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DIFFERENTIAL INCLUSIONS WITH SECOND ORDER TANGENT VECTORS

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Abstract. In this paper some inclusions for generators of Feller semigroups and evolution families are considered. Existence of solutions of these inclusions is proved with the use of stochastic flows having the same generators as the semigroups and evolution families.

1. Introduction and Preliminaries

The second order tangent vector in $\mathbb{R}^n$ or on a manifold is a second order differential operator without constant term whose matrix of coefficients at the second order derivatives is symmetric and positive semi-definite.

Examples of the second order tangent vector is a generator of the Feller semigroup or of the Feller evolution family and a generator of the stochastic flow (these notions are recalled below, see details, say, in [1, 2, 3, 4]). It is known that under some natural conditions, say, of smoothness and boundedness type, from such a generator one can recover the corresponding semigroup, evolution family and flow. This relation can be considered as an equation with second order tangent vectors. Here we deal with the case where the set-valued field of genmerators is given and so the equation is turned into the inclusion. For investigation of these inclusions we apply some methods from [5, 6].

The set of symmetric positive-definite $n\times n$ matrices we denote by $S_+(n)$ and its closure, the set of positive semi-definite matrices, by $\bar{S}_+(n)$. Below, for simplicity, instead of using the words "second order tangent vector" we call the set of single valued differential operator semi-elliptic if its second order part takes values in the space $\bar{S}_+(n)$ (i.e. the matrices may degenerate) and elliptic if they lie in $S_+(n)$ (positive definite symmetric square matrices).

Let us recall some notions from the Theory of Set-Valued Maps (see e.g. [8]).

Definition 1.1. A set-valued map F from the metric space X into the metric space Y is called upper semicontinuous at $x \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every x' from δ-neighbourhood of x the image $F(x')$ belongs to ε-neighbourhood of $F(x)$. It is called upper semi-continuous if it is upper semi-continuous at each x.

Definition 1.2. A set-valued map F from the metric space X into the metric space Y is called lower semicontinuous at $x \in X$ if for each $\varepsilon > 0$ there exists

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$\delta > 0$ such that for every x' from δ-neighbourhood of x the image $F(x)$ belongs to ε-neighbourhood of $F(x')$. It is called lower semi-continuous if it is lower semi-continuous at each x.

Definition 1.3. Let X and Y be normed spaces. Set-valued map F from X to Y is called Lipschitz continuous at $x \in X$ if there exist $k > 0$ and neighborhood U of x such that

$$\forall x_1, x_2 \in U, \quad F(x_1) \subset F(x_2) + k\|x_1 - x_2\| B_Y,$$

where B_Y is the unit ball in Y. It is called Lipschitz continuous if it is Lipschitz continuous at each point $x \in X$ and the constant k is independent of x.

We denote the norm of a set-valued map in a standard way

$$\|F(x)\| = \sup_{y \in F(x)} \|y\|.$$

Consider the one-point compactification $\mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$.

By definition, a function f on $\mathbb{R}^n$ belongs to $C_0(\mathbb{R}^n)$ if it is continuous on $\mathbb{R}^n \cup \{\infty\}$, takes zero value at $\{\infty\}$ and if for every $\varepsilon > 0$ there exists a compact subset K_ε of $\mathbb{R}^n$ such that $\|f(x)\|_{\mathbb{R}^n} < \varepsilon$ for each x from K_ε.

2. Autonomous case and Feller semigroups

Definition 2.1. A family $\{U(t) : t \geq 0\}$ of operators defined on $L^\infty(\mathbb{R}^n)$ is a Feller semigroup on $C_0(\mathbb{R}^n)$ if the following properties holds.

(1) $U(t)C_0(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$, $\forall t \geq 0$.
(2) semigroup property: $U(t + \tau) = U(t)U(\tau)$, $\forall t, \tau \geq 0$, $U(0) = I$.
(3) $\|U(t)f\|_\infty \leq \|f\|_\infty$.
(4) positivity: $f \geq 0$ implies $S(t)f \geq 0$.
(5) $\lim\limits_{t \downarrow 0} [U(t)f](x) = f(x)$.

Definition 2.2. The infinitesimal generator G of the semigroup $U(t)$ is the operator G such that

$$Gf(x) = \lim_{t \downarrow 0} \frac{U(t)f(x) - f(x)}{t}.$$

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be some probability space. Consider a stochastic differential equation of the Ito type

$$d\xi_{s,x}(t) = a(\xi_{s,x}(t))dt + A(\xi_{s,x}(t))dw(t) \tag{2.1}$$

for some Wiener process $w(t) \in \mathbb{R}^n$ and $\xi_{s,x}(s) = x$. The solution of (2.1) defines the Feller semigroup $U(t)$ by its action on $f \in C^2(\mathbb{R}^n)$:

$$[U(t)f](x) = E^x[f(\xi(t))]. \tag{2.2}$$

Consider Euclidean space $\mathbb{R}^n$ and let L be a second order semi-elliptic autonomous differential operator without constant term on that space. Consider differential equation

$$\frac{\partial u}{\partial t} - Lu = 0, \tag{2.3}$$

with the condition $u(0) = f(x)$, $t \in [0, T]$. The solution of it can be defined thought the semigroup $U(t)$ as follows

$$u(t, x) = U(t)f(x). \tag{2.4}$$

Then the generator of this family is L.

Note that L coincides with the generator of the equation (2.1), i.e. it is an operator of the form:

$$L(x) = a^i(x)\frac{\partial}{\partial x^i} + \frac{1}{2}\sigma^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j},$$

where $\sigma(x) = A(x)A^*(x)$.

On the other hand, each Feller semigroup with generator L defines a strong Markov process such that (2.2) holds and solution of equation (2.3) for the operator L that has form (2.4).

We look for the solution of the following problem. Take the set-valued second order semi-elliptic differential operator $\mathbf{L}(x)$ and find a semigroup whose generator satisfies the inclusion

$$L(x) \in \mathbf{L}(x). \tag{2.5}$$

We use the following notation:

$$\|\mathbf{L}(x)\| = \sup_{y \in \mathbf{L}(x)} \{\|y(x)\|\}, \quad \|y(x)\| = \sup_{f \in C^2} \left\{ \frac{\|y(x)f\|}{\|f\|_{C^2}} \right\},$$

$$\|y(x)f\| = \|y_1(x)f\| + \|y_2(x)f\|, \text{ where } y(x) = y_1(x)\frac{\partial}{\partial q^i} + y_2(x)\frac{\partial^2}{\partial q^i \partial q^j}.$$

Theorem 2.3. *Let $\mathbf{L}(x)$ be an upper semicontinuous set-valued operator with closed convex values such that for each $x \in \mathbb{R}^n$ the following inequality holds*

$$\|\mathbf{L}(x)\| \le K(1 + \|x\|)^2. \tag{2.6}$$

Then there exists a Feller semigroup $U(t)$ with generator $L(x)$ such that for all $x \in \mathbb{R}^n$ inclusion (2.5) is satisfied.

Proof. Consider a sequence of positive numbers $\varepsilon_i \to 0$. By [5, Theorem 2] and [6, Theorem 4] for every number ε_i there exists a single-valued continuous operator $L_i(t, x)$ whose graph belongs to the ε_i-neighborhood of graph of $\mathbf{L}(t, x)$ in $[0, T] \times \mathbb{R}^n$ such that the sequence $L_i(t, x)$ point-wise converges to a Borel measurable selector $L(t, x)$ of $\mathbf{L}(t, x)$ as i goes to ∞.

The sequence $\{L_i\}_{i=1}^{\infty}$ gives us sequences $\{a_i\}_{i=1}^{\infty}$ and $\{\sigma_i\}_{i=1}^{\infty}$ where the latter is positive semi-definite. Introduce $\tilde{\sigma}_i = \sigma_i + \varepsilon_i I$ so that $\tilde{\sigma}_i$ is positive definite. Since each a_i and $\tilde{\sigma}_i$ are continuous, they can be approximated by smooth ones. Without loss of generality we preserve the notations for these smooth approximations. Thus by the construction we get that $L_i(x)$ is a $2\varepsilon_i$-approximation of $\mathbf{L}(x)$ and the sequence $L_i(x)$ point-wise converges to $\mathbf{L}(x)$.

Consider $\Omega = C^0([0, T], \mathbb{R}^n)$ equipped with the σ-algebra $\mathcal{F}$ generated by cylinder sets. Denote by $\mathcal{P}_t$ the σ-subalgebra of $\mathcal{F}$ generated by cylinder sets with bases over $[0, t] \subset [0, T]$.

For each i we get decomposition $\tilde{\sigma}_i(x) = A_i(x)A_i^*(x)$ and the stochastic differential equation of the form

$$d\xi(t) = a_i(t, \xi(t))dt + A_i(t, \xi(t))dw(t)$$

with the initial condition $\xi_i(0)|_x = x$. Since the coefficients of this equation are smooth and condition (2.6) is fulfilled, it has unique strong solution (see for example, [7]). The solution defines measure μ_i on $(\Omega, \mathcal{F})$. The whole set of these measures (that we get for each i) is weakly compact. Thus we can choose subsequence $\{\mu_q\}_{q=1}^{\infty}$ that weakly converges to some measure μ.

The coordinate process on the probability space $(\Omega, \mathcal{F}, \mu)$, say $\xi(t)$, has the infinitesimal generator $L(x)$.

Then by [1, Theorem 2] it defines a Feller semigroup $U(t)$ such that

$$U(t)f(x) = E^x[f(\xi(t))].$$

Also by this Theorem semigroup $U(t)$ has the generator $L(x)$ which by the construction lies in $\mathbf{L}(x)$. $\square$

3. Non-autonomous case and Feller evolution families

In this section we study the solvability of inclusions given in terms of generators of Feller evolution families. We consider the cases where the set-valued field of non-autonomous generators has a selection generating unique Markov process. A more general case will be considered in future works.

Recall that the family of operators $U(s,t)$ (take $t \geq s$) on $C_0(\mathbb{R}^n)$ is called the Feller evolution family if the following properties hold:

1. the evolution property $U(s,\tau)U(\tau,t) = U(s,t)$ $(s \leq \tau \leq t)$and $U(s,s) = I$;
2. operators $U(s,t)$ acts in $C_0(\mathbb{R}^n)$: $U(s,t)(C_0(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$;
3. operators $U(s,t)$ are strongly continuous jointly in both parameters;
4. for any $f \in C_0(\mathbb{R}^n)$, $0 \leq f \leq 1$ and $t \geq s \geq 0$ the inequality $0 \leq U(s,t)f \leq 1$ holds.

The infinitesimal generator of such a family is an operator $G(s,x)$ such that its action on every function from $C_0(\mathbb{R}^n)$ is given by the formula

$$G(s,x)f(x) = \lim_{t \downarrow s} \frac{U(t,s)f(x) - f(x)}{t-s}.$$

For more details see e.g. [1].

The same second order tangent vectors may also be generators of stochastic flows. Take mappings $a(t,x)$ and $A(t,x)$ from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$ and the set of linear operators on $\mathbb{R}^n$ respectively. Consider in $\mathbb{R}^n$ the stochastic dynamical system governed by the following equation of Itô type

$$\begin{cases} d\xi_{s,x}(t) = a(t, \xi_{s,x}(t))dt + A(t, \xi_{s,x}(t))dw(t), \\ \xi_{s,x}(s) = x. \end{cases} \tag{3.1}$$

given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$, where $w(t)$ is a Wiener process on that space with values in $\mathbb{R}^n$; $0 \leq s \leq t \leq T$. Recall that the infinitesimal generator of this stochastic evolution family of processes is an operator $G(s,x)$ which acts on the functions $f \in C(\mathbb{R}^n, \mathbb{R})$ in the following way

$$G(s,x)f(x) = \lim_{t \downarrow s} \frac{\mathsf{E}[f(\xi_{s,x}(t))] - f(x)}{t-s},$$

where E is the expectation. Then for $f \in C^2(\mathbb{R}^n, \mathbb{R})$ it can be written in the form

$$G(s,x) = a^i(s,x)\frac{\partial}{\partial q^i} + \frac{1}{2}(AA^*)^{ij}(s,x)\frac{\partial^2}{\partial q^i \partial q^j},$$

where q^i, $i = 1, 2, \ldots, n$ are coordinates in $\mathbb{R}^n$.

Denote by $\mu_{s,x}$ the measures on the space of sample paths corresponding to the solutions $\xi_{s,x}(t)$ of (3.1).

Consider a set-valued semi-elliptic differential operator $\mathbf{G}(s,x)$ on $\mathbb{R} \times \mathbb{R}^n$. It can be written in the form

$$\mathbf{G}(s,x) = \mathbf{a}^i(s,x)\frac{\partial}{\partial q^i} + \boldsymbol{\alpha}^{ij}(s,x)\frac{\partial^2}{\partial q^i \partial q^j},$$

where $\mathbf{a}(s,x)$ and $\boldsymbol{\alpha}(s,x)$ are set-valued mappings from $\mathbb{R}\times\mathbb{R}^n$ to $\mathbb{R}^n$ and to $\bar{S}_+(n)$, respectively.

In $\mathbb{R}^n$ it is convenient to introduce the norm of the semi-elliptic differential operator $\boldsymbol{G}(s,x)$ as the sum of norms of its components, i.e., the norm of the vector of first order part plus the norm of the matrix of second order part.

Denote by I the closed interval $[0,T]$ in $\mathbb{R}$.

Theorem 3.1. *Suppose that the measurable set-valued semi-elliptic differential operator $\boldsymbol{G}(s,x)$ has the single valued second order term $\alpha(s,x)$ that is C^2-smooth in x and satisfies the inequality*

$$\|a(s,x)\|^2 + \|\operatorname{tr}\alpha(s,x)\| < K(s)(1+\|x\|^2), \tag{3.2}$$

for some function $K(s)$ on I. Then there exists a Feller evolution family with generator $G(s,x)$ such that the inclusion

$$G(s,x) \in \boldsymbol{G}(s,x) \tag{3.3}$$

holds for all (s,x) from $I \times \mathbb{R}^n$.

Proof. Recall that there exists a measurable selection $G(s,x)$ of the set-valued map $\boldsymbol{G}(s,x)$ (see e.g. [8]). Since $\boldsymbol{G}(s,x)$ is a semi-elliptic differential operator, its selection $G(s,x)$ can be written in the form

$$G(s,x) = a^i(s,x)\frac{\partial}{\partial q^i} + \alpha^{ij}(s,x)\frac{\partial^2}{\partial q^i \partial q^j}, \tag{3.4}$$

where $a(s,x)$ is measurable and $\alpha(s,x)$ is the above-mentioned positive semi-definite symmetric C^2-smooth matrix. By [9, Theorem 1] there exists Lipschitz continuous $A(s,x)$ such that $\alpha(s,x) = A(s,x)A^*(s,x)$. Since estimate (3.2) holds, the stochastic equation without drift

$$\mathrm{d}\tilde{\xi}_{s,x}(t) = A(t, \tilde{\xi}_{s,x}(t)\mathrm{d}w(t),$$

where $w(t)$ is a Wiener process in $\mathbb{R}^n$, has a unique strong solution for each initial data $\tilde{\xi}_{s,x}(s) = x$. Then by results of [7] the equation with drift $a(t,x)$

$$\mathrm{d}\xi_{s,x}(t) = a(t, \xi_{s,x}(t)\mathrm{d}t + A(t, \xi_{s,x}(t)\mathrm{d}w(t),$$

has unique weak solution for each initial data. Hence there is a unique week solution $\xi(t)$, $t \in I$, of equation

$$\mathrm{d}\xi(t) = a(t, \xi(t))\mathrm{d}t + A(t, \xi(t))\mathrm{d}w(t),$$

such that for $t \in I$, $t \geq s$ it coincides with $\xi_{s,x}(t)$ with probability 1.

This solution (see, e.g. [1]) is a Markov process. Thus we get the dynamical system of form (3.1), and the operators $U(t,s)$ defined as $U(t,s)f(x) = \mathsf{E}(f(\xi_{s,x}(t)))$, form a Feller evolution family with the generator $G(s,x)$. So, inclusion (3.3) holds for each $(s,x) \in I \times \mathbb{R}^n$. □

Theorem 3.2. *Suppose that a set-valued elliptic differential operator $\boldsymbol{G}(s,x)$ is such that $\mathbf{a}(s,x)$ and $\boldsymbol{\alpha}(s,x)$ are Lipschitz continuous (in set-valued sense) and their values belong to the sets of nonempty closed convex subsets of $\mathbb{R}^n$ and $S_+(n)$, respectively. Let also inequality* (3.2) *hold. Then there exists a Feller evolution family such that inclusion* (3.3) *holds for all (s,x) from $I \times \mathbb{R}^n$.*

Proof. By [8, Theorem 9.4.3] there exists a Lipschitz continuous selection $G(s,x)$ of the set-valued map $\boldsymbol{G}(s,x)$. Obviously it satisfies estimate (3.2). From the fact that $\boldsymbol{\alpha}(s,x)$ is non-degenerate, it follows that there exists Lipschitz continuous $A(s,x)$ such that $A(s,x)A^*(s,x) = \sigma(s,x)$.

By the existence theorem of strong solutions (see e.g. [7]) the equation

$$\mathrm{d}\xi_{s,x}(t) = a(t, \xi_{s,x}(t)\mathrm{d}t + A(t, \xi_{s,x}(t)\mathrm{d}w(t),$$

has unique strong solution $\xi_{s,x}(t)$ starting at the moment s from the point x.

As in Theorem 3.1, construct the Feller evolution family by the rule $U(t,s)f(x) = \mathsf{E}(f(\xi_{s,x}(t)))$. Its generator satisfies inclusion (3.3). □

Suppose that a set-valued elliptic differential operator $\boldsymbol{G}(s,x)$ is lower semicontinuous and has convex closed images. Then by Michael's selection theorem (see e.g. [8]) it has a continuous selection $G(s,x)$. Since $G(s,x)$ is elliptic, i.e., the matrix $\alpha(s,x)$ of second order part is non-degenerate, there exists continuous matrix $A(s,x)$ such that $\alpha(s,x) = A(s,x)A^*(s,x)$ (see, e.g., [10]). This matrix $A(s,x)$ together with the first order term $a(s,x)$ of $G(s,x)$ determine the stochastic differential equation of form (3.1) with continuous coefficients. Let inequality (3.2) hold. Then the above equation has a weak solution for each initial data for $t \in I$, $t \geq s$. Assume also that this solution is weakly unique.

Theorem 3.3. *Under the above assumptions there exists a Feller evolution family such that inclusion* (3.3) *holds for all (s,x) from $I \times \mathbb{R}^n$.*

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