

## Periodic Solutions of Second-order Nonautonomous Impulsive Differential Equations

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**Abstract.** The main purpose of this paper is to study the existence of periodic solutions of second order impulsive differential equations with superlinear nonlinear terms. Our result generalizes one of Paul H. Rabinowitz's existence results of periodic solutions of second order ordinary differential equations to impulsive cases. Mountain Pass Lemma is applied in order to prove our main results.

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### Introduction

Impulsive differential equations arising from real world describe the dynamics of processes in which sudden, discontinuous jumps occur. Such processes are naturally seen in control theory [15, 19, 20], population dynamics [21], chemotherapeutic treatment in medicine [22], and some physics problems [18, 23]. For its significance, a lot of effort has been done in the theory of impulsive differential equations. See, for instance [16, 17, 25, 26]. For general aspects of impulsive differential equations, monographs [11, 10, 12] are recommended.

Since 1990s, lots of significant results concerning the existence of periodic solutions of impulsive differential equations have been proved. The main tools used are fixed point theory, topological degree theory (including continuation method) and comparison method (including upper and lower solutions methods and monotone iterative method), c.f. [25-33] and references therein. However, compared to the ordinary differential equations, the results are still relatively rare. One of the reasons is that

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some tools which are widely used in the ODE theory are still not effectively applied in impulsive differential equation theory. Critical point theory(including direct methods, minimzx methods and Morse theory) is just such an example. It has been extensively applied in study of the existence of periodic solutions of second order ODEs  $u''(t) = \nabla F(t, u(t))$  with superlinear or sublinear nonlinear terms  $\nabla F$ , c.f. [3, 5, 9, 8, 13, 14] and references therein. However, to the best of our knowledge, there are still no similar results for impulsive differential equations. In this paper, we generalize one of Paul H. Rabinowitz's existence results of periodic solutions of second order ordinary differential equations in [1] to impulsive cases.

More precisely, this paper studies the following differential equations

$$-u''(t) + a_k(t)u(t) = f_k(t, u(t)), \quad \text{for } t \in (t_k, t_{k+1}). \quad (0.1)$$

$$u(s_k^+) = u(s_k^-) + c_k u(s_k^-), \quad (0.2)$$

$$u'(s_k^+) = u'(s_k^-) + d_k u'(s_k^-) + g_k(u(s_k^-)), \quad (0.3)$$

where  $u(t^+) = \lim_{s \rightarrow t^+} u(s)$ ,  $u(t^-) = \lim_{s \rightarrow t^-} u(s)$ ,  $a_k, f_k, g_k, s_k, c_k$  and  $d_k$  satisfy the following conditions.

- $a_k \in C(\mathbb{R}, \mathbb{R}^+)$ ,  $f_k \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$  and  $g_k \in C(\mathbb{R}, \mathbb{R})$  for all integers  $k$ ,  $\{s_k\}$ ,  $\{c_k\}$  and  $\{d_k\}$  are real sequences, there exist a positive integer  $m$  and
- (A) a positive number  $T$  such that  $0 = s_0 < s_1 < \dots < s_m = T$ ,  $s_{k+m} = s_k + T$ ,  $c_{k+m} = c_k$ ,  $d_{k+m} = d_k$ ,  $g_{k+m} = g_k$ ,  $a_{k+m}(t+T) = a_k(t)$  and  $f_{k+m}(t+T, x) = f_k(t, x)$  for any integer  $k$  and  $(t, x) \in \mathbb{R} \times \mathbb{R}$ .

$u(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  is a solution of (0.1)-(0.3) if it's a piecewise continuous function which has discontinuous of the first kind at  $s_k$  and satisfying (0.1)-(0.3). For convenience and historical reason, we also suppose  $u$  is continuous from the left, that is, we set  $u(s_k) = \lim_{s \rightarrow s_k^-} u(s)$  for each  $s_k$ .  $u$  is said to be a  $T$ -periodic solution of equations (0.1)-(0.3) if it is a solution of (0.1)-(0.3) and satisfies

$$u(0) - u(T) = u'(0) - u'(T) = 0, \quad (0.4)$$

where  $u'(0) = \lim_{t \rightarrow 0^-} u'(t)$  and  $u'(T) = \lim_{t \rightarrow T^-} u'(t)$ . Under hypothesis (A), it's easy to show that finding a  $T$ -periodic solution of (0.1)-(0.3) is equivalent to finding a solution of the boundary value problem the BVP (0.1)-(0.4).

Our main result will be given in Section 2 and it's a generalization of Theorem 6.10 of [1](see Remark 2.3).

There are three difficulties encountered when applying the critical point theory to prove our main result. The first two are how to construct an appropriate Hilbert space and that how to define a functional on that Hilbert space whose critical points correspond to the solutions of (0.1)-(0.4). The common definitions of such Hilbert space and functional for second order ODE without impulse can not be used any more, since the impulse conditions (0.2) and (0.3) must be taken into account. We deal with these difficulties in section 1 by giving explicit definition of the Hilbert space (denoted by  $H_{IT}$ , see (1.1)) and the functional (denoted by  $I$ , see (1)). The equivalence between solutions of the BVP (0.1)-(0.4) and critical points of functional  $I$  is also proved in that section.

The next difficulty concerns how to find the critical points of the functional  $I$ . We handle this in section 2. Via Mountain Pass Lemma, an existence result of solutions of the BVP (0.1)-(0.4) is proved in that section.

This paper takes little space to talk about the detail of critical point theory. For general theory about it, references[1, 2, 3, 4] are recommended.

## 1. Variational Structure

Let  $\Gamma_k = (s_{k-1}, s_k)$  for  $k = 1, 2, \dots, m$ . Let  $u^{(k)} = u|_{\Gamma_k}$ ,  $\forall k = 1, 2, \dots, m$ . Define the function space

$$H_{IT} = \{u : [0, T] \rightarrow \mathbb{R} \mid u \text{ is continuous from left at each } s_k, u^{(k)} \text{ is absolutely continuous, the weak derivative of } u^{(k)} \text{ is in } L^2(\Gamma_k), u \text{ satisfies the condition (0.2) for all } k = 1, \dots, m, \text{ and } u(0) = u(T)\}. \quad (1.1)$$

$$\hat{H}_{IT} = \{u : \mathbb{R} \rightarrow \mathbb{R} \mid u \text{ is } T\text{-periodic, } u|_{[0, T]} \in H_{IT}\} \quad (1.2)$$

It's easy to prove that the space  $H_{IT}$  and  $\hat{H}_{IT}$  are isomorphic Hilbert spaces with the following inner product

$$\langle u, v \rangle = \sum_{k=1}^m \int_{\Gamma_k} u(s)v(s)ds + \sum_{k=1}^m \int_{\Gamma_k} u'(s)v'(s)ds.$$

In this paper, we identify these two spaces and denote the norm induced by the above inner product by  $\|\cdot\|$ .

**Remark 1.1.** Obviously, for any  $u \in H_{IT}$ ,  $u^{(k)} \in W^{1,2}(\Gamma_k)$ . By Sobolev-Rellich-Kondrachov Imbedding Theorem(c.f. [6, 7]), the imbeddings  $W^{1,2}(\Gamma_k) \subset C(\overline{\Gamma_k})$  and  $W^{1,2}(\Gamma_k) \subset L^2(\Gamma_k)$  are continuous and compact.

Consider the functional  $I$  defined on  $H_{IT}$  as follows.

$$I(u) = \sum_{k=1}^m \int_{\Gamma_k} \frac{1}{2}|u'(s)|^2 + \frac{a_k(s)}{2}u^2(s) - F_k(s, u(s))ds + \sum_{k=1}^m \int_0^{u(s_k^-)} \bar{g}_k(x)dx,$$

where  $F_k(t, x) = \int_0^x f_k(t, y)dy$ , and

$$\bar{g}_k = (1 + c_k)g_k, \quad \forall k = 1, 2, \dots, m. \quad (1.3)$$

We will prove that the critical points of  $I$  are solutions of the BVP (0.1)-(0.4). First we show that under some conditions, the functional  $I$  is Fréchet differentiable.

**Lemma 1.1.** *Assume  $f_k$  and  $g_k$  are continuous, then the functional  $I$  given by (1) is Fréchet differentiable, and*

$$I'(u)v = \sum_{k=1}^m \int_{\Gamma_k} u'v' + a_k uv - f_k(s, u)v ds + \sum_{k=1}^m \bar{g}_k(u(s_k^-))v(s_k^-), \quad (1.4)$$

for all  $u, v \in H_{IT}$ .

*Proof.* Note that for each  $k = 1, 2, \dots, m$ ,  $u^{(k)} \in W^{(1,2)}(\Gamma_k)$ , with little modification to the proof of Theorem 1.4 in [3], it's not difficult to prove that for each  $k = 1, 2, \dots, m$ ,

$$J_k(u) = \int_{\Gamma_k} \frac{1}{2}|u'|^2 + \frac{a_k}{2}u^2 - F_k(s, u) ds$$

is Fréchet differentiable and

$$J'_k(u)v = \int_{\Gamma_k} u'v' + a_k uv - f_k(s, u)v ds$$

for all  $u, v \in H_{IT}$ .

Since  $\bar{g}_k$  is continuous and that the imbedding  $W^{1,2}(\Gamma_k) \subset C(\overline{\Gamma_k}, \mathbb{R})$  is continuous, it's easy to check  $H_k(u) = \int_0^{u(s_k^-)} \bar{g}_k(x) dx$  is differentiable and  $J'_k(u)v = \bar{g}_k(u(s_k^-))v(s_k^-)$ .  $\square$

**Lemma 1.2.** *Suppose hypothesis (A) holds and  $d_k = (1+c_k)^{-1} - 1$ , then the following two statements are equivalent*

- (1)  $u$  is critical point of  $I(u)$ ;
- (2)  $u$  is a classical solution of the BVP (0.1)-(0.4) .

*Proof.* First suppose  $u$  is a critical point of  $I$ . It needs to verify that  $u$  is a classical solution of (0.1)-(0.4) .

$u$  is a critical point of  $I$  implies  $\forall v \in H_{IT}$

$$\sum_{k=1}^m \int_{\Gamma_k} u'v' + a_k uv - f_k(s, u)v ds + \sum_{k=1}^m \bar{g}_k(u(s_k^-))v(s_k^-) = 0. \quad (1.5)$$

Choosing  $v \in H_{IT}$ ,  $v|_{\overline{\Gamma_j}} = 0$  for  $j = 1, \dots, k-1, k+1, \dots, m$ , (1.5) shows that

$$\int_{\Gamma_k} u'v' + a_k uv - f_k(s, u)v ds = 0. \quad (1.6)$$

This means, for any  $w \in W_0^{1,2}(\Gamma_k)$ , the following equation holds.

$$\int_{\Gamma_k} u'w' + a_k uw - f_k(s, u)w ds = 0. \quad (1.7)$$

Thus  $u^{(k)}$  is a weak solution of equation

$$-u''(t) + a_k u(t) = f_k(t, u(t)), \quad t \in \Gamma_k. \quad (1.8)$$

Let  $h(t) = f_k(t, u(t))$ . By the fact  $u^{(k)} \in W^{1,2}(\Gamma_k) \subset C(\Gamma_k)$  and  $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $h(t)$  is continuous. Following from the regularity of solutions of linear elliptic differential equations(c.f. [6]),  $u^{(k)} \in W^{2,2}(\Gamma_k)$ , for all  $k = 1, 2, \dots, m$ . Also by  $u^{(k)} \in C(\Gamma_k)$  and  $h \in C(\mathbb{R}, \mathbb{R})$ , (1.8) implies  $u^{(k)} \in C^2(\Gamma_k, \mathbb{R})$ , which means  $u^{(k)}$  is a classical solution of (1.6) for all  $k = 1, 2, \dots, m$ . So the following equations hold for  $u$  and  $v \in H_{IT}$

$$\int_{\Gamma_k} u'v' + u''v ds = u'v|_{s_{k-1}^+}^{s_k^-}, \quad \forall k = 1, 2, \dots, m. \quad (1.9)$$

Combining (1.5) and (1.9) shows that

$$\sum_{k=1}^m \int_{\Gamma_k} (u'' - a_k u + f_k(s, u))v ds = \sum_{k=1}^m u'v|_{s_{k-1}^+}^{s_k^-} + \sum_{k=1}^m \bar{g}_k(u(s_k^-))v(s_k^-), \quad (1.10)$$

where  $u'v|_{s_{k-1}^+}^{s_k^-} = u'(s_k^-)v(s_k^-) - u'(s_{k-1}^+)v(s_{k-1}^+)$ . Choosing  $v \in H_{IT}$  and  $v(s_k^-) = v(s_k^+) = 0, \forall k = 1, 2, \dots, m$ , (1.10) shows that

$$\sum_{k=1}^m \int_{\Gamma_k} (u'' - a_k u + f_k(s, u))v ds = 0 \quad (1.11)$$

for all  $v \in H_{IT}, v(s_k^-) = v(s_k^+) = 0, k = 1, 2, \dots, m$ . By the fact  $C_0^\infty(\Gamma_k)$  is dense in  $L^2(\Gamma_k)$ , (1.11) holds for all  $v \in H_{IT}$ . It follows that

$$\sum_{k=1}^m u'v|_{s_{k-1}^+}^{s_k^-} + \sum_{k=1}^m \bar{g}_k(u(s_k^-))v(s_k^-) = 0 \quad (1.12)$$

holds for all  $v \in H_{IT}$ . By (0.1) and the fact  $u^{(k)} \in C(\bar{\Gamma}_k)$ ,  $u''(t)$  is bounded for  $t \in [0, T] \setminus \{s_1, \dots, s_m\}$ , which means  $u'(s_k^\pm)$  exist for all  $k = 1, 2, \dots, m$ .

It follows from (1.12) that

$$\begin{aligned} 0 &= \sum_{k=1}^{m-1} (u'(s_k^-)v(s_k^-) - u'(s_k^+)v(s_k^+) + \bar{g}_k(u(s_k^-))v(s_k^-)) \\ &\quad + u'(T^-)v(T^-) - u'(0^+)v(0^+) + \bar{g}_T(u(T^-))v(T^-) \\ &= \sum_{k=1}^{m-1} (u'(s_k^-) - (1 + c_k)u'(s_k^+) + \bar{g}_k(u(s_k^-)))v(s_k^-) \\ &\quad + (u'(T^-) - (1 + c_0)u'(0^+) + \bar{g}_0(u(T^-)))v(T^-) \end{aligned} \quad (1.13)$$

for any  $v \in H_{IT}$ . Since  $v \in H_{IT}$  is arbitrary, (1.13) implies  $u'(s_k^+) = (1 + c_k)^{-1}u'(s_k^-) + g_k(u(s_k^-)) = u'(s_k^-) + d_k u'(s_k^-) + g_k(u(s_k^-)), \forall k = 1, 2, \dots, m - 1$  and

$$u'(0^+) = (1 + c_0)^{-1}u'(T^-) + g_0(u(T^-)). \quad (1.14)$$

Since  $u$  is  $T$ -periodic,  $u'(0^-) = u'(T^-)$ , which means  $u'(T^+) = (1+c_m)^{-1}u'(T^-) + g(u(T^-)) = u'(T^-) + d_m u'(T^-) + g(u(T^-))$ . So condition (3) holds for  $u$ . Thus  $u$  is a classical solution of the BVP (0.1)-(0.4).

If  $u$  is a solution of (0.1)-(0.4), it's not difficult to show  $u$  satisfies both (1.11) and (1.13) for all  $v \in H_{IT}$ . This implies equation (1.5) holds for all  $v \in H_{IT}$  which means  $u$  is a critical point of functional  $I$ .  $\square$

## 2. Existence of Nonzero Periodic Solutions

To facilitate statements in the remainder of the paper, we introduce some notations.

H1.  $\lim_{x \rightarrow 0} \frac{f_k(t, x)}{x} = 0$  uniformly with respect to  $t \in [0, T]$  for all  $k = 1, 2, \dots, m$ ;

H2. there are constants  $\mu_k > 2$  and  $r > 0$  such that for  $|x| \geq r$ ,

$$x f_k(t, x) \geq \mu_k \int_0^x f_k(t, y) dy > 0 \tag{2.1}$$

uniformly with respect to  $t \in [0, T]$  for all  $k = 1, 2, \dots, m$ ;

H3.  $\lim_{x \rightarrow 0} \frac{g_k(x)}{x} = 0$  for all  $k = 1, 2, \dots, m$ ;

H4. suppose  $\mu_k$  is the constant given above,

$$\liminf_{|x| \rightarrow \infty} x^{-2} \left( \int_0^x \bar{g}_k(y) dy - \frac{1}{\mu_k} \bar{g}_k(x)x \right) \geq 0 \tag{2.2}$$

holds for all  $k = 1, 2, \dots, m$ , where  $\bar{g}_k$  are defined by (1.3).

**Remark 2.1.** Integrating equation (2.1) shows that there exists constants  $a, b > 0$  such that

$$F_k(t, x) \geq a|x|^{\mu_k} - b, \forall x \in \mathbb{R}. \tag{2.3}$$

Thus  $F_k(t, x)$  grows at a "superquadratic" rate and  $f_k(t, x)$  grows at a "superlinear" rate uniformly with respect to  $t \in [0, T]$  as  $|x| \rightarrow \infty$

**Remark 2.2.** Condition H4 implies there exists a continuous nonincreasing function  $\psi : \mathbb{R} \rightarrow [0, +\infty)$  such that

$$\int_0^x \bar{g}_k(y) dy - \frac{1}{\mu_k} \bar{g}_k(x)x \geq -\psi(x)|x|^2, \quad \forall k = 1, 2, \dots, m \tag{2.4}$$

and  $\lim_{|x| \rightarrow \infty} \psi(x) = 0$ .

**Theorem 2.1.** *Suppose hypothesis (A) holds and there exists a positive constant  $A$  such that  $a_k(t) \geq A$  for all integers  $k$  and  $t \in \mathbb{R}$ ,  $d_k = (1 + c_k)^{-1} - 1$ ,  $f_k$  satisfies the H1 and H2,  $g_k$  satisfies H3 and H4. Then the BVP (0.1)-(0.4) possesses at least one nonzero solution.*

**Remark 2.3.** *Theorem 1 is a generalization of Theorem 6.10 of Rabinowitz [1]. In fact Theorem 6.10 of [1] follows from Theorem 1 by letting  $c_k = 1$  and  $g_k = 0$  for all  $k$ .*

We will use the famous Mountain Pass Lemma to prove this result.

**Lemma 2.1. (Mountain Pass Lemma)** *Let  $E$  be a real Banach space and  $I \in C^1(E, \mathbb{R})$  satisfying (P.S.). Suppose  $I(0) = 0$  and*

- (I<sub>1</sub>) *there are constants  $\rho, \beta > 0$  such that  $I|_{\partial B_\rho} \geq \beta$ , and*
- (I<sub>2</sub>) *there is an  $e \in E \setminus B_\rho$  such that  $I(e) \leq 0$ .*

*Then  $I$  possesses a critical value  $c \geq \beta$ . Moreover  $c$  can be characterized as*

$$c = \inf_{h \in \Gamma} \max_{x \in h([0,1])} I(x),$$

where

$$\Gamma = \{h \in C([0, 1], E) | h(0) = 0, h(1) = e\}.$$

*Proof of Theorem 2.1.* Let

$$J(u) = \sum_{k=1}^m \int_{\Gamma_k} F_k(s, u(s)) ds.$$

By Remark 2.1

$$J(u) \geq \sum_{k=1}^m a \int_{\Gamma_k} |u(s)|^{\mu_k} ds - bT. \tag{2.5}$$

Choosing  $u \in H_{IT} \setminus \{0\}$ , such that  $u(s_k^+) = u(s_k^-) = 0$ . (2.5) implies

$$\begin{aligned} I(tu) &\leq \frac{t^2}{2} \sum_{k=1}^m \int_{\Gamma_k} |u'(s)|^2 + a_k |u(s)|^2 ds + \sum_{k=1}^m \int_0^{tu(s_k^-)} \bar{g}_k(x) dx \\ &\quad - t^\mu \sum_{k=1}^m \int_{\Gamma_k} a |u(s)|^{\mu_k} ds + bT \\ &\leq \frac{t^2}{2} \sum_{k=1}^m \int_{\Gamma_k} |u'(s)|^2 + a_k |u(s)|^2 ds - t^\mu \sum_{k=1}^m \int_{\Gamma_k} a |u(s)|^{\mu_k} ds + bT \\ &\leq \frac{t^2}{2} \sum_{k=1}^m \int_{\Gamma_k} |u'(s)|^2 + a_k |u(s)|^2 ds - t^\mu \sum_{k=1}^m \int_{\Gamma_k} a |u(s)|^{\mu_k} ds + bT \\ &\rightarrow -\infty \quad (\text{as } t \rightarrow +\infty), \end{aligned}$$

where  $\mu = \min\{\mu_k\}$ . So the condition  $(I_2)$  of Mountain Pass Lemma holds.

By condition H1, for any  $\epsilon > 0$ , there exist  $\delta_1 > 0$  such that for any  $|x| \leq \delta_1$ ,

$$|F_k(t, x)| \leq \frac{1}{2}\epsilon|x|^2 \text{ and } |g(x)| \leq \epsilon|x|, \forall k = 0, 1, \dots, m. \quad (2.6)$$

Chose  $\epsilon \leq \min\{\frac{A}{2}, \frac{1}{2C^2}, \frac{A}{4C^2}\}$ . By Remark 1.1,  $\forall k = 1, 2, \dots, m$ , there exists  $C_k > 0$ , such that

$$\|u^{(k)}\|_\infty \leq C_k \|u^{(k)}\|_{W^{(1,2)}(\Gamma_k)} \quad (2.7)$$

Let  $C = \max\{C_k\}$ ,  $\delta = \min\{\delta_1, \delta_1/C\}$ . Then for any  $u \in H_{IT}$ ,  $\|u\| \leq \delta$ ,

$$|F(u(s))| \leq \frac{1}{2}\epsilon|u(s)|^2. \quad (2.8)$$

Then

$$\begin{aligned} I(u) &\geq \sum_{k=1}^m \frac{1}{2} \int_{\Gamma_k} |u'(s)|^2 + A|u(s)|^2 ds - \sum_{k=1}^m \frac{\epsilon}{2} \int_{\Gamma_k} |u(s)|^2 ds - \sum_{k=1}^m \int_0^{u(s_k^-)} \frac{\epsilon}{2} |x| dx \\ &\geq \sum_{k=1}^m \frac{1}{2} \int_{\Gamma_k} |u'(s)|^2 + \frac{A}{2}|u(s)|^2 ds - \frac{1}{2} \sum_{k=1}^m \epsilon |u(s_{k+1}^-)|^2. \end{aligned} \quad (2.9)$$

$$\begin{aligned} &\geq \frac{1}{2} \min\{1, \frac{A}{2}\} \|u\|^2 - \frac{1}{2} \epsilon \|u\|_\infty^2 \\ &\geq \frac{1}{2} \min\{1, \frac{A}{2}\} \|u\|^2 - \frac{1}{2} C^2 \epsilon \|u\|^2 \\ &\geq \frac{1}{4} \min\{1, \frac{A}{2}\} \|u\|^2 \end{aligned} \quad (2.10)$$

Let  $\rho = \delta$ . (2.7) and (2.10) mean that  $\forall u \in \partial B_\rho$ ,

$$I(u) \geq \min\{\frac{1}{4}, \frac{A}{8}\} \rho^2 \quad (2.11)$$

Let  $\beta = \min\{\frac{1}{4}, \frac{A}{8}\} \rho^2$ . (2.11) implies condition  $(I_1)$  of Mountain Pass Lemma holds too.

Now we verify the  $(P.S.)$  condition.

Suppose  $\{u_n\}$  is a sequence in  $H_{IT}$  such that  $|I(u_n)| \leq M$  for some positive number  $M$ , and  $\|I'(u_n)\| \rightarrow 0$  ( $k \rightarrow \infty$ ). We need to show that  $\{u_n\}$  has a convergent subsequence. Now fix  $k$  and let  $T_n^k = \frac{1}{\mu_k} (f_k(s, u_n^{(k)}(s)) u_n^{(k)}(s) - \mu_k F_k(s, u_n^{(k)}(s)))$  and

$$I_k(u^{(k)}) = \int_{\Gamma_k} \left| \frac{du^{(k)}}{ds} \right|^2 + a_k |u^{(k)}|^2 - F_k(s, u^{(k)}) ds + \int_0^{u_n^{(k)}(s_k^-)} \bar{g}_k(x) dx,$$



then

$$\begin{aligned} I_k(u_n^{(k)}) - \frac{1}{\mu_k} I_k'(u_n^{(k)})u_n^{(k)} &= \int_{\Gamma_k} \left( \frac{1}{2} - \frac{1}{\mu_k} \right) \left( \left| \frac{du_n^{(k)}}{ds} \right|^2 + a_k |u_n^{(k)}|^2 \right) ds \\ &\quad + \int_0^{u_n^{(k)}(s_k^-)} \bar{g}_k(x) dx - \frac{1}{\mu_k} \bar{g}_k(u_n^{(k)}(s_k^-)) u_n^{(k)}(s_k^-) \\ &\quad + \int_{\Gamma_k} T_n^k ds. \end{aligned}$$

By Remark 2.2,

$$\begin{aligned} I_k(u_n^{(k)}) - \frac{1}{\mu_k} I_k'(u_n^{(k)})u_n^{(k)} &\geq M_1 \|u_n^{(k)}\|^2 - \psi(u_n^{(k)}(s_k^-)) u_n^{(k)}(s_k^-)^2 \\ &\quad + \int_{\Gamma_{k,r}^+} T_n^k ds + \int_{\Gamma_{k,r}^-} T_n^k ds, \end{aligned}$$

where  $M_1 = \min\{1, A\}(\frac{1}{2} - \frac{1}{\mu_k})$ ,  $\Gamma_{k,r}^+ = \{s \in \Gamma_k \mid |u_n^{(k)}(s)| \geq r\}$ , and  $\Gamma_{k,r}^- = \{s \in \Gamma_k \mid |u_n^{(k)}(s)| < r\}$ . By H2,  $\int_{\Gamma_{k,r}^+} T_n^k ds$  is nonnegative and  $\int_{\Gamma_{k,r}^-} T_n^k ds$  is bounded by some number  $M_2$ . So for all  $n$ ,

$$M + \mu_k^{-1} \|u_n^{(k)}\| \geq M_1 \|u_n^{(k)}\|^2 - \psi(u_n^{(k)}(s_k^-)) u_n^{(k)}(s_k^-)^2 - M_2. \quad (2.12)$$

If  $\{u_n^{(k)}(s_k^-)\}$  is bounded, the boundedness of  $\{u_n^{(k)}\}$  can be derived from (2.12) easily. Assume that  $\{u_n^{(k)}(s_k^-)\}$  is unbounded, then  $\{u_n^{(k)}\}$  is unbounded in  $W^{1,2}(\Gamma_k)$  sense too. By Remark 2.2, for  $n$  large enough,  $\psi(u_n^{(k)}(s_k^-)) < M_1/(2C)$ , where  $C$  is defined after (2.7). Then for  $n$  large enough,

$$\begin{aligned} M_1 &\leq \frac{M_1 u_n^{(k)}(s_k^-)^2}{2 \|u_n^{(k)}\|^2} + \frac{1}{\mu_k \|u_n^{(k)}\|} + \frac{M + M_2}{\|u_n^{(k)}\|^2} \\ &\leq \frac{M_1}{2} + \frac{1}{\mu_k \|u_n^{(k)}\|} + \frac{M + M_2}{\|u_n^{(k)}\|^2}. \end{aligned} \quad (2.13)$$

A contradiction can be easily derived from (2.13) for  $n$  large enough. So  $\{u_n^{(k)}(s_k^-)\}$  is bounded.

By Remark 1.1, the boundedness of  $\{u_n^{(k)}\} \subset W^{1,2}(\Gamma_k)$  implies  $\{u_n^{(k)}\}$  is precompact both in space  $C(\overline{\Gamma_k})$  and  $L^2(\Gamma_k)$ . Without loss of generality, suppose  $\{u_n^{(k)}\}$  is convergent both in  $C(\overline{\Gamma_k})$  and  $L^2(\Gamma_k)$  for each  $k = 1, 2, \dots, m$ . This implies for each  $k$ ,  $\{u_n^{(k)}(s_k^-)\}$  and  $\{g_k(u_n^{(k)}(s_k^-))\}$  are convergent sequences;  $\{f_k(s, u_n^{(k)})\}$  converges uniformly in  $C(\overline{\Gamma_k})$ , hence it converges in  $L^2(\overline{\Gamma_k})$ .

$$\begin{aligned}
 & \left( I'(u_l^{(k)}) - I'(u_n^{(k)}) \right) (u_l^{(k)} - u_n^{(k)}) \\
 &= \int_{\Gamma_k} \left| \frac{du_l^{(k)}}{ds} - \frac{du_n^{(k)}}{ds} \right|^2 + a_k |u_l^{(k)} - u_n^{(k)}|^2 ds \\
 & - \int_{\Gamma_k} (f_k(s, u_l^{(k)}) - f_k(s, u_n^{(k)})) (u_l^{(k)} - u_n^{(k)}) ds \\
 & + \left( \bar{g}_k(u_l^{(k)}(s_k^-)) - \bar{g}_k(u_n^{(k)}(s_k^-)) \right) (u_l^{(k)}(s_k^-) - u_n^{(k)}(s_k^-))
 \end{aligned}$$

implies

$$\begin{aligned}
 & \min\{1, A\} \|u_l^{(k)} - u_n^{(k)}\| \\
 & \leq \left( I'(u_l^{(k)}) - I'(u_n^{(k)}) \right) \frac{u_l^{(k)} - u_n^{(k)}}{\|u_l^{(k)} - u_n^{(k)}\|} \\
 & + \frac{1}{2} \|u_l^{(k)} - u_n^{(k)}\|^{-1} \left( \|f_k(s, u_l^{(k)}) - f_k(s, u_n^{(k)})\|_{L^2(\bar{\Gamma}_k)} + \|u_l^{(k)} - u_n^{(k)}\|_{L^2(\bar{\Gamma}_k)} \right) \\
 & + \|u_l^{(k)} - u_n^{(k)}\|^{-1} \left( \bar{g}_k(u_l^{(k)}(s_k^-)) - \bar{g}_k(u_n^{(k)}(s_k^-)) \right) (u_l^{(k)}(s_k^-) - u_n^{(k)}(s_k^-)).
 \end{aligned}$$

This shows that  $\{u_n^{(k)}\}$  is a Cauchy sequence in  $W^{1,2}(\Gamma_k)$ . With this fact in mind, it's not difficult to verify  $\{u_n\}$  has a convergent subsequence in  $H_{IT}$ . So the (P.S.) condition holds for  $I$ .

At last, by Mountain Pass Lemma,  $I$  possesses at least one nonzero critical point, which means the boundary value problems (0.1)-(0.4) has at least one nonzero classical solution.  $\square$

Next, we consider a special case of equations (0.1)-(0.3).

$$-u''(t) + a_k(t)u(t) = f_k(t, u(t)), \quad t \in (s_k, s_{k+1}), \quad (2.14)$$

$$u'(s_k^+) = u'(s_k^-) + g_k(u(s_k^-)), \quad (2.15)$$

**Corollary 2.1.** *Suppose there exist positive numbers  $T, A$  and a positive integer  $m$  such that  $0 = s_0 < s_1 < \dots < s_m = T, s_{k+m} = s_k + T, a_{k+m}(t+T) = a_k(t), f_{k+m}(t+T) = f_k(t)$  and  $a_k(t) \geq A$  for all integers  $k$  and  $t \in \mathbb{R}$ ,  $f_k$  satisfies the H1 and H2,  $g_k$  satisfies H3 and H4. Then (2.14)-(2.15) possesses at least one nonzero  $T$ -periodic solution.*

For the more simple equations

$$-u''(t) + au(t) = f(u(t)), \quad t \in (s_k, s_{k+1}), \quad (2.16)$$

$$u'(s_k^+) = u'(s_k^-) + g(u(s_k^-)), \quad (2.17)$$

we have the following results.

H1'.  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0;$

H2'. there are constants  $\mu > 2$  and  $r > 0$  such that for  $|x| \geq r,$

$$xf(x) \geq \mu \int_0^x f(y)dy > 0.$$

H3'.  $\lim_{x \rightarrow 0} \frac{g(x)}{x} = 0.$

H4'. if  $\mu$  is the constant given above, then

$$\liminf_{|x| \rightarrow \infty} x^{-2} \left( \int_0^x g(y)dy - \frac{1}{\mu}g(x)x \right) \geq 0.$$

**Corollary 2.2.** *Suppose  $a > 0$  and there exist a positive numbers  $T$  and a positive integer  $m$  such that  $0 = s_0 < s_1 < \dots < s_m = T, s_{k+m} = s_k + T$  for all integers  $k,$   $f$  satisfies H1' and H2',  $g$  satisfies H3' and H4'. Then (2.16)-(2.17) possess at least one nonzero  $T$ -periodic solution.*

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