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EXACT SOLUTIONS OF WAVE-TYPE EQUATIONS BY THE ABOODH DECOMPOSITION METHOD

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Abstract. This paper deals with the derivation of exact solutions of linear and nonlinear wave-type equations by employing the Aboodh transform coupled to the Adomian decomposition method. The new method is based on the derivation of convergent series and, with respect to the existing methods of the pertinent literature, does not require discretization, perturbation and linearization.

1. Introductions

Several mathematical methods have been proposed to treat wave-type equations among which is the Adomian decomposition method [1-2]. The Adomian decomposition method is an efficient method for solving linear and nonlinear, homogeneous and nonhomogeneous, ordinary and partial, integro-differential and fractional differential equations [10]. Kaya [3] and Kaya and Inc [4] used the Adomian decomposition method to solve nonlinear wave equations, while Momani [5] determined the analytical approximate solutions for fractional wave-like equations with variable coefficients using the decomposition method. Ghoreishi *et al* [6] solved nonlinear wave-like equations with variable coefficients using the Adomian decomposition method, while Ramadan and Al-Luhaibi [11] determined the solution of nonlinear wave-like equations using Sumudu decomposition method. Furthermore, in 2013, an integral transform called the Aboodh transform has been proposed by K.S. Aboodh [7] in his effort to devise more methods for solving ordinary and partial differential equations. He also applied it to solve some partial differential equations in [8], and further coupled the new integral transform with the homotopy perturbation method to solve some fourth order parabolic partial differential equations [9], and finally Nuruddeen and Nass [13] solved some linear and nonlinear heat equations using the Aboodh decomposition method.

This paper deals with the derivation of exact solutions of linear and nonlinear wave-type equations by employing the Aboodh transform coupled to the Adomian decomposition method. The new method is based on the derivation of convergent series and, with respect to the existing methods of the pertinent literature, does not require discretization, perturbation and linearization.

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The paper is organized as follows: In Section 2, we present the concept of the Aboodh transform and some of its properties. Section 3 presents the Aboodh decomposition method. In Section 4, we apply the Aboodh decomposition method to solve some wave-type equations in order to show the potential of the method proposed in Section 2. Finally, Section 5 is devoted to the conclusion.

2. The Aboodh Transform

The Aboodh transform is a new integral transform similar to the Laplace transform and other integral transforms that are defined in the time domain $t \geq 0$, such as the Sumudu transform [14], the Natural transform [15] and the Elzaki transform [16], respectively.

An Aboodh transform is defined for functions of exponential order. We consider functions in the set F defined by;

$$F = \{u(t) : |u(t)| < Me^{-vt},\ if \ \ t \in [0, \infty), M, k_1, k_2 > 0, k_1 \leq v \leq k_2\ \}.$$

For a given function in the set F, the constant M must be finite number and k_1, k_2 may be finite or infinite with variable v define as $k_1 \leq v \leq k_2$.

Then, the Aboodh integral transform denoted by the operator $A(.)$ is defined (by [7]) by the integral equation:

$$A\{u(t)\} = \frac{1}{v}\int_0^\infty u(t)e^{-vt}dt, \quad t \geq 0, \quad k_1 \leq v \leq k_2. \tag{2.1}$$

The Aboodh transforms of some elementary functions are given in Table 1.

Table 1. The Aboodh transforms of some elementary functions

$u(t)$	$A\{u(t)\}$
1	$\frac{1}{v^2}$
$t^n, \quad n = 1, 2, 3, ...$	$\frac{n!}{v^{n+2}}$
e^{-at}	$\frac{1}{v^2+av}$
$\sin(at)$	$\frac{a}{v(v^2+a^2)}$

Furthermore, the Aboodh transform is linear, i.e., if a and b are any constants and $u(t)$ and $w(t)$ are functions defined over the set F above, then

$$A\{au(t) + bw(t)\} = aA\{u(t)\} + bA\{w(t)\}.$$

Also, for any given function $u(t)$ defined over the set F, the Aboodh transform of derivatives are given as follows

(1) $A\{u^{'}(t)\} = vA\{u(t)\} - \frac{u(0)}{v}$,
(2) $A\{u^{''}(t)\} = v^2A\{u(t)\} - \frac{u'(0)}{v} - u(0)$,
(3) $A\{u^n(t)\} = v^nA\{u(t)\} - \sum_{k=0}^{n-1} \frac{u^k(0)}{v^{2-n+k}}$.

3. The Method of the Solutions

This section deals with the presentation of the method for second order nonhomogeneous partial differential equation of the form

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t), \tag{3.1}$$

with the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in [a,b],\ 0 < t \leq T, \tag{3.2}$$

where L is the second-order invertible derivative $\frac{\partial^2}{\partial^2 t}$, R is the remaining linear operator of order one, $Nu(x,t)$ includes the nonlinear terms, and $h(x,t)$ is the nonhomogeneous term.

By taking the Aboodh transform of equation (3.1) with respect to t, we get

$$A\{Lu(x,t)\} + A\{Ru(x,t)\} + A\{Nu(x,t)\} = A\{h(x,t)\}, \tag{3.3}$$

and from the differentiation property of the Aboodh transform, equation (3.3) becomes

$$v^2 A\{u(x,t)\} - \frac{1}{v}u_t(x,0) - u(x,0) + A\{Ru(x,t)\} + A\{Nu(x,t)\} = A\{h(x,t)\}, \tag{3.4}$$

which can be simplified as

$$\begin{aligned} A\{u(x,t)\} - \frac{1}{v^3}u_t(x,0) - \frac{1}{v^2}u(x,0) + \frac{1}{v^2}A\{Ru(x,t)\} \\ + \frac{1}{v^2}A\{Nu(x,t)\} = \frac{1}{v^2}A\{h(x,t)\}. \end{aligned} \tag{3.5}$$

The solution $u(x,t)$ is assumed to be the sum of the following series

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t), \tag{3.6}$$

and the nonlinear term $Nu(x,t)$ to be replaced by the infinite series of the Adomian polynomials [1-2] given by

$$Nu(x,t) = \sum_{m=0}^{\infty} A_m(u_0, u_1, u_2, ...), \quad m = 0,1,2,..., \tag{3.7}$$

where,

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\Big[N\Big(\sum_{i=0}^{\infty}\lambda^i u_i\Big)\Big]_{\lambda=0}, \quad m = 0,1,2,... \tag{3.8}$$

Using equations (3.6) and (3.7) in equation equation (3.5) we obtain

$$\begin{aligned} A\{\sum_{m=0}^{\infty} u_m(x,t)\} = \frac{1}{v^2}u(x,0) + \frac{1}{v^3}u_t(x,0) + \frac{1}{v^2}A\{h(x,t)\} \\ -\frac{1}{v^2}A\Big(R\sum_{m=0}^{\infty} u_m(x,t) + \sum_{m=0}^{\infty} A_m\Big), \end{aligned} \tag{3.9}$$

$$\sum_{m=0}^{\infty} A\{u_m(x,t)\} = \frac{1}{v^2}f(x) + \frac{1}{v^3}g(x) + \frac{1}{v^2}A\{h(x,t)\} - \frac{1}{v^2}\sum_{m=0}^{\infty} A\Big(Ru_m(x,t) + A_m\Big). \tag{3.10}$$

Comparing both sides of equation (3.10) and then taking the inverse Aboodh transform in each case, we thus obtain the general recursive relation

$$\begin{cases} u_0(x,t) = f(x) + tg(x) + A^{-1}\{\frac{1}{v^2}A\{h(x,t)\}\}, & m = 0 \\ u_{m+1}(x,t) = -A^{-1}\{\frac{1}{v^2}A\{Ru_m(x,t) + A_m\}\}, & m \ \geq 0. \end{cases} \tag{3.11}$$

4. Applications and Results

In this section we present some numerical results of our proposed method for the wave-type equations consisting of linear, wave-type and nonlinear wave propagation equations.

4.1. Example One. Consider the linear nonhomogeneous wave equation [Al-Mazmumy & Al-Malki [12]]

$$u_{tt} = u_{xx} + 2(x^2 - t^2), \tag{4.1}$$

with the initial conditions

$$u(x,0) = \sinh(x), \quad u_t(x,0) = \cosh(x). \tag{4.2}$$

Taking the Aboodh transform of both sides of equation (4.1), we obtain

$$A\{u_{tt}(x,t)\} = A\{u_{xx}\} + A\{2(x^2 - t^2)\}. \tag{4.3}$$

Using the differentiation property of the Aboodh transform, we get

$$v^2 A\{u(x,t)\} - u(x,0) - \frac{1}{v}u_t(x,0) = A\{u_{xx}\} + A\{2(x^2 - t^2)\}, \tag{4.4}$$

which can further be expressed as

$$A\{u(x,t)\} = \frac{1}{v^2}u(x,0) + \frac{1}{v^3}u_t(x,0) + \frac{1}{v^2}A\{u_{xx}\} + \frac{1}{v^2}A\{2(x^2 - t^2)\}. \tag{4.5}$$

Replacing $u(x,t)$ by an infinite series

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t),$$

and then taking the inverse Aboodh transform, we thus obtain the general recursive relation,

$$\begin{cases} u_0(x,t) = u(x,0) + tu_t(x,0) + x^2t^2 - \frac{t^4}{3!}, & m = 0 \\ u_{m+1}(x,t) = A^{-1}\{\frac{1}{v^2}A\{u_{m_{xx}}\}\}, & m \geq 0. \end{cases} \tag{4.6}$$

We now obtain some few terms from equation (4.6) as follows

$$u_0(x,t) = \sinh(x) + t\cosh(x) + x^2t^2 - \frac{t^4}{3!}, \tag{4.7}$$

$$\begin{aligned}
u_1(x,t) &= A^{-1}\{\frac{1}{v^2}A\{u_{0_{xx}}\}\},\\
&= A^{-1}\{\frac{1}{v^2}A\{\sinh(x)+t\cosh(x)+2t^2\}\},\\
&= \frac{t^2\sinh(x)}{2!}+\frac{t^3\cosh(x)}{3!}+\frac{t^4}{4!},
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
u_2(x,t) &= A^{-1}\{\frac{1}{v^2}A\{u_{1_{xx}}\}\},\\
&= A^{-1}\{\frac{1}{v^2}A\{\frac{t^2\sinh(x)}{2!}+\frac{t^3\cosh(x)}{3!}\}\},\\
&= \frac{t^4\sinh(x)}{4!}+\frac{t^5\cosh(x)}{5!},
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
u_3(x,t) &= A^{-1}\{\frac{1}{v}A\{u_{2_{xx}}\}\}\\
&= A^{-1}\{\frac{1}{v}A\{\frac{t^4\sinh(x)}{4!}+\frac{t^5\cosh(x)}{5!}\}\}\\
&= \frac{t^6\sinh(x)}{6!}+\frac{t^7\cosh(x)}{7!},
\end{aligned} \tag{4.10}$$

and so on. Summing these iterations yield a series solution

$$\begin{aligned}
u(x,t) &= \sum_{m=0}^{\infty} u_m(x,t)\\
&= x^2t^2+\sinh(x)\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+...\right)\\
&+\cosh(x)\left(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+...\right),
\end{aligned}$$

which leads to the exact solution

$$u(x,t) = x^2t^2+\sinh(x+t). \tag{4.11}$$

4.2. Example Two. Consider the wave-type equation [Momani [5]]

$$u_{tt} = \frac{1}{2}x^2u_{xx}, \tag{4.12}$$

with the initial conditions

$$u(x,0) = x, \quad u_t(x,0) = x^2. \tag{4.13}$$

Proceeding as discussed, we get the recursive relation

$$\begin{cases} u_0(x,t) = u(x,0)+tu_t(x,0) \quad , \quad m=0 \\ u_{m+1}(x,t) = A^{-1}\{\frac{1}{v^2}A\{\frac{1}{2}x^2u_{m_{xx}}\}\} \quad , \quad m \geq 0. \end{cases} \tag{4.14}$$

Thus, we get the following iterations

$$u_0(x,t) = x+tx^2, \tag{4.15}$$

$$
\begin{aligned}
u_1(x,t) &= A^{-1}\{\frac{1}{v^2}A\{\frac{1}{2}x^2u_{0_{xx}}\}\},\\
&= A^{-1}\{\frac{1}{v^2}A\{x^2t\}\},\\
&= \frac{x^2t^3}{3!},
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
u_2(x,t) &= A^{-1}\{\frac{1}{v^2}A\{\frac{1}{2}x^2u_{1_{xx}}\}\},\\
&= A^{-1}\{\frac{1}{v^2}A\{\frac{x^2t^3}{3!}\}\\
&= \frac{x^2t^5}{5!},
\end{aligned} \tag{4.17}
$$

$$
\begin{aligned}
u_3(x,t) &= A^{-1}\{\frac{1}{v^2}A\{\frac{1}{2}x^2u_{2_{xx}}\}\},\\
&= A^{-1}\{\frac{1}{v^2}A\{\frac{x^2t^5}{5!}\}\},\\
&= \frac{x^2t^7}{7!},
\end{aligned} \tag{4.18}
$$

and so on. Accordingly, the solution in a series form is given by

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = x + x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + ...\right),$$

which leads to the exact solution

$$u(x,t) = x + x^2\sinh(t). \tag{4.19}$$

4.3. Example Three. Consider the one-dimensional nonlinear wave equation [Kaya [3]]

$$u_t + u_x^2 = 0, \tag{4.20}$$

with initial condition

$$u(x,0) = -x^2. \tag{4.21}$$

As described above, we get the recursive relation

$$
\begin{cases}
u_0(x,t) = u(x,0), \quad m = 0\\
u_{m+1}(x,t) = -A^{-1}\{\frac{1}{v}A\{A_m\}\}, \quad m\ \geq 0,
\end{cases} \tag{4.22}
$$

where A_m's are the Adomian polynomials given in equation (3.8) with some few terms given by

$$
\begin{aligned}
A_0 &= u_{0_x}^2,\\
A_1 &= 2u_{0_x}u_{1_x},\\
A_2 &= 2u_{0_x}u_{2_x} + u_{1_x}^2,\\
&\vdots
\end{aligned} \tag{4.23}
$$

Therefore, by replacing these Adomian polynomials into the general recursive relation given in equation (4.22) we get the following iterations as

$$u_0(x,t) = -x^2, \tag{4.24}$$

$$\begin{aligned} u_1(x,t) &= -A^{-1}\{\frac{1}{v}A\{A_0\}\}, \\ &= -A^{-1}\{\frac{4x^2}{v^3}\}, \\ &= -4x^2t, \end{aligned} \tag{4.25}$$

$$\begin{aligned} u_2(x,t) &= -A^{-1}\{\frac{1}{v}A\{A_1\}\}, \\ &= -A^{-1}\{\frac{32x^2}{v^4}\}, \\ &= -16x^2t^2, \end{aligned} \tag{4.26}$$

$$\begin{aligned} u_3(x,t) &= -A^{-1}\{\frac{1}{v}A\{A_2\}\}, \\ &= -A^{-1}\{\frac{1}{v}A\{\frac{384x^2}{v^5}\}, \\ &= -64x^2t^3, \end{aligned} \tag{4.27}$$

and so on. Consequently, the solution in a series form is given by

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = -x^2\left(1 + 4t + 16t^2 + 64t^3 + ...\right),$$

which leads to the exact solution

$$u(x,t) = \frac{x^2}{4t+1}. \tag{4.28}$$

5. Conclusion

Many methods has been developed to find the exact solutions of linear and nonlinear partial differential equations [3, 4, 5, 6]. In this article, we successfully apply the Aboodh decomposition method to find the exact solutions of wave-type differential equations. The method was based upon employing the Aboodh transform coupled to the Adomian decomposition method and it revealed remarkable exact solutions without any need of discretization, perturbation or linearization. The method can be used in solving higher order nonlinear partial differential equations as its solution is obtained in form of rapid convergent series with easily computable components.

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