k(t, x)$  satisfies the equation

$$\begin{aligned} w_k(t', x') &= \int_{\mathbb{R}^3} dy p_{t'}^{k^{1/2} \lambda_k \alpha}(x', y) \varphi(y) - \eta \int_0^{t'} ds \int_{\mathbb{R}^3} dy p_{t'-s}^{k^{1/2} \lambda_k \alpha}(x', y) w_k^{1+\beta}(s, y), \\ t' > 0, x' \in \mathbb{R}^3. \end{aligned}$$

By uniqueness of solutions of the log-Laplace equation (1.17) and by (1.15), claim (2.1) follows.  $\square$

**2.2. Proof of Theorem 1.2.** Fix  $\varphi \in \mathcal{C}_{\text{com}}^+(\mathbb{R}^3)$ . Using formula (1.18) for the first moment of  $X^\alpha$  and substitution, we obtain

$$\begin{aligned} k^{-1/2} \mathbb{E}_\mu \langle X_{kt}^{\lambda_k \alpha}, \varphi(k^{-1/2} \cdot) \rangle &= k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy p_{kt}^{\lambda_k \alpha}(x, y) \varphi(k^{-1/2}y) \\ &= k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy k^{3/2} p_{kt}^{\lambda_k \alpha}(x, k^{1/2}y) \varphi(y). \end{aligned} \quad (2.8)$$

By Lemma 1.1 this is equal to

$$k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy p_t^{k^{1/2} \lambda_k \alpha}(k^{-1/2}x, y) \varphi(y). \quad (2.9)$$

Using formula (1.5) for the expression of  $p^\alpha$ , we get three terms, we will deal with separately.

1° (*First term*). The first term equals,

$$k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy p_t(k^{-1/2}x, y) \varphi(y). \quad (2.10)$$

This double integral is finite and vanishes as  $k \uparrow \infty$ . To see this, let us restrict the outer integral first to  $|x| > K$  where we specify  $K \geq 1$  later. We call this restricted integral  $I_K$ . We use  $\varphi \leq C\phi$  (since  $\varphi \in \Phi$ ) and, with  $S$  denoting the free heat flow,

$$S_t \phi \leq C\phi, \quad t \geq 0, \quad (2.11)$$

with changed constant  $C$  (see [6, Lemma 2.1]) to arrive at

$$I_K \leq Ck^{-1/2} \int_{|x|>K} \mu(dx) \phi(k^{-1/2}x) = C \int_{|x|>K} \mu(dx) \phi(x), \quad (2.12)$$

the last step by the particular form of  $\phi$ . The latter integral can be made arbitrarily small (uniformly in  $k$ ) by choosing  $K$  sufficiently large (by our assumption on  $\mu$ ).

It remains to deal with the case  $|x| \leq K$  for fixed  $K$ . We split the internal integral in (2.10) as follows. First, if  $|k^{-1/2}x - y| \geq |y|/2$ , then

$$p_t(k^{-1/2}x, y) \leq p_t(|y|/2), \quad (2.13)$$

which leads to the bound

$$k^{-1/2} \int_{|x| \leq K} \mu(dx) \int_{\mathbb{R}^3} dy p_t(|y|/2) \varphi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty \quad (2.14)$$

(the  $\mu(dx)$ -integral is finite as  $\langle \mu, \phi \rangle < \infty$ ). On the other hand, if  $|k^{-1/2}x - y| < |y|/2$ , then  $-k^{-1/2}|x| + |y| < |y|/2$  which implies  $|y| < 2k^{-1/2}|x| \leq 2K$ . Hence as  $p_t(k^{-1/2}x, y) \leq Ct^{-3/2}$  and  $\varphi \leq C\phi$ , we get the upper estimate

$$C_t k^{-1/2} \int_{|x| \leq K} \mu(dx) \int_{|y| < 2K} dy \phi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty \quad (2.15)$$

(the  $dy$ -integral is finite, since we are in dimension three).

2° (*Second term*). The second term reads

$$\int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{2t}{|x||y|} p_t(k^{-1/2}|x| + |y|) \varphi(y) =: II_k. \quad (2.16)$$

Observe that,

$$p_t(k^{-1/2}|x| + |y|) \uparrow p_t(|y|) \quad \text{as } k \uparrow \infty. \quad (2.17)$$

We can apply the monotone convergence theorem to obtain the limit,

$$\lim_{k \uparrow \infty} II_k = \int_{\mathbb{R}^3} \mu(dx) \frac{2t}{|x|} \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) \varphi(y), \quad (2.18)$$

where finiteness follows from  $\langle \mu, \phi \rangle < \infty$  and  $\varphi \leq C\phi$ . Hence, this summand gives the first part of the kernel  $\vartheta^{\alpha^*}$ .

3° (*Third term*). It remains to insert the scaled third term from representation (1.5) into (2.9) which reads as

$$k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{-8\pi t k^{1/2} \lambda_k \alpha}{k^{-1/2}|x||y|} \int_0^\infty du e^{-4\pi k^{1/2} \lambda_k \alpha u} p_t(u + k^{-1/2}|x| + |y|) \varphi(y). \quad (2.19)$$

We distinguish several cases: If  $\lambda_k \alpha = 0$  for all sufficiently large  $k$ , then the third term disappears and we are done. From now on assume  $\lambda_k \alpha \neq 0$  for all  $k$ . Substituting  $u \mapsto 4\pi k^{1/2} |\lambda_k \alpha| u$  into (2.19) yields,

$$\int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{-2t \operatorname{sign}(\lambda_k \alpha)}{|x||y|} \int_0^\infty du e^{-\operatorname{sign}(\lambda_k \alpha) u} p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \varphi(y). \quad (2.20)$$

Now let  $k^{1/2} |\lambda_k \alpha| \rightarrow \infty$ . We may consider a monotone subsequence of  $k^{1/2} \lambda_k \alpha$ . Clearly,

$$p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \uparrow p_t(|y|) \quad \text{as } k \uparrow \infty, \quad (2.21)$$



and by monotone convergence, (2.20) converges along the subsequence to

$$-\int_{\mathbb{R}^3} \mu(dx) \frac{2t \operatorname{sign}(\alpha^*)}{|x|} \int_0^\infty du e^{-\operatorname{sign}(\alpha^*)u} \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) \varphi(y), \quad (2.22)$$

which is independent of the choice of the subsequence. Note that

$$\operatorname{sign}(\alpha^*) \int_0^\infty du e^{-\operatorname{sign}(\alpha^*)u} = \begin{cases} 1 & \text{if } \operatorname{sign}(\alpha^*) = 1, \\ +\infty & \text{if } \operatorname{sign}(\alpha^*) = -1. \end{cases} \quad (2.23)$$

In the first case the second and the third limiting terms cancel.

Next, let  $k^{1/2}\lambda_k\alpha \rightarrow 0$ . Note, that

$$p_t\left(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|} + k^{-1/2}|x| + |y|\right) \leq p_t\left(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|}\right) e^{-|y|^2/4t}. \quad (2.24)$$

In this case the double integral (2.20) is in absolute value bounded by

$$\int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{2t}{|x||y|} e^{-|y|^2/4t} \varphi(y) \int_0^\infty du e^u p_t\left(\frac{u}{4\pi k^{1/2}|\lambda_k\alpha|}\right), \quad (2.25)$$

which tends to 0 as  $k \uparrow \infty$  as the  $\mu(dx)$  and  $dy$ -integrals are finite and in the  $du$ -integral the  $p_t$ -term compensates the  $e^u$ .

It remains to deal with the case  $k^{1/2}\lambda_k\alpha \rightarrow \alpha^* \in \dot{\mathbb{R}}^1$ . Note, that we only have to justify to change the limit and integration in (2.20), as substituting  $u \mapsto (4\pi|\alpha^*|)^{-1}u$  leads to the desired expression. To justify the interchange, we estimate as in (2.24). The resulting  $\mu(dx)$  and  $dy$ -integrals are independent of  $k$  and finite, whereas to dominate in the second integral we use  $k^{1/2}|\lambda_k\alpha| \leq |\alpha^*| + 1$  for all sufficiently large  $k$ .  $\square$

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