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FILTERING OF CONTINUOUS TIME PERIODICALLY CORRELATED ISOTROPIC RANDOM FIELDS

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Abstract. The problem of optimal estimation of functionals depending on the unknown values of a mean-square continuous periodically correlated with respect to time argument and isotropic on the unit sphere with respect to spatial argument random field $\zeta(t,x)$ is considered. Estimates are based on observations of the field $\zeta(t,x)+\theta(t,x)$ at points $(t,x): t\leq 0, x\in S_n$, where $\theta(t,x)$ is uncorrelated with $\zeta(t,x)$ mean-square continuous periodically correlated with respect to time argument and isotropic on the sphere with respect to spatial argument random field. Formulas for calculating the mean square errors and spectral characteristics of the optimal linear estimates of functionals are derived in the case of spectral certainty where spectral densities of the fields are exactly known. Formulas that determine the least favourable spectral densities and the minimax (robust) spectral characteristics are proposed in the case where the spectral densities are not exactly known while a class of admissible spectral densities is given.

1. Introduction

The Einstein Cosmological Principle: the Universe is, in the large, homogeneous and isotropic (J. G. Bartlett [3]). Last decades indicate growing interest to the spatio-temporal data measured on the surface of a sphere. These data includes cosmic microwave background (CMB) anisotropies (J. G. Bartlett [3], W. Hu and S. Dodelson [21], N. Kogo and N. Komatsu [29], T. Okamoto and W. Hu [48], P. Adshead and W. Hu [1]), medical imaging (R. Kakarala [25]), global and land-based temperature data (P. D. Jones [23], T. Subba Rao and G. Terdik [52]), gravitational and geomagnetic data, climate model (G. R. North and R. F. Cahalan [47]). Basic results and references of the theory of isotropic random fields on a sphere can be found in books by M. I. Yadrenko [57] and A. M. Yaglom [58, 59]. For more recent applications and results see new books by C. Gaetan and X. Guyon [16], N. Cressie and C. K. Wikle [4], D. Marinucci and G. Peccati [31] and several papers covering a number of problems in general for spatial temporal isotropic observations (T. Subba Rao and G. Terdik [53], G. Terdik [54]).

Periodically correlated processes and fields are not homogeneous but have numerous properties similar to properties of stationary processes and homogeneous fields. They describe appropriate models of numerous physical and man-made

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processes. A comprehensive list of the existing references up to the year 2005 on periodically correlated processes and their applications was proposed by E. Serpedin, F. Panduru, I. Sari and G. B. Giannakis [51]. See also reviews by J. Antoni [2] and A. Napolitano [46]. For more details see a survey paper by W. A. Gardner [18] and book by H. L. Hurd and A. Miamee [22]. Note, that in the literature periodically correlated processes are named in multiple different ways such as cyclostationary, periodically nonstationary or cyclic correlated processes.

The mean square optimal estimation problems for periodically correlated with respect to time isotropic on a sphere random fields are natural generalization of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and homogeneous random fields. Effective methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and random fields were developed under the condition of certainty where spectral densities of processes and fields are known exactly (see, for example, selected works of A. N. Kolmogorov [30], survey article by T. Kailath [24], books by Yu. A. Rozanov [50], N. Wiener [56], A. M. Yaglom [58, 59], M. I. Yadrenko [57], articles by M. P. Moklyachuk and M. I. Yadrenko [43] - [44]).

The traditional approach to the problems of interpolation, extrapolation and filtering of stochastic processes and random fields is based on the assumption that the spectral densities of processes and fields are known. In practice, however, complete information about the spectral density is impossible in most cases. To overcome this complication one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then applies the traditional estimation method provided that the estimated or selected densities are the true ones. This procedure can result in a significant increasing of the value of error as K. S. Vastola and H. V. Poor [55] have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error of estimates. Such problems arise when considering problems of automatic control theory, coding and signal processing in radar and sonar, pattern recognition problems of speech signals and images. A comprehensive survey of results up to the year 1985 in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor [28]. J. Franke [14], J. Franke and H. V. Poor [15] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities. The paper by Ulf Grenander [20] should be marked as the first one where the minimax approach to extrapolation problem for the functionals from stationary processes was developed. For more details see, for example, survey articles M. P. Moklyachuk [36], [37], [40], books by M. Moklyachuk [38], M. Moklyachuk and O. Masytka [42], M. Moklyachuk and I. Golichenko [41]. In papers by I. I. Dubovets'ka, O.Yu. Masyutka and M.P. Moklyachuk [5], I. I. Dubovets'ka and M. P. Moklyachuk [6] - [9] the minimax-robust estimation problems (extrapolation, interpolation and filtering) are investigated for linear functionals which depend on unknown values of periodically correlated stochastic processes. Methods of solution the minimax-robust estimation problems for time-homogeneous isotropic random fields on a sphere were developed by M. P. Moklyachuk [33] - [35]. In papers by I. I. Dubovets'ka, O.Yu. Masyutka and M.P. Moklyachuk [10] - [12] results of investigation of minimax-robust estimation problems for periodically correlated isotropic random fields are proposed.

In this article we deal with the problem of mean square optimal linear estimation of the functional

$$A\zeta = \int_0^\infty \int_{S_n} a(t,x)\zeta(-t,x)\, m_n(dx)dt$$

which depends on unknown values of a periodically correlated (cyclostationary with period T) with respect to time isotropic on the unit sphere S_n in Euclidean space $\mathbb{E}^n$ random field $\zeta(t,x)$, $t \leq 0$, $x \in S_n$. Estimates are based on observations of the field $\zeta(t,x) + \theta(t,x)$ at points (t,x), $t \leq 0$, $x \in S_n$, where $\theta(t,x)$ is an uncorrelated with $\zeta(t,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field. Formulas are derived for computing the value of the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A\zeta$ in the case of spectral certainty, where spectral densities of the fields are known. Formulas are proposed that determine the least favourable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional $A\zeta$ for concrete classes of spectral densities in the case of spectral uncertainty, where spectral densities are not known exactly while classes $D = D_f \times D_g$ of admissible spectral densities are specified.

The contents of the article is the following.

In section 2 spectral properties of periodically correlated with respect to time parameter isotropic on a sphere random fields are described.

In section 3 the Hilbert space projection method of mean square optimal linear estimation of functionals which depend on unknown values of a periodically correlated (cyclostationary with period T) with respect to time parameter isotropic on the unit sphere S_n random field $\zeta(t,x)$, $t \leq 0$, $x \in S_n$ is analysed.

In section 4 the minimax-robust method of mean square optimal linear estimation of functionals is described. Basic definitions and statements which determine the least favourable spectral densities and the minimax-robust spectral characteristics of optimal estimates of functionals are presented.

In section 5 relations are derived which determine the least favourable spectral densities and the minimax-robust spectral characteristics for concrete classes of spectral densities.

2. Spectral properties of periodically correlated isotropic on a sphere random fields

Let S_n be a unit sphere in the n-dimensional Euclidean space $\mathbb{E}^n$, let $m_n(dx)$ be the Lebesgue measure on S_n, and let

$$S_m^l(x),\ l = 1, ..., h(m,n);\ m = 0, 1, ...$$

be the orthonormal spherical harmonics of degree m, where $h(m,n)$ is the number of orthonormal spherical harmonics (see books by A. Erdelyi et al. [13] and C. Müller [45] for more details).

A mean-square continuous random field $\zeta(t,x)$, $t\in\mathbb{R}$, $x\in S_n$, $\zeta(t,x)\in H = L_2(\Omega,F,\mathbb{P})$, where $L_2(\Omega,F,\mathbb{P})$ denotes the Hilbert space of random variables ζ with zero first moment, $\mathbb{E}\zeta=0$, and finite second moment, $\mathbb{E}|\zeta|^2<\infty$, is called periodically correlated (cyclostationary with period T) with respect to time isotropic on the sphere S_n if for all $t,s\in\mathbb{R}$ and $x,y\in S_n$ the following property holds true

$$\mathbb{E}\left(\zeta(t+T,x)\overline{\zeta(s+T,y)}\right) = B\left(t,s,\cos\vartheta\right),$$

where $\cos\vartheta=(x,y)$, ϑ is the angular distance between points $x,y\in S_n$.

The correlation function $B\left(t,s,\cos\vartheta\right)$ of the mean-square continuous random field $\zeta(t,x)$ is continuous. It can be represented in the form of the series

$$B\left(t,s,\cos\vartheta\right)=\frac{1}{\omega_n}\sum_{m=0}^{\infty}h(m,n)\frac{C_m^{(n-2)/2}(\cos\vartheta)}{C_m^{(n-2)/2}(1)}\ B_m^{\zeta}(t,s),$$

where $\omega_n=(2\pi)^{n/2}\Gamma(n/2)$, $C_m^l(z)$ are the Gegenbauer polynomials (see book by M. I. Yadrenko [57]).

It follows from the Karhunen theorem that the random field $\zeta(t,x)$ itself can be represented in the form of the mean square convergent series (see K. Karhunen [27], I. I. Gikhman and A. V. Skorokhod [17])

$$\zeta(t,x)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}S_m^l(x)\zeta_m^l(t), \tag{2.1}$$

where

$$\zeta_m^l(t)=\int_{S_n}\zeta(t,x)S_m^l(x)\,m_n(dx).$$

In this representation

$$\zeta_m^l(t),\ l=1,\dots,h(m,n);\ t\in\mathbb{R},m=0,1,\dots$$

are mutually uncorrelated periodically correlated stochastic processes with the correlation functions $B_m^{\zeta}(t,s)$:

$$\mathbb{E}\left(\zeta_m^l(t+T)\overline{\zeta_u^v(s+T)}\right)=\delta_m^u\delta_l^v\ B_m^{\zeta}(t,s),$$

$$l,v=1,\dots,h(m,n);\ m,u=0,1,\dots;\ t,s\in\mathbb{R},$$

where δ_l^v are the Kroneker delta-functions.

Consider two mutually uncorrelated periodically correlated with respect to time isotropic random fields $\zeta(t,x)$ and $\theta(t,x)$. We construct the following sequences of stochastic functions

$$\{\zeta_m^l(j,u)=\zeta_m^l(u+jT),u\in[0,T),j\in\mathbb{Z}\}, \tag{2.2}$$

$$\{\theta_m^l(j,u)=\theta_m^l(u+jT),u\in[0,T),j\in\mathbb{Z}\} \tag{2.3}$$

which correspond to the random fields $\zeta(t,x)$ and $\theta(t,x)$. The sequences (2.2) and (2.3) form the $L_2([0,T);H)$-valued stationary sequences $\{\zeta^l_m(j), j\in\mathbb{Z}\}$ and $\{\theta^l_m(j), j\in\mathbb{Z}\}$, respectively, with the correlation functions

$$R^\zeta_m(k,j)=\int_0^T \mathbb{E}\left[\zeta^l_m(u+kT)\overline{\zeta^l_m(u+jT)}\right]du$$
$$=\int_0^T B^\zeta_m(u+(k-j)T,u)du=R^\zeta_m(k-j),$$
$$R^\theta_m(k,j)=\int_0^T \mathbb{E}\left[\theta^l_m(u+kT)\overline{\theta^l_m(u+jT)}\right]du$$
$$=\int_0^T B^\theta_m(u+(k-j)T,u)du=R^\theta_m(k-j).$$

To describe properties of the stationary sequences $\{\zeta^l_m(j), j\in\mathbb{Z}\}$ and $\{\theta^l_m(j), j\in\mathbb{Z}\}$ we define in the space $L_2([0,T);\mathbb{R})$ the following orthonormal basis

$$\left\{\widetilde{e}_k=\frac{1}{\sqrt{T}}e^{2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}u/T}, k=1,2,\dots\right\},\ \langle\widetilde{e}_j,\widetilde{e}_k\rangle=\delta^j_k.$$

Making use of the introduced basis the stationary sequences $\{\zeta^l_m(j), j\in\mathbb{Z}\}$ and $\{\theta^l_m(j), j\in\mathbb{Z}\}$ can be represented as follows

$$\zeta^l_m(j)=\sum_{k=1}^\infty \zeta^l_{mk}(j)\widetilde{e}_k, \tag{2.4}$$

$$\zeta^l_{mk}(j)=\langle\zeta^l_m(j),\widetilde{e}_k\rangle=\frac{1}{\sqrt{T}}\int_0^T \zeta^l_m(j,v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv,$$

$$\theta^l_m(j)=\sum_{k=1}^\infty \theta^l_{mk}(j)\widetilde{e}_k, \tag{2.5}$$

$$\theta^l_{mk}(j)=\langle\theta^l_m(j),\widetilde{e}_k\rangle=\frac{1}{\sqrt{T}}\int_0^T \theta^l_m(j,v)e^{-2\pi i\{(-1)^k\left[\frac{k}{2}\right]\}v/T}dv.$$

Components of the constructed vector-valued stationary sequences $\{\zeta^l_m(j)=(\zeta^l_{mk}(j), k=1,2,\dots), j\in\mathbb{Z}\}$ and $\{\theta^l_m(j)=(\theta^l_{mk}(j), k=1,2,\dots), j\in\mathbb{Z}\}$ have the following properties [26], [32]

$$\mathbb{E}\zeta^l_{mk}(j)=0,\quad \|\zeta^l_m(j)\|^2_H=\sum_{k=1}^\infty \mathbb{E}|\zeta^l_{mk}(j)|^2=R^\zeta_m(0),$$
$$\mathbb{E}\zeta^l_{mk}(j_1)\overline{\zeta^l_{mn}(j_2)}=\langle K^\zeta_m(j_1-j_2)e_k,e_n\rangle,$$
$$\mathbb{E}\theta^l_{mk}(j)=0,\quad \|\theta^l_m(j)\|^2_H=\sum_{k=1}^\infty \mathbb{E}|\theta^l_{mk}(j)|^2=R^\theta_m(0),$$
$$\mathbb{E}\theta^l_{mk}(j_1)\overline{\theta^l_{mn}(j_2)}=\langle K^\zeta_m(j_1-j_2)e_k,e_n\rangle,$$

where $\{e_k, k=1,2,\dots\}$ is a basis in the space ℓ_2. The correlation functions $K^\zeta_m(j)$ and $K^\theta_m(j)$ of the stationary sequences $\{\zeta^l_m(j)=(\zeta^l_{mk}(j), k=1,2,\dots), j\in\mathbb{Z}\}$ and $\{\theta^l_m(j)=(\theta^l_{mk}(j), k=1,2,\dots), j\in\mathbb{Z}\}$ are correlation operator functions in ℓ_2.

The vector-valued stationary sequences $\{\zeta^l_m(j)=(\zeta^l_{mk}(j),k=1,2,\dots),j\in\mathbb{Z}\}$ and $\{\theta^l_m(j)=(\theta^l_{mk}(j),k=1,2,\dots),j\in\mathbb{Z}\}$ have the spectral density functions

$$F_m(\lambda)=\left\{f^{kn}_m(\lambda)\right\}^{\infty}_{k,n=1},\quad G_m(\lambda)=\left\{g^{kn}_m(\lambda)\right\}^{\infty}_{k,n=1},$$

that are operator-valued functions of variable $\lambda\in[-\pi,\pi)$ in the space ℓ_2 if their correlation functions $K^{\zeta}_m(j)$ and $K^{\theta}_m(j)$ can be represented in the form

$$\langle K^{\zeta}_m(j)e_k,e_n\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ij\lambda}\langle F_m(\lambda)e_k,e_n\rangle d\lambda,$$

$$\langle K^{\theta}_m(j)e_k,e_n\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ij\lambda}\langle G_m(\lambda)e_k,e_n\rangle d\lambda,$$

For almost all $\lambda\in[-\pi,\pi)$ the spectral densities $F_m(\lambda)$ and $G_m(\lambda)$ are kernel operators with integrable kernel norm

$$\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle F_m(\lambda)e_k,e_k\rangle d\lambda=\sum_{k=1}^{\infty}\langle K^{\zeta}_m(0)e_k,e_k\rangle=\|\zeta^l_m(j)\|^2_H=R^{\zeta}_m(0),$$

$$\sum_{k=1}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\langle G_m(\lambda)e_k,e_k\rangle d\lambda=\sum_{k=1}^{\infty}\langle K^{\theta}_m(0)e_k,e_k\rangle=\|\theta^l_m(j)\|^2_H=R^{\theta}_m(0).$$

In the following sections we explore the described spectral properties of random fields to find solution of the estimation problems.

3. Hilbert space projection method of filtering

Consider the problem of mean square optimal linear estimation of the functional

$$A\zeta=\int_0^{\infty}\int_{S_n}a(t,x)\zeta(-t,x)\,m_n(dx)dt$$

which depends on the unknown values of a periodically correlated with respect to time isotropic on the unit sphere S_n in Euclidean space $\mathbb{E}^n$ random field $\zeta(t,x)$, $t\leq 0$, $x\in S_n$. Estimates are based on observations of the field $\zeta(t,x)+\theta(t,x)$ at points (t,x), $t\leq 0$, $x\in S_n$, where $\theta(t,x)$ is an uncorrelated with $\zeta(t,x)$ periodically correlated with respect to time isotropic on the sphere S_n random field.

It follows from representation (2.1) that the functional $A\zeta$ can be represented in the form

$$A\zeta=\int_0^{\infty}\int_{S_n}a(t,x)\zeta(-t,x)m_n(dx)dt=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\int_0^{\infty}a^l_m(t)\zeta^l_m(-t)dt=$$

$$=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\sum_{j=0}^{\infty}\int_0^{T}a^l_m(j,u)\zeta^l_m(-j,-u)du,$$

$$a^l_m(t)=\int_{S_n}a(t,x)S^l_m(x)\,m_n(dx),$$

$$a^l_m(j,u)=a^l_m(u+jT),\ u\in[0,T),$$

$$\zeta^l_m(-j,-u)=\zeta^l_m(-u-jT),\ u\in[0,T).$$

Taking into account decomposition (2.4) of stationary sequence $\{\zeta_m^l(j), j \in \mathbb{Z}\}$, the functional $A\zeta$ can be represented in the following form

$$A\zeta = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} a_{mk}^l(j) \zeta_{mk}^l(-j) = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \vec{a}_m^l(j)^\top \vec{\zeta}_m^l(-j),$$

$$\vec{\zeta}_m^l(-j) = (\zeta_{mk}^l(-j), k = 1, 2, \dots)^\top,$$
$$\vec{a}_m^l(j) = (a_{mk}^l(j), k = 1, 2, \dots)^\top =$$
$$= (a_{m1}^l(j), a_{m3}^l(j), a_{m2}^l(j), \dots, a_{m(2k+1)}^l(j), a_{m(2k)}^l(j), \dots)^\top,$$
$$a_{mk}^l(j) = \langle a_m^l(j), \widetilde{e}_k \rangle = \frac{1}{\sqrt{T}} \int_0^T a_m^l(j, v) e^{-2\pi i \{(-1)^k \left[\frac{k}{2}\right]\} v/T} dv.$$

We will assume that coefficients $\{\vec{a}_m^l(j), j = 0, 1, \dots\}$ which form this representation satisfy the following conditions

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} \|\vec{a}_m^l(j)\| < \infty, \quad \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \sum_{j=0}^{\infty} (j+1) \|\vec{a}_m^l(j)\|^2 < \infty, \tag{3.1}$$

$$\|\vec{a}_m^l(j)\|^2 = \textstyle\sum_{k=1}^{\infty} |a_{mk}^l(j)|^2.$$

Under these condition the functional $A\zeta$ has finite second moment and operators defined below with the help of the coefficients $\{\vec{a}_m^l(j), j = 0, 1, \dots\}$ are compact.

Denote by $L_2(F)$ the Hilbert space of complex vector functions

$$h(\lambda) = \left\{ h_m^l(\lambda) : m = 0, 1, \dots; l = 1, 2, \dots, h(m,n) \right\}, \; h_m^l(\lambda) = \left\{ h_{mk}^l \right\}_{k=1}^{\infty},$$

that satisfy condition

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} (h_m^l(\lambda))^\top F_m(\lambda) \overline{h_m^l(\lambda)} d\lambda < \infty.$$

We denote by $L_2^-(F)$ the subspace of $L_2(F)$ generated by the functions

$$e^{ij\lambda} \delta_k, \delta_k = \{\delta_k^n\}_{n=1}^{\infty}, k = 1, 2, \dots, j \leq 0,$$

where $\delta_k^k = 1, \delta_k^n = 0$, $k \neq n$.

Every linear estimate $\hat{A}\zeta$ of the functional $A\zeta$ from observations of the sequence $\{\zeta_m^l(j) + \theta_m^l(j), j \in \mathbb{Z}\}$ at points $j \leq 0$ is defined by the spectral characteristic $h(\lambda) \in L_2^-(F+G)$ and is of the form

$$\hat{A}\zeta = \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \int_{-\pi}^{\pi} (h_m^l(\lambda))^\top Z_m^{l\,\zeta+\theta}(d\lambda), \tag{3.2}$$

where $Z_m^{l\,\zeta+\theta}(\Delta) = \{Z_{mk}^{l\,\zeta+\theta}(\Delta)\}_{k=1}^{\infty}$ is the orthogonal stochastic measure of sum of sequences $\zeta_m^l(j)$ and $\theta_m^l(j)$.

Suppose that spectral densities of stationary sequence $\{\zeta_m^l(j)\}, \{\theta_m^l(j)\}$ admit the canonical factorizations (G. Kallianpur and V. Mandrekar [26], M. P. Moklyachuk [32])

$$F_m(\lambda) = \varphi_m(\lambda)(\varphi_m(\lambda))^*, \; \varphi_m(\lambda) = \sum_{u=0}^{\infty} \varphi_m(u) e^{-iu\lambda}, \tag{3.3}$$

$$G_m(\lambda)=\psi_m(\lambda)(\psi_m(\lambda))^*,\ \psi_m(\lambda)=\sum_{u=0}^{\infty}\psi_m(u)e^{-iu\lambda}, \tag{3.4}$$

$$F_m(\lambda)+G_m(\lambda)=d_m(\lambda)(d_m(\lambda))^*,\ d_m(\lambda)=\sum_{u=0}^{\infty}d_m(u)e^{-iu\lambda}, \tag{3.5}$$

where matrices

$$d_m(u)=\{d^r_{mk}(u)\}_{k=\overline{1,\infty}}^{r=\overline{1,M}},\ \ \varphi_m(u)=\{\varphi^r_{mk}(u)\}_{k=\overline{1,\infty}}^{r=\overline{1,M_1}},\ \ \psi_m(u)=\{\psi^r_{mk}(u)\}_{k=\overline{1,\infty}}^{r=\overline{1,M_2}}$$

are coefficients of the canonical factorizations, M_1 is the multiplicity of $\zeta^l_m(j)$, M_2 is the multiplicity of $\theta^l_m(j)$ and M is the multiplicity of $\zeta^l_m(j)+\theta^l_m(j)$.

The mean square error $\Delta(h;F,G)$ of the linear estimate $\hat{A}\zeta$ with the spectral characteristic $h^l_m(\lambda)=\sum_{j=0}^{\infty}\vec{h}^l_m(j)e^{-ij\lambda}$ can be represented in the form

$$\Delta(h;F,G)=E|A\zeta-\hat{A}\zeta|^2=$$
$$=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}(\|\mathbf{\Psi^l_m a^l_m}\|^2+\|\mathbf{D_m(a^l_m-h^l_m)}\|^2-$$
$$-\langle\mathbf{\Psi_m(a^l_m-h^l_m)},\mathbf{\Psi_m a^l_m}\rangle-\langle\mathbf{\Psi_m a^l_m},\mathbf{\Psi_m(a^l_m-h^l_m)}\rangle),$$

where operators $\mathbf{\Psi}$, $\mathbf{D}$ are defined as follows

$$\|\mathbf{\Psi_m a^l_m}\|^2=\sum_{q=0}^{\infty}\|(\mathbf{\Psi_m a^l_m})_q\|^2,\ (\mathbf{\Psi_m a^l_m})_q=\sum_{j=0}^{q}(\psi_m(q-j))^{\top}\vec{a}^l_m(j),$$

$$\|\mathbf{D_m(a^l_m-h^l_m)}\|^2=\sum_{q=0}^{\infty}\|(\mathbf{D_m(a^l_m-h^l_m)})_q\|^2,$$

$$(\mathbf{D_m(a^l_m-h^l_m)})_q=\sum_{j=0}^{q}(d_m(q-j))^{\top}(\vec{a}^l_m(j)-\vec{h}^l_m(j)),$$

$$\langle\mathbf{\Psi_m(a^l_m-h^l_m)},\mathbf{\Psi_m a^l_m}\rangle=\sum_{q=0}^{\infty}\langle(\mathbf{\Psi_m(a^l_m-h^l_m)})_q,(\mathbf{\Psi_m a^l_m})_q\rangle.$$

The spectral characteristic $h(F,G)$ of the optimal linear estimate $\hat{A}\zeta$ of the functional minimizes the value of the mean square error

$$\Delta(F,G)=\Delta(h(F,G);F,G)=$$
$$=\min_{h\in L_2^-(F+G)}\Delta(h;F,G)=\min_{\hat{A}\zeta}E|A\zeta-\hat{A}\zeta|^2. \tag{3.6}$$

In the case where the spectral densities $G_m(\lambda)$ and $F_m(\lambda)+G_m(\lambda)$ admit factorizations (3.4) and (3.5), the spectral characteristic $h(F,G)$, which is a solution of the optimization problem (3.6), and the mean square error $\Delta(F,G)$ of the optimal estimate $\hat{A}\zeta$ are determined by formulas

$$h^l_m(F,G)=A^l_m(\lambda)-(b_m(\lambda))^{\top}C^l_m(G)(\lambda), \tag{3.7}$$

$$\Delta(F,G)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\|\mathbf{\Psi_m a^l_m}\|^2-\|\mathbf{B^*_m\Psi^*_m\Psi_m a^l_m}\|^2\right], \tag{3.8}$$

where

$$b_m(\lambda) = \{b_{mr}^k(\lambda)\}_{r=\overline{1,M}}^{k=\overline{1,\infty}}, \quad b_m(\lambda) = \sum_{u=0}^{\infty} b_m(u)e^{-iu\lambda}, \quad b_m(\lambda)d_m(\lambda) = I_M,$$

$$C_m^l(G)(\lambda) = \sum_{j=0}^{\infty}(C_m^l(G))_j e^{-ij\lambda}, \quad A_m^l(\lambda) = \sum_{j=0}^{\infty} \vec{a}_m^l(j)e^{-ij\lambda},$$

$$(C_m^l(G))_j = (\mathbf{B}^*\mathbf{\Psi}^*\mathbf{\Psi}\mathbf{a})_j = \sum_{q=0}^{\infty} \overline{b_m(q)}(\mathbf{\Psi_m^*\Psi_m a_m^l})_{j+q},$$

$$(\mathbf{\Psi_m^*\Psi_m a_m^l})_q = \sum_{u=0}^{\infty} \overline{\psi_m(u)}(\mathbf{\Psi_m a_m^l})_{u+q},$$

$$\|\mathbf{B_m^*\Psi_m^*\Psi_m a_m^l}\|^2 = \sum_{q=0}^{\infty} \|(\mathbf{B_m^*\Psi_m^*\Psi_m a_m^l})_q\|^2.$$

In the case where the spectral densities $F_m(\lambda)$ and $F_m(\lambda)+G_m(\lambda)$ admit factorizations (3.3) and (3.5), the spectral characteristic $h(F,G)$ and the mean square error $\Delta(F,G)$ of the optimal estimate $\hat{A}\zeta$ are defined by formulas

$$h_m^l(F,G) = (b_m(\lambda))^\top C_m^l(F)(\lambda), \tag{3.9}$$

$$\Delta(F,G) = \sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)} \left[\|\mathbf{\Phi_m a_m^l}\|^2 - \|\mathbf{B_m^*\Phi_m^*\Phi_m a_m^l}\|^2\right], \tag{3.10}$$

$$C_m^l(F)(\lambda) = \sum_{j=0}^{\infty}(C_m^l(F))_j e^{-ij\lambda},$$

$$(C_m^l(F))_j = (\mathbf{B_m^*\Phi_m^*\Phi_m a_m^l})_j = \sum_{q=0}^{\infty} \overline{b_m(q)}(\mathbf{\Phi_m^*\Phi_m a_m^l})_{j+q},$$

$$(\mathbf{\Phi_m^*\Phi_m a_m^l})_q = \sum_{u=0}^{\infty} \overline{\varphi_m(u)}(\mathbf{\Phi_m a_m^l})_{u+q},$$

$$(\mathbf{\Phi_m a_m^l})_q = \sum_{j=0}^{q}(\varphi_m(q-j))^\top \vec{a}_m^l(j), \quad \|\mathbf{\Phi_m a_m^l}\|^2 = \sum_{q=0}^{\infty}\|(\mathbf{\Phi_m a_m^l})_q\|^2,$$

$$\|\mathbf{B_m^*\Phi_m^*\Phi_m a_m^l}\|^2 = \sum_{q=0}^{\infty} \|(\mathbf{B_m^*\Phi_m^*\Phi_m a_m^l})_q\|^2.$$

Let us summarize our results and present them in the form of a theorem.

Theorem 3.1. *Let $\{\zeta(t,x), t\in\mathbb{R}, x\in S_n\}$ and $\{\theta(t,x), t\in\mathbb{R}, x\in S_n\}$ be mutually uncorrelated random fields, which are periodically correlated with respect to time argument $t\in\mathbb{R}$ and isotropic on the unit sphere S_n with respect to spatial argument $x\in S_n$. Let the stationary sequences $\{\zeta_m^l(j), j\in\mathbb{Z}\}$ and $\{\theta_m^l(j), j\in\mathbb{Z}\}$ constructed with the help of relations (2.2), (2.3), respectively, have spectral densities $F_m(\lambda)$ and $G_m(\lambda)$ that admit the canonical factorizations (3.3), (3.5) (or (3.4), (3.5)). Let coefficients $\{\vec{a}_m^l(j), j=0,1,\dots\}$ that determine the functional $A\zeta$ satisfy conditions (3.1). Then the spectral characteristic $h(F,G)$ and the mean* *square error $\Delta(F,G)$ of the optimal estimate of the functional $A\zeta$ from observations of the field $\zeta(t,x)+\theta(t,x)$ at points (t,x), $t\leq 0$, $x\in S_n$ are given by formulas (3.9), (3.10) (or (3.7), (3.8)), respectively. The optimal estimate $\hat{A}\zeta$ of the functional $A\zeta$ is calculated by formula (3.2).*

4. Minimax-robust method of filtering

Formulas (3.7) – (3.10) can be applied for calculating the spectral characteristic and the mean square error of the optimal linear estimate of the functional $A\zeta$ in the case where the spectral densities $F_m(\lambda)$ and $G_m(\lambda)$ of stationary sequences $\{\zeta_m^l(j), j\in\mathbb{Z}\}$ and $\{\theta_m^l(j), j\in\mathbb{Z}\}$ constructed by relations (2.2), (2.3), are known. If these spectral densities are not exactly known while a set of admissible densities $D=D_F\times D_G$ is specified, then the minimax approach to estimation of the functional is reasonable. That is we find the estimate which minimizes the mean square error for all spectral densities from a given set $D=D_F\times D_G$ simultaneously.

Definition 4.1. For a given class of spectral densities $D=D_F\times D_G$ the spectral densities $F_m^0(\lambda)\in D_F$ and $G_m^0(\lambda)\in D_G$ are called the least favorable in D for the optimal estimation of functional $A\zeta$ if

$$\Delta(F^0,G^0)=\Delta(h(F^0,G^0);F^0,G^0)=\max_{(F,G)\in D}\Delta(h(F,G);F,G).$$

Definition 4.2. For a given class of spectral densities $D=D_F\times D_G$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\zeta$ is called minimax-robust if the following relations hold true

$$h^0(\lambda)\in H_D=\bigcap_{(F,G)\in D} L_2^-(F+G),$$
$$\min_{h\in H_D}\max_{(F,G)\in D}\Delta(h;F,G)=\max_{(F,G)\in D}\Delta(h^0;F,G).$$

Taking into account the introduced definitions and derived relations (3.3) – (3.10) we can verify that the following lemmas hold true.

Lemma 4.3. *Spectral densities $F_m^0(\lambda)\in D_F$ and $G_m^0(\lambda)\in D_G$ which admit the canonical factorizations (3.3) - (3.5) are the least favorable in the class $D=D_F\times D_G$ for the optimal linear estimation of the functional $A\zeta$ if coefficients of factorizations define a solution of the constrained optimization problem*

$$\begin{gathered}\Delta(F,G)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\|\mathbf{\Phi_m a_m^l}\|^2-\|\mathbf{B_m^*\Phi_m^*\Phi_m a_m^l}\|^2\right]\to sup,\\ F_m(\lambda)=\varphi_m(\lambda)(\varphi_m(\lambda))^*\in D_F,\\ G_m(\lambda)=d_m(\lambda)(d_m(\lambda))^*-\varphi_m(\lambda)(\varphi_m(\lambda))^*\in D_G,\end{gathered}\tag{4.1}$$

or the constrained optimization problem

$$\begin{gathered}\Delta(F,G)=\sum_{m=0}^{\infty}\sum_{l=1}^{h(m,n)}\left[\|\mathbf{\Psi_m a_m^l}\|^2-\|\mathbf{B_m^*\Psi_m^*\Psi_m a_m^l}\|^2\right]\to sup,\\ G_m(\lambda)=\psi_m(\lambda)(\psi_m(\lambda))^*\in D_G,\\ F_m(\lambda)=d_m(\lambda)(d_m(\lambda))^*-\psi_m(\lambda)(\psi_m(\lambda))^*\in D_F.\end{gathered}\tag{4.2}$$

Lemma 4.4. *Let the spectral density $F_m(\lambda)$ be given and admits the factorization (3.3). Then spectral density $G^0_m(\lambda)$ is the least favorable in D_G for the optimal estimation of the functional $A\zeta$ if*

$$F_m(\lambda) + G^0_m(\lambda) = d^0_m(\lambda)(d^0_m(\lambda))^*,$$

where $d^0_m(\lambda) = \sum_{u=0}^{\infty} d^0_m(u)e^{-iu\lambda}$ and coefficients $\{d^0_m(u), u = 0, 1, \dots\}$ are determined by solution of the constrained optimization problem

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left[\|\mathbf{B}^*_{\mathbf{m}}\mathbf{\Phi}^*_{\mathbf{m}}\mathbf{\Phi}_{\mathbf{m}}\mathbf{a}^{\mathbf{l}}_{\mathbf{m}}\|^2\right] \to inf, \qquad G_m(\lambda) = d_m(\lambda)(d_m(\lambda))^* - F_m(\lambda) \in D_G. \tag{4.3}$$

Lemma 4.5. *Let the spectral density $G_m(\lambda)$ be given and admits the factorization (3.4). Then the spectral density $F^0_m(\lambda)$ is the least favorable in D_F for optimal estimation of the functional $A\zeta$ and admits canonical factorizations (3.3), (3.5) if*

$$F^0_m(\lambda) + G_m(\lambda) = d^0_m(\lambda)(d^0_m(\lambda))^*,$$

where $d^0_m(\lambda) = \sum_{u=0}^{\infty} d^0_m(u)e^{-iu\lambda}$ and coefficients $\{d^0_m(u), u = 0, 1, \dots\}$ are determined by solution of the constrained optimization problem

$$\sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left[\|\mathbf{B}^*_{\mathbf{m}}\mathbf{\Psi}^*_{\mathbf{m}}\mathbf{\Psi}_{\mathbf{m}}\mathbf{a}^{\mathbf{l}}_{\mathbf{m}}\|^2\right] \to inf, \qquad F_m(\lambda) = d_m(\lambda)(d_m(\lambda))^* - G_m(\lambda) \in D_F. \tag{4.4}$$

For more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics we observe that the least favorable spectral densities $F^0_m(\lambda) \in D_F$, $G^0_m(\lambda) \in D_G$ and the minimax spectral characteristic $h^0 = h(F^0, G^0)$ form a saddle point of the function $\Delta(h; F, G)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta(h^0; F, G) \leq \Delta(h^0; F^0, G^0) \leq \Delta(h; F^0, G^0),$$
$$\forall h \in H_D, \quad \forall F \in D_F, \quad \forall g \in D_G$$

hold if $h^0 = h(F^0, G^0)$, $h(F^0, G^0) \in H_D$ and (F^0, G^0) is a solution of the constrained optimization problem

$$\Delta(h(F^0, G^0); F, G) \to sup, \quad (F, G) \in D, \tag{4.5}$$

where the functional

$$\Delta(h(F^0, G^0); F, G) =$$
$$= \sum_{m=0}^{\infty} \sum_{l=1}^{h(m,n)} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} (C^l_m(G^0)(\lambda)^{\top} b^0_m(\lambda) F_m(\lambda) \left(b^0_m(\lambda)\right)^* \overline{C^l_m(G^0)(\lambda)} d\lambda + \right.$$
$$\left. + \frac{1}{2\pi}\int_{-\pi}^{\pi} (C^l_m(F^0)(\lambda))^{\top} b^0_m(\lambda) G_m(\lambda) \left(b^0_m(\lambda)\right)^* \overline{C^l_m(F^0)(\lambda)} d\lambda\right]. \tag{4.6}$$

The constrained optimization problem (4.5) is equivalent to the following unconstrained optimization problem

$$\Delta_D(F, G) = -\Delta(h(F^0, G^0); F, G) + \delta((F, G)|D) \to inf, \tag{4.7}$$

where $\delta((F,G)|D)$ is the indicator function of the set D. Solution $(F^0(\lambda),G^0(\lambda))$ to the optimization problem (4.7) is determined by the condition $0\in\partial\Delta_D(F^0,G^0)$ which is necessary for the point (F^0,G^0) to belong to the set of minimums of a convex functional. Here $\partial\Delta_D(F^0,G^0)$ is a subdifferential of the convex functional $\Delta_D(F,G)$ at point $(F,G)=(F^0,G^0)$ (see R. T. Rockafellar [49], M. P. Moklyachuk [39] for more details).

The form (4.6) of the functional $\Delta(h(F^0,G^0);F,G)$ is convenient for application the method of Lagrange multipliers for finding solution to the problem (4.7). Making use of the method of Lagrange multipliers and the form of subdifferentials of the indicator functions $\delta((F,G)|D)$ we describe relations that determine the least favourable spectral densities in some special classes of spectral densities (see books by M. Moklyachuk [38], M. Moklyachuk and O. Masytka [42], M. Moklyachuk and I. Golichenko [41] for more details).

5. The least favorable spectral densities in the class $D_0\times D_V^U$

Consider the problem of minimax estimation of the functional $A\zeta$ depending on the unknown values of the random field $\{\zeta(t,x),t\in\mathbb{R},x\in S_n\}$, which is periodically correlated with respect to the time argument $t\in\mathbb{R}$ and isotropic on the sphere S_n with respect to spatial argument $x\in S_n$ based on observations of the random field $\zeta(t,x)+\theta(t,x)$ at points $(t,x):t\le 0,x\in S_n$, under the condition that spectral densities $F_m(\lambda)$, $G_m(\lambda)$ of stationary sequences $\{\zeta_m^l(j),j\in\mathbb{Z}\}$ and $\{\theta_m^l(j),j\in\mathbb{Z}\}$ constructed with the help of relations (2.2), (2.3), respectively, are not known exactly while the following pairs of sets of admissible spectral densities are specified.

The first pair is

$$D_0^1=\left\{F(\lambda)|\,\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}\mathrm{Tr}\,F_m(\lambda)d\lambda=p\right\},$$

$$D_V^{U\,1}=\Big\{G(\lambda)|TrV_m(\lambda)\le\mathrm{Tr}\,G_m(\lambda)\le TrU_m(\lambda),$$
$$\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}\mathrm{Tr}\,G_m(\lambda)d\lambda=q\Big\}.$$

The second pair of sets of admissible spectral densities is

$$D_0^2=\left\{F(\lambda)|\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}F_m^{kk}(\lambda)d\lambda=p_k,k=1,2,\dots\right\},$$

$$D_V^{U\,2}=\Big\{G(\lambda)|V_m^{kk}(\lambda)\le G_m^{kk}(\lambda)\le U_m^{kk}(\lambda),$$
$$\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}G_m^{kk}(\lambda)d\lambda=q_k,k=1,2,\dots\Big\}.$$

The third pair of sets of admissible spectral densities is

$$D_0^3=\left\{F(\lambda)|\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}\langle B,F_m(\lambda)\rangle\,d\lambda=p\right\},$$

$$D_V^{U\,3}=\Big\{G(\lambda)|\,\langle B_2,V_m(\lambda)\rangle\leq\langle B_2,G_m(\lambda)\rangle\leq\langle B_2,U_m(\lambda)\rangle\,,$$

$$\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}\langle B_2,G_m(\lambda)\rangle\,d\lambda=q\Big\}.$$

The forth pair of sets of admissible spectral densities is

$$D_0^4=\left\{F(\lambda)|\,\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}F_m(\lambda)d\lambda=P\right\},$$

$$D_V^{U\,4}=\Big\{G(\lambda)|V_m(\lambda)\leq G_m(\lambda)\leq U_m(\lambda),$$

$$\frac{1}{2\pi\omega_n}\sum_{m=0}^{\infty}h(m,n)\int_{-\pi}^{\pi}G_m(\lambda)d\lambda=Q\Big\}.$$

Here $V_m(\lambda),U_m(\lambda)$ are given matrices of spectral densities, $p,q,p_k,q_k,k=1,2,\dots$ are given numbers, B_1,B_2,P,Q are given positive-definite Hermitian matrices.

From the condition $0\in\partial\Delta_D(F^0,G^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair $D_0^1\times D_V^{U\,1}$ we have equations

$$\sum_{l=1}^{h(m,n)}C_m^l(G^0)(\lambda)(C_m^l(G^0)(\lambda))^*=\alpha_m^2 d_m^0(\lambda)^\top\overline{d_m^0(\lambda)},\tag{5.1}$$

$$\sum_{l=1}^{h(m,n)}C_m^l(F^0)(\lambda)(C_m^l(F^0)(\lambda))^*=(\beta_m^2+\gamma_{m_1}(\lambda)+\gamma_{m_2}(\lambda))d_m^0(\lambda)^\top\overline{d_m^0(\lambda)},\tag{5.2}$$

where

$$\gamma_{m_1}(\lambda)\leq 0 \text{ and } \gamma_{m_1}(\lambda)=0 \text{ if } \operatorname{Tr}G_m^0(\lambda)>\operatorname{Tr}V_m(\lambda),$$
$$\gamma_{m_2}(\lambda)\geq 0 \text{ and } \gamma_{m_2}(\lambda)=0 \text{ if } \operatorname{Tr}G_m^0(\lambda)<\operatorname{Tr}U_m(\lambda),$$

and α_m^2,β_m^2 are unknown Lagrange multipliers.

For the second pair $D_0^2\times D_V^{U\,2}$ we have equations

$$\sum_{l=1}^{h(m,n)}C_m^l(G^0)(\lambda)(C_m^l(G^0)(\lambda))^*=d_m^0(\lambda)^\top\left\{\alpha_{mk}^2\delta_{kl}\right\}_{k,l=1}^{\infty}\overline{d_m^0(\lambda)},\tag{5.3}$$

$$\sum_{l=1}^{h(m,n)}C_m^l(F^0)(\lambda)(C_m^l(F^0)(\lambda))^*=$$
$$=d_m^0(\lambda)^\top\left\{(\beta_{mk}^2+\gamma_{m_1k}(\lambda)+\gamma_{m_2k}(\lambda))\delta_k^l\right\}_{k,l=1}^{\infty}\overline{d_m^0(\lambda)},\tag{5.4}$$

where

$$\gamma_{m_1k}(\lambda)\leq 0 \text{ and } \gamma_{m_1k}(\lambda)=0 \text{ if } G_m^{0kk}(\lambda)>V_m^{kk}(\lambda),$$
$$\gamma_{m_2k}(\lambda)\geq 0 \text{ and } \gamma_{m_2k}(\lambda)=0 \text{ if } G_m^{0kk}(\lambda)<U_m^{kk}(\lambda),$$

and $\alpha_{mk}^2,\beta_{mk}^2$ are unknown Lagrange multipliers.

For the third pair $D_0^3 \times D_V^{U3}$ we have equations

$$\sum_{l=1}^{h(m,n)} C_m^l(G^0)(\lambda)(C_m^l(G^0)(\lambda))^* = \alpha_m^2 d_m^0(\lambda)^\top B_1 \overline{d_m^0(\lambda)}, \tag{5.5}$$

$$\sum_{l=1}^{h(m,n)} C_m^l(F^0)(\lambda)(C_m^l(F^0)(\lambda))^* = (\beta_m^2 + \gamma_{m_1}(\lambda) + \gamma_{m_2}(\lambda)) d_m^0(\lambda)^\top B_2 \overline{d_m^0(\lambda)}; \tag{5.6}$$

where

$$\gamma_{m_1}(\lambda) \leq 0 \text{ and } \gamma_{m_1}(\lambda) = 0 \text{ if } \left\langle B_2, G_m^0(\lambda)\right\rangle > \left\langle B_2, V_m(\lambda)\right\rangle,$$
$$\gamma_{m_2}(\lambda) \geq 0 \text{ and } \gamma_{m_2}(\lambda) = 0 \text{ if } \left\langle B_2, G_m^0(\lambda)\right\rangle < \left\langle B_2, U_m(\lambda)\right\rangle,$$

and α_m^2, β_m^2 are unknown Lagrange multipliers.

For the forth pair $D_0^4 \times D_V^{U4}$ we have equations

$$\sum_{l=1}^{h(m,n)} C_m^l(G^0)(\lambda)(C_m^l(G^0)(\lambda))^* = d_m^0(\lambda)^\top \vec{\alpha_m} \cdot \vec{\alpha_m}^* \overline{d_m^0(\lambda)}, \tag{5.7}$$

$$\sum_{l=1}^{h(m,n)} C_m^l(F^0)(\lambda)(C_m^l(F^0)(\lambda))^* = d_m^0(\lambda)^\top (\vec{\beta} \cdot \vec{\beta}^* + \Gamma_{m_1}(\lambda) + \Gamma_{m_2}(\lambda)) \overline{d_m^0(\lambda)}. \tag{5.8}$$

where $\Gamma_{m_1}(\lambda), \Gamma_{m_2}(\lambda)$ are Hermitian matrices,

$$\Gamma_{m_1}(\lambda) \leq 0 \text{ and } \Gamma_{m_1}(\lambda) = 0 \text{ if } G_m^0(\lambda) > V_m(\lambda),$$
$$\Gamma_{m_2}(\lambda) \geq 0 \text{ and } \Gamma_{m_2}(\lambda) = 0 \text{ if } G_m^0(\lambda) < U_m(\lambda),$$

and $\vec{\alpha_m}, \vec{\beta_m}$ are unknown Lagrange multipliers.

Theorem 5.1. *The least favorable spectral densities $F_m^0(\lambda)$, $G_m^0(\lambda)$ in the classes $D_0 \times D_V^U$ for the optimal estimate of the functional $A\zeta$ are determined by relations* (5.1), (5.2) *for the first pair $D_0^1 \times D_V^{U1}$ of sets of admissible spectral densities (*(5.3), (5.4) *for the second pair $D_0^2 \times D_V^{U2}$ of sets of admissible spectral densities,* (5.5), (5.6) *for the third pair $D_0^3 \times D_V^{U3}$ of sets of admissible spectral densities,* (5.7), (5.8) *for the fourth pair $D_0^4 \times D_V^{U4}$ of sets of admissible spectral densities), factorizations (3.3), (3.4), (3.5), constrained optimization problem (4.1) or (4.2), and restrictions on densities from the corresponding classes $D_0 \times D_V^U$. The minimax spectral characteristic $h(F^0, G^0)$ of the optimal estimate $\hat{A}\zeta$ is calculated by (3.9) or (3.7). The mean square error $\Delta(F^0, G^0)$ is calculated by (3.10) or (3.8).*

In the case where one of spectral densities $F_m(\lambda)$ or $G_m(\lambda)$ from the corresponding classes is known we have the following corollary from the theorem.

Corollary 5.2. *If the spectral density $F_m(\lambda) \in D_0$ is known and admits the canonical factorization (3.3), then the least favorable spectral densities $G_m^0(\lambda)$ in the classes D_V^{Uk}, $k = 1, 2, 3, 4$ are determined by relations (3.4), (3.5), (4.3), equations* (5.2), (5.4), (5.6), (5.8) *correspondingly to $k = 1, 2, 3, 4$ and by restrictions on densities from classes D_V^{Uk}, $k = 1, 2, 3, 4$. If the spectral density $G_m(\lambda) \in D_V^U$ is known and admits the canonical factorization (3.4), then the least favorable spectral densities $F_m^0(\lambda)$ in the classes D_0^k, $k = 1, 2, 3, 4$ are determined by relations (3.3),* *(3.5), (4.4), equations* (5.1), (5.3), (5.5), (5.7)*) correspondingly to* $k = 1, 2, 3, 4$ *and by restrictions on densities from classes* $D_V^{U^k}$*,* $k = 1, 2, 3, 4$*. The minimax spectral characteristic* $h(F^0, G^0)$ *of the optimal estimate* $\hat{A}\zeta$ *is calculated by (3.9) or (3.7). The mean square error* $\Delta(F^0, G^0)$ *is calculated by (3.10) or (3.8).*

6. Conclusions

In this paper we propose formulas for calculating the mean square error and the spectral characteristic of the optimal linear estimate of the functional

$$A\zeta = \int_0^\infty \int_{S_n} a(t,x)\zeta(-t,x)\, m_n(dx)dt$$

depending on unknown values of a mean-square continuous periodically correlated (cyclostationary with period T) with respect to time argument and isotropic on the unit sphere S_n in Euclidean space $\mathbb{E}^n$ random field $\zeta(t,x)$, $t \in \mathbb{R}$, $x \in S_n$. Estimates are based on observations of the field $\zeta(t,x) + \theta(t,x)$ at points (t,x), $t \leq 0$, $x \in S_n$, where $\theta(t,x)$ is an uncorrelated with $\zeta(t,x)$ mean-square continuous periodically correlated with respect to time argument and isotropic on the sphere S_n random field. The problem is investigated in the case of spectral certainty where matrices of spectral densities of random fields are known exactly and in the case of spectral uncertainty where matrices of spectral densities of random fields are not known exactly while some classes of admissible spectral density matrices are given. We derive formulas for calculating the spectral characteristic and the mean-square error of the optimal linear estimate of the functional $A\zeta$ in the case of spectral certainty, where spectral densities $F_m(\lambda), G_m(\lambda)$ of the stationary sequences that generate the random fields $\zeta(t,x)$, $\theta(t,x)$ are known exactly.

We propose a representation of the mean square error in the form of a linear functional in the $L_1 \times L_1$ space with respect to spectral densities (F, G), which allows us to solve the corresponding constrained optimization problem and describe the minimax (robust) estimates of the functional $A\zeta$ for concrete classes of spectral densities under the condition that spectral densities are not known exactly while classes $D = D_f \times D_g$ of admissible spectral densities are specified.

In the forthcoming articles we will propose solution to the interpolation and extrapolation problems for functionals depending on unknown values of mean-square continuous periodically correlated with respect to time argument and isotropic on the unit sphere random fields in the case of spectral certainty as well as in the case of spectral uncertainty. The minimax-robust estimation technique will be developed for the interpolation and extrapolation problems.

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