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On the Almost Periodic Solution of a Class of Singularly Perturbed Differential Equations with Piecewise Constant Argument

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Abstract. The present paper is concerned with the module containment of almost periodic solution for a nonautonomous, singularly perturbed differential equations with piecewise constant argument. The present result is basically a complete extension of the some known (periodic, quasi-periodic) results.

AMS Subject Classifications: 34K15

Keywords: Almost periodicity; Quasi-periodicity; Module containment; Singular perturbation; Piecewise constant argument

1. Introduction

The present paper is concerned with the following singularly perturbed systems of differential equations with piecewise constant argument

$$\begin{cases} x'(t) = F(t, x(t), \{x([t+i])\}_{-N}^{N}, y(t), \{y([t+i])\}_{-N}^{N}, \epsilon), \\ \epsilon y'(t) = G(t, x(t), \{x([t+i])\}_{-N}^{N}, y(t), \{y([t+i])\}_{-N}^{N}, \epsilon) \end{cases} \tag{1.1}$$

in the case that F and G are almost periodic for t uniformly on $\mathbb{R}^{2N+2} \times \mathbb{R}^{2N+2}$, where $\epsilon > 0$ is a small parameter, $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $[\cdot]$ denotes the greatest integer function. The existence problem of almost periodic solution has been studied in a series of papers ([1, 4, 8, 9, 18, 20, 21]). To our knowledge, there are no discussions about the spectrum of almost periodic solution to (1.1) up to now. The main purpose of this paper is to study the spectrum or module of almost periodic

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solution. Approximation theorem and quasi-periodicity are employed. Differential equations with piecewise constant argument (EPCA, for short) have received extensive investigations (see, e.g., [2,5,10, 19-24] and references therein). In these equations, the derivatives of the unknown functions depend on not just the time t at which they are determined, but on constant values of the unknown functions in certain intervals of the time t before t. These equations have the structure of continuous dynamical systems in intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals implies a recursion relation for the values of the solution at such points. Therefore, they combine the properties of differential equations and difference equations. The equations are thus similar in structure to those found in certain "sequential-continuous" models of disease dynamics as treated by Busenberg and Cooke in [2]. Another potential application of EPCAs is in the stabilization of hybrid control systems with feedback delays. By a hybrid dynamical system we mean one with a continuous plant and with a discrete (sampled) controller. The hybrid dynamical systems have widely been studied (see [24] and its references therewith). Some of these systems may be described by EPCA.

As is well known, the existence problem of periodic solutions and almost periodic solutions has been one of the most attracting topics in the qualitative theory of ordinary or functional differential equations for its significance in the physical sciences. There have been many remarkable works ([3,4,6-10,12-15,17-18, 20-23] and the references cited therein) concerning the existence of almost periodic solutions. For a special form of singularly perturbed differential equation (1.1):

$$\begin{cases} x'(t) = F(t, x(t), y(t), \epsilon), \\ \epsilon y'(t) = G(t, x(t), y(t), \epsilon), \end{cases} \tag{1.2}$$

the existence of periodic solutions has been studied by Flatto & Levinson [8], and Anosov [1], etc.. Hale & Seifert [9] and Chang [4] generalized their results and investigated the existence of almost periodic solutions to Eq.(1.2). Recently, Smith [18] also showed the existence and stability of almost periodic solutions to Eq.(1.2). Clearly, Eq.(1.1) is a generalization form of Eq.(1.2).

Suppose that $f(t)$ and $g(t)$ are almost periodic. Then the module containment property:

$$\mathrm{mod}(g) \subset \mathrm{mod}(f)$$

can be characterized in several ways (see [6, 7, 12, 13]). For periodic functions this inclusion just means that the minimal period of $g(t)$ is a multiple of the minimal period of $f(t)$. Therefore the discussion about module containment is very important. We also note that there was a discussion about the spectrum containment of almost periodic solution to ordinary differential equations in [3], i.e., Cartwright compared the basic frequencies of almost periodic differential equations $\dot{x} = \psi(x, t),\ x \in \mathbb{R}^n$ with those of its unique almost periodic solution. Mallet-Paret [14] gave an extension of such Cartwright's theorem to functional differential equations. But there are no papers to singularly perturbed systems.

It is assumed that the degenerate system

$$\begin{cases} x'(t) = F(t, x(t), \{x([t+i])\}_{-N}^{N}, y(t), \{y([t+i])\}_{-N}^{N}, 0), \\ \quad 0 = G(t, x(t), \{x([t+i])\}_{-N}^{N}, y(t), \{y([t+i])\}_{-N}^{N}, 0) \end{cases} \tag{1.3}$$

has an almost periodic " outer " solution which we take to be the trivial solution, that is, we suppose

$$F(t, 0, \cdots, 0, 0) \equiv G(t, 0, \cdots, 0, 0) \equiv 0$$

so that $(x, y) = (0, 0)$ satisfies (1.3). Our aim is to seek for almost periodic solutions of Eq.(1.1) near the "outer" solution and to study the spectrum of the almost periodic solutions. Expanding (1.1) about the trivial solution gives

$$\begin{cases} x'(t) = a(t,\epsilon)x(t) + \sum\limits_{i=-N}^{N} a_i(t,\epsilon)x([t+i]) + b(t,\epsilon)y(t) + \sum\limits_{i=-N}^{N} b_i(t,\epsilon)y([t+i]) \\ \qquad\qquad + f(t, x(t), \{x([t+i])\}_{-N}^{N}, y(t), \{y([t+i])\}_{-N}^{N}, \epsilon), \\ \epsilon y'(t) = c(t,\epsilon)x(t) + \sum\limits_{i=-N}^{N} c_i(t,\epsilon)x([t+i]) + d(t,\epsilon)y(t) + \sum\limits_{i=-N}^{N} d_i(t,\epsilon)y([t+i]) \\ \qquad\qquad + g(t, x(t), \{x([t+i])\}_{-N}^{N}, y(t), \{y([t+i])\}_{-N}^{N}, \epsilon). \end{cases} \tag{1.4}$$

One can think of, e.g., $a(t, \epsilon)$ as $\partial F/\partial x(t, 0, \cdots, 0, \epsilon)$. In the present paper, in fact, we will consider the following general singularly perturbed systems

$$\begin{cases} x'(t) = a(t,\epsilon)x(t) + \sum\limits_{i=-N}^{N} a_i(t,\epsilon)x([t+i]) + b(t,\epsilon)y(t) + \sum\limits_{i=-N}^{N} b_i(t,\epsilon)y([t+i]) \\ \qquad\qquad + f(t, x_t, , y_t, \epsilon), \\ \epsilon y'(t) = c(t,\epsilon)x(t) + \sum\limits_{i=-N}^{N} c_i(t,\epsilon)x([t+i]) + d(t,\epsilon)y(t) + \sum\limits_{i=-N}^{N} d_i(t,\epsilon)y([t+i]) \\ \qquad\qquad + g(t, x_t, y_t, \epsilon), \end{cases} \tag{1.5}$$

where $f, g : \mathbb{R} \times \mathcal{C} \times \mathcal{C} \times [0, \epsilon_0) \to \mathbb{R}$, here $\mathcal{C} = C([-1, 0], \mathbb{R})$. We define $x_t \in \mathcal{C}$ as $x_t(\theta) = x(t + \theta), -1 \le \theta \le 0$ and $|x_t| = \sup_{-1 \le \theta \le 0} |x(t + \theta)|$.

We say that a function $(x, y) : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is a solution of Eq.(1.5) (or Eq.(1.1)) if the following conditions are satisfied

(i) (x, y) is continuous on $\mathbb{R}$,

(ii) the derivative (x', y') of (x, y) exists on $\mathbb{R}$ except possibly at the point $t = n, n \in \mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$ where one-sided derivative exists,

(iii) (x, y) satisfies Eq.(1.5) (or Eq.(1.1)) in the intervals $(n, n + 1),\ n \in \mathbb{Z}$.

The following hypotheses are assumed to hold throughout the paper.

(H1) $a(t,\epsilon),\ a_i(t,\epsilon),\ b(t,\epsilon),\ b_i(t,\epsilon),\ c(t,\epsilon),\ c_i(t,\epsilon),\ d(t,\epsilon),\ d_i(t,\epsilon),\ i=0,\pm1,\cdots,$ $\pm N$, are almost periodic functions in t. They are continuous in ϵ, uniformly in $t\in\mathbb{R}$. Let M denote a common bound for these functions on $(t,\epsilon)\in\mathbb{R}\times[0,\epsilon_0]$.

(H2) $a(t,0)=a^0$, $a_i(t,0)=a_i^0$, $c(t,0)=0$, $c_i(t,0)=0$, $d(t,0)=d^0$, $d_i(t,0)=d_i^0$, $i=0,\pm1,\cdots,\pm N$, are constants and $d^0<0$.

(H3) All roots of algebraic equation

$$\sum_{i=-N}^{N} A_i\xi^i=0$$

are not on $S^1 \triangleq \{z\in\mathbb{C};\ |z|=1\}$, where

$$\begin{aligned} &A_0=e^{a^0}+{a^0}^{-1}a_0^0(e^{a^0}-1), \qquad A_1={a^0}^{-1}a_1^0(e^{a^0}-1)-1,\\ &A_i={a^0}^{-1}a_i^0(e^{a^0}-1), \qquad i=-1,\pm2,\cdots,\pm N. \end{aligned}$$

(H4) All roots of algebraic equation

$$\sum_{i=-N}^{N} D_i\mu^i=0 \tag{1.6}$$

are not on S^1, where

$$\begin{aligned} &D_0=-{d^0}^{-1}d_0^0, \qquad D_1=-{d^0}^{-1}d_1^0-1,\\ &D_i=-{d^0}^{-1}d_i^0, \qquad i=-1,\pm2,\cdots,\pm N. \end{aligned}$$

(H5) f, g are almost periodic in t uniformly on $(\phi,\psi)\in\mathcal{C}\times\mathcal{C}$. Furthermore, there exist nondecreasing functions $M(\epsilon)$ and $\eta(\rho,\epsilon)$, $0\le\epsilon\le\epsilon_0$, $0\le\rho\le\rho_0$ satisfying $\lim_{\epsilon\to0}M(\epsilon)=0$, $\lim_{(\rho,\epsilon)\to(0,0)}\eta(\rho,\epsilon)=0$, such that

$$|f(t,0,0,\epsilon)|\le M(\epsilon),\ |g(t,0,0,\epsilon)|\le M(\epsilon), \qquad t\in\mathbb{R},\ 0\le\epsilon\le\epsilon_0,$$

and

$$\begin{aligned} &|f(t,\phi,\psi,\epsilon)-f(t,\bar\phi,\bar\psi,\epsilon)|\le\eta(\rho,\epsilon)[|\phi-\bar\phi|+|\psi-\bar\psi|],\\ &|g(t,\phi,\psi,\epsilon)-g(t,\bar\phi,\bar\psi,\epsilon)|\le\eta(\rho,\epsilon)[|\phi-\bar\phi|+|\psi-\bar\psi|] \end{aligned}$$

hold for all $t\in\mathbb{R}$, $|\phi|$, $|\bar\phi|$, $|\psi|$, $|\bar\psi|\le\rho_0$, $0\le\epsilon\le\epsilon_0$.

This paper is organized as follows. In section 2, we discuss quasi-periodic functions, almost periodic functions. We also discuss the Bohr exponents (or spectrum) for almost periodic sequence. In section 3, we discuss a quasi-periodic differential equations with piecewise constant argument and the spectrum's relations between almost periodic sequence (solution) and almost periodic function (solution). Quasi-periodic DPCA with a parameter will be discussed in section 4. Quasi-periodic and almost periodic singularly perturbed DPCA will be discussed in sections 5 and 6, respectively. Approximation ideas are employed to discuss the spectrum. We formulate a nonlinear result in section 7.

2. Almost Periodic Functions / Sequences

Definition 1. **([6, 7, 12, 13])** *A function $f:\mathbb{R}\to\mathbb{R}$ is called an almost periodic function introduced by Bohr, if it is continuous and for any $\epsilon>0$, the ϵ-translation set of f*

$$T(f,\epsilon)=\{\tau\in\mathbb{R};\ |f(t+\tau)-f(t)|<\epsilon,\ t\in\mathbb{R}\}$$

is a relative dense set on $\mathbb{R}$.

The theory of almost periodic functions is sometimes called Bohr's theory and can be found in some remarkable books. There is a deep connection between almost periodic functions and periodic functions of several variables (including a countable number). It is well known that for every almost periodic function $f(t)$, the mean value

$$M_t\{f\}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T+a}^{T+a}f(t)dt$$

exists uniformly with respect to a. Let $\{\lambda_j\}$ denote the set of all real numbers such that

$$a(\lambda_j;f):=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)\exp(-i\lambda_j t)dt\neq 0.$$

It is well known that the set of numbers $\{\lambda_j\}$ in the above formula is countable. With each almost periodic function f we associate the Fourier series

$$f(t)\sim\sum_k a_k\exp(i\lambda_k t),$$

where $a_k=a(\lambda_k;f)$. The elements a_k are called the Fourier coefficients and the numbers λ_k the Fourier exponents of f. Denoted by Λ_f the set of all Fourier exponents $\{\lambda_k\}$ of f, which is called the spectrum of f. The set $\{\sum_1^N n_j\lambda_j\}$ for all integers N and integers n_j is called the module of $f(t)$, denoted by $\mathrm{mod}(f)$, which is the least additive subgroup of the integral numbers containing the Fourier exponents of $f(t)$.

A finite or countable set of real numbers $\beta_1,\beta_2,\cdots,\beta_n,\cdots$ is said to be rationally independent, if the equality $r_1\beta_1+r_2\beta_2+\cdots+r_n\beta_n=0$ ($r_1,\cdots,r_n$ are rational and n is an arbitrary natural number) implies that all of $r_1,r_2,\cdots,r_n$ are zero. A finite or countable set of rationally independent real numbers $\beta_1,\beta_2,\cdots,\beta_n,\cdots$ is called a rational basis of a countable set of real numbers $\lambda_1,\lambda_2,\cdots,\lambda_n,\cdots$ if every λ_n is representable as a finite linear combination of the β_j with rational coefficients, that is,

$$\lambda_n=r_1^{(n)}\beta_1+r_2^{(n)}\beta_2+\cdots+r_{m_k}^{(n)}\beta_{m_k}\quad(n=1,2,\cdots).$$

where the $r_j^{(n)}$ are rational numbers. If all the $r_j^{(n)}$ are integers, then the basis is called an integer basis. If a basis consists of a finite number of terms, then it is called a finite basis.

Suppose that f is almost periodic in the Bohr's sense. When a basis in $\mathrm{mod}(f)$ is both integer and finite, f is called to be quasi-periodic. Let $\omega_1,\omega_2,\cdots,\omega_p$ be rationally independent real numbers. If each element in $\mathrm{mod}(f)$ is linear combinations of

ω_j with integral coefficients, we say that f is a quasi-periodic function with frequencies $(\frac{\omega_1}{2\pi}, \frac{\omega_2}{2\pi}, \cdots, \frac{\omega_p}{2\pi})$. This definition is virtually given by Bogoliubov. Quasi-periodic functions form a special case of almost periodic functions. In fact, P.Bohl and E. Esclangon first considered the quasi-periodic function as a generalization of the notion of periodic function. Some of the methods of P. Bohl and E. Esclangon have been used by Bohr to elaborate his theory. In Bohr's theory, any denumerable set of real numbers is admitted for the frequencies of an almost periodic function.

For convenience and explicitness, the quasi-periodic function is often defined in some literatures and books as follows. Suppose $F(\theta_1, \theta_2, \cdots, \theta_r)$ is 1-periodic in each $\theta_1, \theta_2, \cdots, \theta_r$ and Lebesgue integral. The set of such functions is denoted by $L(\mathbb{T}^r)$, where $\mathbb{T}^r = \mathbb{R}^r/2\pi\mathbb{Z}^r$. If $\omega_1, \cdots, \omega_r$ are rational independent, a function $x : \mathbb{R} \to \mathbb{R}, t \to x(t)$, is said to be quasi-periodic with frequencies $(\omega_1, \cdots, \omega_r)$ if x is continuous and there exists a continuous function $F \in L(\mathbb{T}^r)$ such that

$$x(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_r t), \qquad \forall t \in \mathbb{R}. \tag{2.1}$$

It is easy to prove that when F is continuous, $x(t)$ in (2.1) is almost periodic in Bohr's sense and quasi-periodic in Bogoliubov's sense (e.g., see [12]). If we denote by $\mathcal{P}$, $\mathcal{QP}$, $\mathcal{AP}$, $\mathcal{AA}$ the set of periodic functions, quasi-periodic functions, almost periodic functions, almost automorphic functions, respectively, it is well known that the following relation holds:

$$\mathcal{P} \subset \mathcal{QP} \subset \mathcal{AP} \subset \mathcal{AA}.$$

In the present paper, we also use several Banach spaces as follows. Assume that $\omega_1, \omega_2, \cdots, \omega_r \in \mathbb{R}$. We denote $\omega = (\omega_1, \cdots, \omega_r)$, $m = (m_1, \cdots, m_r) \in \mathbb{Z}^r$, and $< m, \omega >= m_1\omega_1 + \cdots + m_r\omega_r$. Set

$$\mathcal{QP}(\omega) = \big\{ f(t) = \sum_m f_m e^{i2\pi< m,\omega >t},\ t \in \mathbb{R} \Big| \sum_m |f_m| < +\infty, f_{-m} = \bar{f}_m \big\}.$$

Setting $|f| = \sum_m |f_m|$, it is easily shown that $(\mathcal{QP}(\omega), |\cdot|)$ is a Banach space. Set

$$\mathcal{QP}(\omega;\mathbb{Z}) = \Big\{ \phi(n) = \sum_k \phi_k e^{i2\pi< k,\omega >n};\ n \in \mathbb{Z} \Big| \phi_{-k} = \bar{\phi}_k, \sum_k |\phi_k| < +\infty \Big\}.$$

Every sequence in $\mathcal{QP}(\omega;\mathbb{Z})$ can be seen as that of a function in $\mathcal{QP}(\omega)$ taken values at integer points $t = n \in \mathbb{Z}$. Setting $|\phi| = \sum_k |\phi_k|$, it is easy to prove that $(\mathcal{QP}(\omega;\mathbb{Z}), |\cdot|)$ is a Banach space. We define

$$\begin{aligned} \mathcal{GQP}(1,\omega) = \Big\{ \phi(t) \in C(\mathbb{R}) \Big| \phi(t) = \sum_k \phi_k(\{t\}) e^{i2\pi< k,\omega >[t]}; \phi_{-k}(s) = \bar{\phi}_k(s), \\ \forall k,\ \phi_k(s) \in C[0,1]; \sum_k \sup_{0 \le s \le 1} |\phi_k(s)| < +\infty \Big\}, \end{aligned}$$

where $C(\mathbb{R})$ denotes the set of continuous functions defined on $\mathbb{R}$. It is easy to see that $\mathcal{GQP}(1,\omega)$ is a linear space on $\mathbb{R}$ (or $\mathbb{C}$). Clearly, if $\phi(t) \in \mathcal{QP}(\omega)$, then

$\phi(t) \in \mathcal{GQP}(1, \omega)$, since we can write

$$\phi(t) = \sum_k \phi_k e^{i2\pi<k,\omega>t} = \sum_k \phi_k(\{t\}) e^{i2\pi<k,\omega>[t]},$$

where $\phi_k(s) = \phi_k e^{i2\pi<k,\omega>s}$, which implies that $\mathcal{GQP}(1, \omega)$ is not empty. We define

$$||\phi||_G = \sum_k \sup_{0 \le s \le 1} |\phi_k(s)|, \; \forall \phi \in \mathcal{GQP}(1, \omega).$$

It is easy to show that $(\mathcal{GQP}(1, \omega), ||\cdot||_G)$ is a Banach space (e.g., see [10, 23]). It should be noted that we do not require in the present paper that $\omega_1, \cdots, \omega_r$ in above mentioned $\mathcal{QP}$ and $\mathcal{GQP}$ is rationally independent. It is easy to see that every function in $\mathcal{QP}(\omega)$ is quasi-periodic.

The Approximation Theorem 1. ([6, 7, 13]) *For every continuous almost periodic function $f : \mathbb{R} \to \mathbb{R}$ and for every $\epsilon > 0$ there is a trigonometric polynomial*

$$P_\epsilon(t) = \sum_{\nu=1}^{N_\epsilon} b_{\nu,\epsilon} \exp(i\lambda_{\nu,\epsilon} t) \quad (b_{\nu,\epsilon} \in \mathbb{C}, \lambda_{\nu,\epsilon} \in \Lambda_f) \tag{2.2}$$

such that

$$\sup_{t \in \mathbb{R}} |f(t) - P_\epsilon(t)| < \epsilon,$$

where $b_{\nu,\epsilon}$ is the product of $a(\lambda_{\nu,\epsilon}; f)$ and certain positive number (depending on ϵ and $\lambda_{\nu,\epsilon}$).

As a simple consequence of this Theorem we can conclude that if $f(t)$ and $g(t)$ are two almost periodic functions and $a(\lambda; f) \equiv a(\lambda; g)$, then $f \equiv g$. It follows that if Λ_f is finite, then $f(t)$ is a trigonometric polynomial, i.e., suppose $\Lambda_f = \{\lambda_1, \cdots, \lambda_r\}$, then there exist $a_1, \cdots, a_r$, such that $f(t) = \sum_{j=1}^r a_j e^{i\lambda_j t}$, $t \in \mathbb{R}$. It could be shown that every almost periodic function $f(t)$ is the diagonal function of a limit-periodic function of a finite or countable number of variables.

Definition 2. ([6, 7, 15]) *A sequence $g := \{g(n)\}_{n\in\mathbb{Z}}$ is called an almost periodic sequence, if for any $\epsilon > 0$, there exists an $l = l(\epsilon) \in \mathbb{N}$ with the property that for any $a \in \mathbb{Z}$ there is a $\tau \in \mathbb{Z}$ satisfying $a \le \tau < a + l$ and $|g(n + \tau) - g(n)| < \epsilon, \quad \forall n \in \mathbb{Z}$.*

Suppose that g is an almost periodic sequence. Then the continuous function $G : \mathbb{R} \to \mathbb{R}$ defined by $G(n) = g(n), n \in \mathbb{Z}$, and $G(n + s) = g(n) + s[g(n + 1) - g(n)]$ for $s \in [0, 1]$ is an almost periodic function (See [6, 7]).

By the Approximation Theorem 1 for almost periodic functions, G can be approximated by some trigonometric polynomials $P_k(t), k = 1, 2, \cdots$, of the form $P_k(t) = \sum_{m=1}^{n_k} x_{k,m} \exp(i\lambda_{k,m} t)$, where $x_{k,m} \in \mathbb{C}$ and $\lambda_{k,m} \in \mathbb{R}$, such that

$$\sup_{t\in\mathbb{R}} |G(t) - P_k(t)| < \frac{1}{k}, \quad k = 1, 2, \cdots.$$

In particular, it follows that $\lim_{k\to\infty}(\sup_{n\in\mathbb{Z}}|\sum_{m=1}^{n_k} x_{k,m}z_{k,m}^n - g(n)|) = 0$, where $z_{k,m} := \exp(i\lambda_{k,m})$, which shows that g is a uniform limit of sequences of the form

$$\sum_{m=1}^{\text{finite}} x_m z_m^n, \quad x_m \in \mathbb{C}, z_m \in S^1 := \{z \in \mathbb{C}; |z| = 1\}.$$

It follows from [6] that the limit

$$a(z;g) = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} z^{-k} g(k) = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N-m}^{N-m} z^{-k} g(k)$$

exists uniformly for $m \in \mathbb{Z}$ when $z \in S^1$. $a(z;g)$ is called the Bohr transform of g. In case that g is trigonometric polynomial, that is, g is of the following form

$$g(n) = \sum_{k=1}^{p} x_k z_k^n, \quad x_k \in \mathbb{C}, z_k \in S^1 (z_k \neq z_l (k \neq l)),$$

it is easy to check that $a(z;g) = x_k$ if $z = z_k$ for some $k = 1, 2, \cdots, p$, and $a(z;g) = 0$ if $z \neq z_k$ for any $k = 1, \cdots, p$. We denote by $\sigma_b(g)$ the set which consists of all numbers $z \in S^1$ satisfying $a(z;g) \neq 0$, and call $\sigma_b(g)$ the Bohr spectrum of g. It is easy to see that $\sigma_b(g)$ is at most countable. We define the module of an almost periodic sequence $\{g(n)\}$ as

$$\mathrm{mod}_s(g) = \{\prod_{j=1}^{N} z_j^{n_j} \,|\, \forall z_j \in \sigma_b(g), n_j \in \mathbb{Z}, N \in \mathbb{N}\}.$$

With each almost periodic sequence $\{g(n)\}$ we also associate the Fourier series

$$g(n) \sim \sum_k x_k z_k^n, \quad z_k \in S^1,$$

where $x_k = a(z_k;g)$. The elements x_k are called the Fourier coefficients and the numbers z_k the Fourier exponents of $\{g(n)\}$. We can formulate the approximation theorem for an almost periodic sequence as follows.

The Approximation Theorem 2. *Suppose that g is an almost periodic sequence. For any $\epsilon > 0$ there is a sequence of approximating polynomials $P_\epsilon(n)$ of the form*

$$P_\epsilon(n) = \sum_{k=1}^{n_\epsilon} x_{k,\epsilon} z_{k,\epsilon}^n, \quad z_{k,\epsilon} \in \sigma_b(g)$$

such that

$$\sup_{\tau\in\mathbb{Z}} |g(\tau) - P_\epsilon(\tau)| < \epsilon,$$

where $x_{k,\epsilon}$ is the product of $a(z_{k,\epsilon};g)$ and certain positive number.

Proof. This Theorem can be established by modifying the arguments in [13, pp.14-21]. We omit its proof. □

It follows from this Theorem that if f and g are almost periodic sequences and if $a(z;f) \equiv a(z;g)$, then $f \equiv g$. We could conclude that if $\sigma_b(g)$ is finite, then $\{g(n)\}$ is a trigonometric polynomial.

3. Quasi-Periodic Differential Equations

In this section, we consider the following differential equations with piecewise constant argument

$$\dot{x}(t) = a^0 x(t) + \sum_{i=-N}^{N} a_i^0 x([t+i]) + f(t), \tag{3.1}$$

where $f(t) \in \mathcal{QP}(\omega)$, $\omega = (\omega_1, \cdots, \omega_r)$. When $\omega_1, \cdots, \omega_r$ are rationally independent, the existence of quasi-periodic solution of (3.1) has been studied in [10]. In this section, for the aim of the present paper (see section 6), we would like to consider the general case for $\omega = (\omega_1, \cdots, \omega_r)$, i.e., we do not require that $\omega_1, \cdots, \omega_r$ are rationally independent (for short, ω is not rationally independent). We begin with some lemmas.

Lemma 1. *If $f(t)$ is an almost periodic function, then the sequence*

$$\{h(n)\}_{n\in\mathbb{Z}} = \left\{ \int_n^{n+1} e^{a(n+1-s)} f(s)ds \right\}_{n\in\mathbb{Z}} \tag{3.2}$$

is an almost periodic sequence with the Bohr spectrum $\sigma_b(h) = e^{i\Lambda_f}$. Furthermore, if $f(t) \in \mathcal{QP}(\omega)$, then $\{h(n)\} \in \mathcal{QP}(\omega;\mathbb{Z})$. Here a is a constant.

Proof. The almost periodicity of $\{h(n)\}$ has been proved in [20, 22]. It suffices to study the spectrum relation. Since $f(t)$ is an almost periodic function, it follows from the Approximation Theorem 1 that for any $k > 0$ there exists $\tilde{f}_k(t) = \sum_{m=1}^{n_k} \tilde{F}_{k,m} e^{i\lambda_m t}$, $\lambda_m \in \Lambda_f$ such that

$$|f(t) - \tilde{f}_k(t)| < \frac{1}{k}, \quad k = 1, 2, \cdots.$$

Define

$$\tilde{h}_k(n) := \int_n^{n+1} e^{a(n+1-s)} \tilde{f}_k(s)ds, \quad \forall n \in \mathbb{Z}, k = 1, 2, \cdots \tag{3.3}$$

A direct calculation shows

$$\tilde{h}_k(n) = \sum_{m=1}^{n_k} \tilde{F}_{k,m} \frac{[e^a - e^{i\lambda_m}]}{a - i\lambda_m} e^{i\lambda_m n} := \sum_{m=1}^{n_k} \tilde{H}_{k,m} e^{i\lambda_m n},$$

where $\tilde{H}_{k,m} := \tilde{F}_{k,m}\frac{[e^a - e^{i\lambda_m}]}{a-i\lambda_m}$, which is the Fourier coefficient corresponding to the Fourier exponent $e^{i\lambda_m}$. $\tilde{h}_k(n)$ is a trigonometric polynomial, $\sigma_b(\tilde{h}_k) = e^{i\Lambda_{\tilde{f}_k}} \subset e^{i\Lambda_f} := \{e^{i\lambda_m},\ \lambda_m \in \Lambda_f\}$. It is easy to see that

$$\sup_{n\in\mathbb{Z}} |\tilde{h}_k(n) - h(n)| < \frac{e^{|a|}}{k}, \quad \forall k = 1, 2, \cdots .$$

This implies that $\sigma_b(h) \subset e^{i\Lambda_f}$. For any fixed $\lambda_{j_0} \in \Lambda_f$, we can assume $n_k > j_0$. So, we can rewrite

$$\tilde{f}_k(t) = \sum_{m=1, m\neq j_0}^{n_k} \tilde{F}_{k,m} e^{i\lambda_m t} + \tilde{F}_{k,j_0} e^{i\lambda_{j_0} t}.$$

It is easy to see that $a(\lambda_{j_0}; f) = \lim_{k\to\infty} a(\lambda_{j_0}; \tilde{f}_k) = \lim_{k\to\infty} \tilde{F}_{k,j_0}$. Thus, we can obtain

$$a(e^{i\lambda_{j_0}}; h) = \lim_{k\to\infty} a(e^{i\lambda_{j_0}}; \tilde{h}_k) = \lim_{k\to\infty} \tilde{H}_{k,j_0} = a(\lambda_{j_0}; f)\frac{e^a - e^{i\lambda_{j_0}}}{a - i\lambda_{j_0}}.$$

It follows that $e^{i\lambda_{j_0}} \in \sigma_b(h)$. Thus, $\sigma_b(h) = e^{i\Lambda_f}$. If $f(t) \in \mathcal{QP}(\omega)$, then $\Lambda_f = \{2\pi < m, \omega > \big| \forall m \in \mathbb{Z}^r\}$ and $\tilde{f}_k(t)$ can be taken as partial sum of $f(t)$. □

Lemma 2. *If $f \in \mathcal{GQP}(1, \omega)$, then $\{h(n)\}$ defined by (3.2) is in $\mathcal{QP}(\omega, \mathbb{Z})$.*
Proof. Since $f \in \mathcal{GQP}(1, \omega)$, we have

$$f(t) = \sum_k f_k(\{t\}) e^{i2\pi<k,\omega>[t]}.$$

Clearly,

$$\int_n^{n+1} e^{a(n+1-s)} f(s)ds = \int_n^{n+1} e^{a(n+1-s)} \sum_k f_k(\{s\}) e^{i2\pi<k,\omega>n} ds$$
$$= \sum_k \int_0^1 e^{as} f_k(1-s)ds \cdot e^{i2\pi<k,\omega>n}.$$

□

Lemma 3. *If $f(t)$, $g(t) \in \mathcal{GQP}(1, \omega)$, then $f(t)g(t) \in \mathcal{GQP}(1, \omega)$.*
Proof. If

$$f(t) = \sum_k f_k(\{t\}) e^{i2\pi<k,\omega>[t]},\ g(t) = \sum_k g_k(\{t\}) e^{i2\pi<k,\omega>[t]},$$

it follows from Cauchy's Theorem that we can write

$$h(t) = f(t)g(t) = \sum_k f_k(\{t\}) e^{i2\pi<k,\omega>[t]} \cdot \sum_j g_j(\{t\}) e^{i2\pi<j,\omega>[t]}$$
$$= \sum_m h_m(\{t\}) e^{i2\pi<m,\omega>[t]}.$$

and $\sum_m |h_m| < +\infty$. □

Lemma 4. *If* $f(t) \in \mathcal{GQP}(1, \omega)$, *then*

$$\int_n^t e^{a(t-s)} f(s)ds = \sum_k \tilde{f}_k(\{t\}) e^{i2\pi<k,\omega>n}, \; n \le t < n+1, \tag{3.4}$$

where $\tilde{f}_k(\tau)$ *is continuous for each* k.
Proof. Since $f(t) \in \mathcal{GQP}(1, \omega)$, we have

$$f(t) = \sum_k f_k(\{t\}) e^{i2\pi<k,\omega>[t]}, \; \sum_k \sup_{\tau \in [0,1]} |f_k(\tau)| < +\infty.$$

$$\begin{aligned} \int_n^t e^{a(t-s)} f(s)ds &= \int_n^t e^{a(t-s)} \sum_k f_k(\{s\}) e^{i2\pi<k,\omega>[s]} ds \\ &= \sum_k e^{a(t-n)} \int_0^{t-n} e^{-as} f_k(s)ds \cdot e^{i2\pi<k,\omega>n} \\ &= \sum_k \tilde{f}_k(\{t\}) e^{i2\pi<k,\omega>n}, \; n \le t < n+1, \end{aligned}$$

where

$$\tilde{f}_k(\tau) = e^{a\tau} \int_0^\tau e^{-as} f_k(s)ds.$$

Clearly, $\sum_k \sup_{\tau \in [0,1]} |\tilde{f}_k(\tau)| < +\infty$. □

Since the concept of almost periodic functions is in terms of functions on $\mathbb{R}$, it is suggested that one should find a solution of Eq.(3.1) on $\mathbb{R}$, and not just on $\mathbb{R}^+$. If $x(t)$ is a solution of Eq.(3.1) on $\mathbb{R}$, we have the following relations:

$$x(t) = e^{a^0(t-n)} x(n) + (e^{a^0(t-n)} - 1) \sum_{i=-N}^{N} {a^0}^{-1} a_i^0 x(n+i) + \int_n^t e^{a^0(t-s)} f(s)ds, \tag{3.5}$$

$n \le t \le n+1$. In view of the continuity of a solution at a point, we arrive at the following difference equation

$$x(n+1) = e^{a^0} x(n) + \sum_{i=-N}^{N} (e^{a^0} - 1) {a^0}^{-1} a_i^0 x(n+i) + \int_n^{n+1} e^{a^0(n+1-s)} f(s)ds, \qquad n \in \mathbb{Z}. \tag{3.6}$$

By using (H3), Eq.(3.6) can be rewritten in the form

$$\sum_{i=-N}^{N} A_i x(n+i) = h(n), \tag{3.7}$$

where

$$h(n) = -\int_n^{n+1} e^{a^0(n+1-s)} f(s)ds.$$

In what follows, our idea is to study the properties of the sequence solution of Eq.(3.7). Through the sequence solution and (3.5), we study the properties of solution of Eq.(3.1). In [22], we proved the following results.

Lemma 5. *Assume that (H3) holds. Then for any* $\{h(n)\} \in \mathcal{QP}(\omega_1, \mathbb{Z}) \oplus \mathcal{QP}(\omega_2, \mathbb{Z})$, *the system (3.7) possesses a unique sequence solution* $\{x^*(n)\} \in \mathcal{QP}(\omega_1, \mathbb{Z}) \oplus \mathcal{QP}(\omega_2, \mathbb{Z})$, *where*

$$\begin{aligned}
&\mathcal{QP}(\omega_1, \mathbb{Z}) \oplus \mathcal{QP}(\omega_2, \mathbb{Z}) \\
&\quad = \Big\{\{u(n)\}\Big| u(n) = \sum_{k_1} u^{(1)}_{k_1} e^{i2\pi <k_1,\omega_1> n} + \sum_{k_2} u^{(2)}_{k_2} e^{i2\pi <k_2,\omega_2> n}; \\
&\qquad \sum_{k_1} |u^{(1)}_{k_1}| + \sum_{k_2} |u^{(2)}_{k_2}| < +\infty,\ u^{(1)}_{-k_1} = \bar{u}^{(1)}_{k_1},\ u^{(2)}_{-k_2} = \bar{u}^{(2)}_{k_2}\Big\}.
\end{aligned}$$

Furthermore, there exists a constant M_A *such that*

$$||x^*|| \le M_A ||h|| \triangleq M_A \sup_{n\in\mathbb{Z}} |h(n)|.$$

Remark 1. It should be pointed out that when ω is rationally independent, Lemmas 2-5 have been shown in [23]. In this paper, we focus on the general case. We can find that its proofs are in an analogous way to those in [23]. We give its proofs only for reader's convenience.

Theorem 1. *Assume that (H3) holds. We have the following conclusions.*

(i). For any almost periodic $f(t)$*, Eq.(3.1) has a unique almost periodic solution* $x(t)$ *and there exists* $U > 0$ *such that* $||x|| \le U||f||$.

(ii). For any $f(t) \in \mathcal{QP}(\omega)$*, Eq.(3.1) has a unique solution* $x(t) \in \mathcal{GQP}(1, \omega)$ *and there exists* $U > 0$ *such that* $||x|| \le U||f||$.

Conclusion (i) has been proven in [20] (or see [22]). Conclusion (ii) has essentially been shown in [10] because we could easily find that ω is not needed to be rationally independent. We recall the proof's main points for convenience. First, we show that Eq.(3.7) possesses a unique bounded sequence solution $x^*(n)$, which is almost periodic sequence in case (i) and in $\mathcal{QP}(\omega, \mathbb{Z})$ in case (ii), respectively. Then, define a solution $x(t)$ of Eq.(3.1) by using (3.5) with $x(n) = x^*(n)$, $n \in \mathbb{Z}$. $x(t)$ is continuous on $\mathbb{R}$. It can be shown that $x(t)$ is almost periodic in case (i) and in $\mathcal{GQP}(1, \omega)$ in case (ii), respectively. It is easy to see that under (H3) the bounded solution on $\mathbb{R}$ of Eq.(3.1) is unique. In view of that $f(t) \in \mathcal{QP}(\omega)$ is almost periodic, it follows that the obtained solution $x(t)$ is almost periodic in case (ii).

Corollary 1. *Assume that (H3) holds and* $1, \omega$ *are rationally independent. Then for any* $f(t) \in \mathcal{QP}(\omega)$*, Eq.(3.1) has a unique quasi-periodic solution* $x(t)$ *with frequencies* $(1, \omega)$.

From Theorem 1, it follows that Eq.(3.1) possesses a unique almost periodic solution $x(t)$ and $x(t) \in \mathcal{GQP}(1, \omega)$. It has been shown in [10] that the unique continuous almost periodic solution $x(t)$ can be written in details as

$$x(t) = F(t, \omega t),$$

where

$$\begin{aligned}F(\theta_1,\theta_2) &= e^{a^0\{\theta_1\}}\sum_k G_k e^{i2\pi<k,\theta_2>}e^{-i2\pi<k,\omega>\{\theta_1\}}\\ &+(e^{a^0\{\theta_1\}}-1)\sum_{j=-N}^{N} a^{0^{-1}}a_j^0\sum_k G_k e^{i2\pi<k,\omega>j}e^{i2\pi<k,\theta_2>}e^{-i2\pi<k,\omega>\{\theta_1\}}\\ &+\sum_k \frac{f_k}{i2\pi<k,\omega>-a^0}e^{i2\pi<k,\theta_2>}\\ &-\sum_k \frac{f_k}{i2\pi<k,\omega>-a^0}e^{i2\pi<k,\theta_2>}e^{-i2\pi<k,\omega>\{\theta_1\}}e^{a^0\{\theta_1\}}\end{aligned}$$

(see [10] for details). Clearly, F is periodic in θ_1 and θ_2 with period 1, and Lebesgue integral. Setting $\hat{\theta}=(\theta_1,\theta_2)$, we have the Fourier expansion

$$F(\theta_1,\theta_2)\sim\sum_{\hat{m}} F_{\hat{m}}e^{i2\pi<\hat{m},\hat{\theta}>},$$

where the Fourier coefficient

$$F_{\hat{m}}=\int_{\mathbb{T}^{1+r}} F(\hat{\theta})e^{-i2\pi<\hat{m},\hat{\theta}>}d\hat{\theta}.$$

We get the Fourier expansion of $x(t)$ as

$$x(t)\sim\sum_{\hat{m}} F_{\hat{m}}e^{i2\pi<\hat{m},\hat{\omega}>t},$$

where $\hat{\omega}=(1,\omega)$. It follows that $\Lambda_x=\{2\pi<\hat{m},\hat{\omega}>\}$. Since $1,\omega$ are rationally independent, it is known that Λ_x has a finite and integer base $1,\omega$. Therefore, $x(t)$ is a quasi-periodic function with frequencies $(1,\omega)$ in Bogoliubov's sense.

4. Quasi-Periodic Differential Equations With Parameter

In this section, we consider EPCA with a parameter

$$\epsilon y'(t)=d^0y(t)+\sum_{i=-N}^{N} d_i^0 y([t+i])+g(t). \tag{4.1}$$

Because an almost periodic function is defined on $\mathbb{R}$, we continue to find a solution of Eq.(4.1) on $\mathbb{R}$, and not just on $\mathbb{R}^+$. If $y(t)$ is a solution of Eq.(4.1) on $\mathbb{R}$, we obtain the following relations:

$$y(t)=e^{d^0(\frac{t}{\epsilon}-\frac{n}{\epsilon})}y(n)+\sum_{i=-N}^{N}(e^{d^0(\frac{t}{\epsilon}-\frac{n}{\epsilon})}-1)d^{0^{-1}}d_i^0y(n+i)+\int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{t}{\epsilon}-\sigma)}g(\epsilon\sigma)d\sigma, \tag{4.2}$$

$n \le t \le n+1$. In view of the continuity of a solution at a point, we arrive at the following difference equation:

$$y(n+1) = e^{\frac{d^0}{\epsilon}} y(n) + \sum_{i=-N}^{N} (e^{\frac{d^0}{\epsilon}} - 1) {d^0}^{-1} d_i^0 y(n+i) + \int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon} - \sigma)} g(\epsilon\sigma) d\sigma. \quad (4.3)$$

Let

$$\begin{aligned} D_0(\epsilon) &= e^{\frac{d^0}{\epsilon}} + {d^0}^{-1} d_0^0 (e^{\frac{d^0}{\epsilon}} - 1), \\ D_1(\epsilon) &= {d^0}^{-1} d_1^0 (e^{\frac{d^0}{\epsilon}} - 1) - 1, \\ D_i(\epsilon) &= {d^0}^{-1} d_i^0 (e^{\frac{d^0}{\epsilon}} - 1), \; i = -1, \pm 2, \cdots, \pm N, \\ e(n, \epsilon) &= - \int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon} - \sigma)} g(\epsilon\sigma) d\sigma. \end{aligned}$$

Then Eq.(4.3) can be written as

$$\sum_{i=-N}^{N} D_i(\epsilon) y(n+i) = e(n, \epsilon). \quad (4.4)$$

Lemma 6. *If $g(t)$ is an almost periodic function, then for any fixed $\epsilon > 0$, the sequence*

$$\{e(n,\epsilon)\}_{n\in\mathbb{Z}} = \Big\{ \int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon} - \sigma)} g(\epsilon\sigma) d\sigma \Big\}_{n\in\mathbb{Z}}$$

is an almost periodic sequence with the Bohr spectrum $\sigma_b(e) = e^{i\Lambda_g}$. Furthermore, if $f(t) \in \mathcal{QP}(\omega)$, then $\{e(n,\epsilon)\} \in \mathcal{QP}(\omega; \mathbb{Z})$.

Proof. The almost periodicity of $\{e(n,\epsilon)\}$ has been proved in [20]. Note that $e(n,\epsilon)$ can be rewritten as

$$e(n,\epsilon) = \int_{n}^{n+1} \frac{1}{\epsilon} e^{\frac{d^0}{\epsilon}(n+1-\sigma)} g(\sigma) d\sigma.$$

The rest parts are the same as the proof of Lemma 1. So we omit it. $\square$

Lemma 7. *If $g(t) \in \mathcal{GQP}(1,\omega)$, then*

$$\int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon} - s)} g(\epsilon s) ds = \sum_k \tilde{g}_k(\{t\}, \epsilon) e^{i2\pi<k,\omega>n}, \; n \le t < n+1,$$

where $\tilde{g}_k(\tau,\epsilon)$ is continuous on τ for each k and $\epsilon > 0$.

Proof. Since $g(t) \in \mathcal{GQP}(1,\omega)$, we have

$$g(t) = \sum_k g_k(\{t\}) e^{i2\pi<k,\omega>[t]}, \; \sum_k \sup_{\tau\in[0,1]} |g_k(\tau)| < +\infty.$$

It is easy to see that

$$\begin{aligned}\int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon}-s)} g(\epsilon s)ds &= \int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon}-s)} \sum_k g_k(\{\epsilon s\}) e^{i2\pi<k,\omega>[\epsilon s]} ds \\ &= \sum_k e^{\frac{d^0}{\epsilon}(t-n)} \int_0^{t-n} \frac{1}{\epsilon} e^{-\frac{d^0}{\epsilon}s} g_k(s)ds \cdot e^{i2\pi<k,\omega>n} \\ &= \sum_k \tilde{g}_k(\{t\},\epsilon) e^{i2\pi<k,\omega>n}, \quad n \le t < n+1,\end{aligned}$$

where

$$\tilde{g}_k(\tau,\epsilon) = \frac{1}{\epsilon} e^{\frac{d^0}{\epsilon}\tau} \int_0^{\tau} e^{-\frac{d^0}{\epsilon}s} g_k(s)ds.$$

It is easy to see $|\tilde{g}_k(\tau,\epsilon)| \le \frac{1}{|d^0|}\sup_{0\le\tau\le 1}|g_k(\tau)|$ for $0 \le \tau \le 1$. □

Theorem 2. *Suppose that (H2) and (H4) hold. Then there exists an $\epsilon_1 > 0$ such that when $0 < \epsilon \le \epsilon_1$, for any $\{e(n,\epsilon)\} \in \mathcal{QP}(\omega,\mathbb{Z})$, the difference equation (4.4) has a unique sequence solution $y^*(n,\epsilon) \in \mathcal{QP}(\omega,\mathbb{Z})$ and there exists a constant $\tilde{V} > 0$ such that*

$$|y^*(n,\epsilon)| \le \tilde{V} \sup_{n\in\mathbb{Z}} |e(n,\epsilon)|, \ 0 < \epsilon \le \epsilon_1, \ n \in \mathbb{Z}. \tag{4.5}$$

Proof. Since $\lim_{\epsilon\to 0} D_i(\epsilon) = D_i, \ i = 0, \pm 1, \cdots, \pm N$, it follows from (H4) that there exists ϵ_1 such that when $0 < \epsilon \le \epsilon_1$, all roots of the algebraic equation

$$\sum_{i=-N}^{N} D_i(\epsilon)\mu^i = 0 \tag{4.6}$$

are not on S^1. For such $\epsilon > 0$, suppose that the different roots of Eq.(4.6) are denoted by $\mu_1(\epsilon), \cdots, \mu_s(\epsilon)$, with multiplicities $\kappa_1, \cdots, \kappa_s$, $\sum_{j=1}^{s}\kappa_j = 2N$. Let $L = \{l|\ |\mu_l(\epsilon)| < 1, \ 1 \le l \le 2N\}$, $L' = \{l|\ |\mu_l(\epsilon)| > 1, \ 1 \le l \le 2N\}$. Note that the multiplicity of characteristic roots will vary with $\epsilon > 0$. We omit ϵ dependence in s, κ_j, L, L' only for simplicity. We define a sequence $\{y(n,\epsilon)\}$ by

$$\begin{aligned} y(n,\epsilon) = &\sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j} \sum_{m\le n-1} [n-(m+1)]^j \mu_l^{n-(m+1)}(\epsilon) e(m,\epsilon) \\ &+ \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j} \sum_{m\ge n} [n-(m+1)]^j \mu_l^{n-(m+1)}(\epsilon) e(m,\epsilon), \end{aligned} \tag{4.7}$$

where the unknown constants $k_{l,j}$, $0 \le j \le \kappa_l - 1, 1 \le l \le s$, are determined later.

Putting the sequence $\{y(n,\epsilon)\}$ defined by (4.7) into Eq.(4.4), we can obtain

$$\begin{cases} \sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l(\epsilon) - \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l(\epsilon) = 0, \\ \cdots\cdots \\ \sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{N-2}(\epsilon)(N-2)^j - \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{N-2}(\epsilon)(N-2)^j = 0, \\ \sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{N-1}(\epsilon)(N-1)^j - \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{N-1}(\epsilon)(N-1)^j = \frac{1}{D_N(\epsilon)}, \\ \sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{N}(\epsilon)N^j - \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{N}(\epsilon)N^j = -\frac{D_{N-1}(\epsilon)}{D_N^2(\epsilon)}, \\ \cdots\cdots \\ \sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{2N}(\epsilon)(2N)^j - \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j}\mu_l^{2N}(\epsilon)(2N)^j \\ \qquad = R(D_{-1}(\epsilon), D_0(\epsilon), \cdots, D_N(\epsilon)), \end{cases} \tag{4.8}$$

where $R(D_{-1}(\epsilon), D_0(\epsilon), \cdots, D_N(\epsilon))$ is a rational function of $D_{-1}(\epsilon), \cdots, D_N(\epsilon)$. If we see $k_l(l\in L), -k_l(l\in L')$ as unknown variables, then the coefficient of the linear system (4.8) is the Casorati matrix. Its determinant is different from zero (see [11] for details). Hence, we can uniquely determine a set of values $(k_1^*(\epsilon), \cdots, k_{2N}^*(\epsilon))$ from Eq.(4.8). Therefore, the sequence $\{y^*(n,\epsilon)\}$ defined by

$$\begin{aligned} y^*(n,\epsilon) = & \sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k_{l,j}^*(\epsilon)\sum_{m\le n-1}[n-(m+1)]^j\mu_l^{n-(m+1)}(\epsilon)e(m,\epsilon) \\ & + \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k_{l,j}^*(\epsilon)\sum_{m\ge n}[n-(m+1)]^j\mu_l^{n-(m+1)}(\epsilon)e(m,\epsilon) \end{aligned} \tag{4.9}$$

is a solution of the difference equation (4.4).

Since $e(n,\epsilon)\in \mathcal{QP}(\omega,\mathbb{Z})$, we can assume

$$e(n,\epsilon) = \sum_m E_m(\epsilon)e^{i2\pi<m,\omega>n}, \quad \sum_m |E_m(\epsilon)| < +\infty,\ 0<\epsilon\le\epsilon_1.$$

At this moment, $y^*(n,\epsilon)$ in (4.9) can be rewritten as

$$y^*(n,\epsilon) = \sum_m Y_m(\epsilon)e^{i2\pi<m,\omega>n},$$

where

$$Y_m(\epsilon) = E_m(\epsilon)[\sum_{l\in L}\sum_{j=0}^{\kappa_l-1} k^*_{l,j}(\epsilon)e^{-i2\pi<m,\omega>}p_j(\mu_l(\epsilon)e^{-i2\pi<m,\omega>})$$
$$+\sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} k^*_{l,j}(\epsilon)(-1)^j\mu_l^{-1}(\epsilon)p_j(\mu_l^{-1}(\epsilon)e^{-i2\pi<m,\omega>})],$$

$$p_j(x) = 1 + x + 2^jx^2 + \cdots + u^jx^u + \cdots, \quad |x| < 1.$$

For each $0 < \epsilon \le \epsilon_1$, we have

$$|Y_m(\epsilon)| \le |E_m(\epsilon)|[\sum_{l\in L}\sum_{j=0}^{\kappa_l-1} |k^*_{l,j}(\epsilon)|p_j(|\mu_l(\epsilon)|) + \sum_{l\in L'}\sum_{j=0}^{\kappa_l-1} |k^*_{l,j}(\epsilon)||\mu_l^{-1}(\epsilon)|p_j(|\mu_l^{-1}(\epsilon)|)].$$

It follows that $\{y^*(n,\epsilon)\} \in \mathcal{QP}(\omega,\mathbb{Z})$ for each $0 < \epsilon \le \epsilon_1$. It suffices to show (4.5). We consider the system

$$\sum_{i=-N}^{N} D_iy(n+i) = 0, \tag{4.10}$$

where D_i are as in (H4). Without loss of generalization, we can assume $D_N \ne 0$. (4.10) becomes

$$y(n+N) = -\frac{D_{N-1}}{D_N}y(n+N-1) - \cdots - \frac{D_0}{D_N}y(n) - \cdots - \frac{D_{-N}}{D_N}y(n-N). \tag{4.11}$$

Clearly, Eq.(4.11) is equivalent to the system

$$u(n+1) = Cu(n), \tag{4.12}$$

where

$$u(n) = \begin{pmatrix} u_0(n) \\ u_1(n) \\ \vdots \\ u_{2N-1}(n) \end{pmatrix} := \begin{pmatrix} y(n-N) \\ y(n-N+1) \\ \vdots \\ y(n+N-1) \end{pmatrix},\ C = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ -\frac{D_{N-1}}{D_N} & -\frac{D_{N-2}}{D_N} & \cdots & -\frac{D_{-N}}{D_N} \end{pmatrix}$$

Similarly, the system

$$\sum_{i=-N}^{N} D_i(\epsilon)y(n+i) = 0 \tag{4.13}$$

is equivalent to the system

$$u(n+1) = C(\epsilon)u(n), \tag{4.14}$$

where

$$C(\epsilon) = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ -\frac{D_{N-1}(\epsilon)}{D_N(\epsilon)} & -\frac{D_{N-2}(\epsilon)}{D_N(\epsilon)} & \cdots & -\frac{D_{-N}(\epsilon)}{D_N(\epsilon)} \end{pmatrix}$$

From (H4), we know that (4.12) admits an exponential dichotomy. From the roughness theorem for exponential dichotomies, we know that there exists $\epsilon_1 > 0$ (without loss of generalization) such that when $0 < \epsilon \leq \epsilon_1$, the system (4.14) also admits an exponential dichotomy. The unique bounded solution $y^*(n, \epsilon)$ of system (4.4) can be estimated by the constants appeared in the exponential dichotomy and the constants can be chosen to be independent of the small perturbation (see [16, Prop. 2.10]) so that we can obtain that there exists a constant $\tilde{V}$ such that

$$|y^*(n, \epsilon) \leq \tilde{V} \sup_{n\in\mathbb{Z}} |e(n, \epsilon)|, \quad 0 < \epsilon \leq \epsilon_1,\ n \in \mathbb{Z}.$$

□

Theorem 3. *Suppose that (H2) and (H4) hold. Then there exists an $\epsilon_1 > 0$ such that when $0 < \epsilon \leq \epsilon_1$, for any $g(t) \in \mathcal{QP}(\omega)$, Eq.(4.1) has a unique solution $y(t, g, \epsilon) \in \mathcal{GQP}(1, \omega)$. The map $g \to y(g, \epsilon)$ defines a bounded linear operator $V_\epsilon g$ satisfying $||V_\epsilon|| \leq V$, $0 < \epsilon \leq \epsilon_1$. The map $\epsilon \to V_\epsilon$ is continuous for $0 < \epsilon \leq \epsilon_1$.*
Proof. From Lemma 6, it follows that

$$e(n, \epsilon) = \int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon} - \sigma)} g(\epsilon\sigma) d\sigma, \quad n \in \mathbb{Z}$$

is in $\mathcal{QP}(\omega, \mathbb{Z})$ for each fixed $\epsilon > 0$. Obviously, it can be assumed that if $0 < \epsilon \leq \epsilon_1$,

$$|e(n, \epsilon)| \leq \frac{2}{|d_0|} \sup_{t\in\mathbb{R}} |g(t)| := \frac{2}{|d_0|} ||g||.$$

Theorem 2 implies that Eq.(4.3) possesses a unique sequence solution $y^*(n, \epsilon) \in \mathcal{QP}(\omega, \mathbb{Z})$. We define by (4.2)

$$y(t, \epsilon) = e^{d^0(\frac{t}{\epsilon} - \frac{n}{\epsilon})} y(n) + \sum_{i=-N}^{N} (e^{d^0(\frac{t}{\epsilon} - \frac{n}{\epsilon})} - 1) {d^0}^{-1} d_i^0 y(n + i) + \int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{t}{\epsilon} - \sigma)} g(\epsilon\sigma) d\sigma,$$

with $y(n) = y^*(n, \epsilon), n \leq t < n + 1$. Since $y^*(n, \epsilon)$ is a solution of Eq.(4.3), it follows that $y(t, \epsilon)$ is continuous on $\mathbb{R}$ and satisfies Eq.(4.1). Since $y^*(n, \epsilon) \in \mathcal{QP}(\omega, \mathbb{Z})$, it follows from Lemma 7 that $y(t, \epsilon)$ can be rewritten in the form

$$y(t, \epsilon) = G(t, \omega t, \epsilon),$$

where $G(\theta_1, \theta_2, \epsilon)$ can be rewritten as

$$\begin{aligned} G(\theta_1, \theta_2, \epsilon) = {} & e^{\frac{d^0}{\epsilon}\{\theta_1\}} \sum_k E_k(\epsilon) e^{i2\pi<k,\theta_2>} e^{-i2\pi<k,\omega>\{\theta_1\}} \\ & + (e^{\frac{d^0}{\epsilon}\{\theta_1\}} - 1) \sum_{j=-N}^{N} {d^0}^{-1} d_j^0 \sum_k E_k(\epsilon) e^{i2\pi<k,\omega>j} e^{i2\pi<k,\theta_2>} e^{-i2\pi<k,\omega>\{\theta_1\}} \\ & + \sum_k \frac{g_k}{i2\pi <k, \omega> \epsilon - d^0} e^{i2\pi<k,\theta_2>} \\ & - \sum_k \frac{g_k}{i2\pi <k, \omega> \epsilon - d^0} e^{i2\pi<k,\theta_2>} e^{-i2\pi<k,\omega>\{\theta_1\}} e^{\frac{d^0}{\epsilon}\{\theta_1\}}, \end{aligned}$$

if we write $g(t) = \sum_k g_k e^{i2\pi<k,\omega>t} \in \mathcal{QP}(\omega)$. Clearly, $y(t,\epsilon) \in \mathcal{GQP}(1,\omega)$. The boundedness and continuity are similar to those in [20]. So we omit it. $\square$

5. Quasi-Periodic Singularly Perturbed Differential Equations

In this section, we consider the following singularly perturbed differential equation

$$\begin{cases} x'(t) = a(t,\epsilon)x(t) + \sum_{i=-N}^{N} a_i(t,\epsilon)x([t+i]) + b(t,\epsilon)y(t) + \sum_{i=-N}^{N} b_i(t,\epsilon)y([t+i]) + f(t), \\ \epsilon y'(t) = c(t,\epsilon)x(t) + \sum_{i=-N}^{N} c_i(t,\epsilon)x([t+i]) + d(t,\epsilon)y(t) + \sum_{i=-N}^{N} d_i(t,\epsilon)y([t+i]) + g(t), \end{cases} \tag{5.1}$$

where $f(t), g(t), a(t,\epsilon), b(t,\epsilon)$, etc., are quasi-periodic. In the following, we sometimes use the notations $a_\epsilon(t) = a(t,\epsilon)$, $a_{i,\epsilon}(t) = a_i(t,\epsilon)$ for convenience, ect.

Theorem 4. *Assume that $a(t,\epsilon)$, $a_j(t,\epsilon)$, $b(t,\epsilon)$, $b_j(t,\epsilon)$, $c(t,\epsilon)$, $c_j(t,\epsilon)$, $d(t,\epsilon)$, $d_j(t,\epsilon) \in \mathcal{QP}(\omega)$. Under assumptions (H1)-(H4), there exist ϵ_1, $0 < \epsilon_1 \le \epsilon_0$, positive functions $\alpha_{i,j}(\epsilon)$, $1 \le i,j \le 2$ defined for $0 < \epsilon \le \epsilon_1$, satisfying*

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \alpha_{1,1}(\epsilon) = U, \qquad \lim_{\epsilon\to 0^+} \alpha_{1,2}(\epsilon) = 2UV(N+1)M, \\ &\lim_{\epsilon\to 0^+} \alpha_{2,1}(\epsilon) = 0, \qquad \lim_{\epsilon\to 0^+} \alpha_{2,2}(\epsilon) = V, \\ &\alpha_{i,j}(\epsilon) \le 2UV(N+1)M + U + V,\ 1 \le i,j \le 2 \end{aligned}$$

such that for any $f(t) \in \mathcal{QP}(\omega_1), g(t) \in \mathcal{QP}(\omega_2), 0 < \epsilon \le \epsilon_1$, system (5.1) possesses a unique solution $(x^*(t,f,g,\epsilon), y^*(t,f,g,\epsilon) \in \mathcal{GQP}(1,(\omega,\omega_1,\omega_2))$. The solution satisfies

$$||x|| \le \alpha_{1,1}(\epsilon)||f|| + \alpha_{1,2}(\epsilon)||g||,\ ||y|| \le \alpha_{2,1}(\epsilon)||f|| + \alpha_{2,2}(\epsilon)||g||. \tag{5.2}$$

The map $(f,g) \to (x(f,g,\epsilon), y(f,g,\epsilon))$ defines a bounded linear operator $\mathcal{L}(\epsilon)$ satisfying $||\mathcal{L}(\epsilon)|| \le 4UV(N+1)M + 2U + 2V$ and $\epsilon \to \mathcal{L}(\epsilon)$ is continuous for $0 < \epsilon \le \epsilon_1$.

Proof. Let $\tilde{\omega} = (\omega, \omega_1, \omega_2)$. For any $(x_0(t), y_0(t)) \in \mathcal{GQP}(1, \tilde{\omega})$, we consider

$$\begin{cases} x'(t) = a^0 x(t) + \displaystyle\sum_{i=-N}^{N} a_i^0 x([t+i]) + (a_\epsilon(t) - a^0) x_0(t) \\ \qquad + \displaystyle\sum_{i=-N}^{N} (a_{i,\epsilon}(t) - a_i^0) x_0([t+i]) + b_\epsilon(t) y(t) + \displaystyle\sum_{i=-N}^{N} b_{i,\epsilon}(t) y([t+i]) + f(t), \\ \epsilon y'(t) = d^0 y(t) + \displaystyle\sum_{i=-N}^{N} d_i^0 y([t+i]) + (d_\epsilon(t) - d^0) y_0(t) \\ \qquad + \displaystyle\sum_{i=-N}^{N} (d_{i,\epsilon}(t) - d_i^0) y_0([t+i]) + c_\epsilon(t) x_0(t) + \displaystyle\sum_{i=-N}^{N} c_{i,\epsilon}(t) x_0([t+i]) + g(t). \end{cases} \tag{5.3}$$

From the variation of constants formula, it follows that

$$\begin{aligned} x(t) &= e^{a^0(t-n)} x(n) + \sum_{i=-N}^{N} (e^{a^0(t-n)} - 1) a^{0^{-1}} a_i^0 x(n+i) \\ &+ \sum_{i=-N}^{N} \int_n^t e^{a^0(t-s)} [a_{i,\epsilon}(s) - a_i^0] ds \cdot x_0(n+i) \\ &+ \sum_{i=-N}^{N} \int_n^t e^{a^0(t-s)} b_{i,\epsilon}(s) ds \cdot y(n+i) \\ &+ \int_n^t e^{a^0(t-s)} [(a_\epsilon(s) - a^0) x_0(s) + b_\epsilon(s) y(s) + f(s)] ds, \\ y(t) &= e^{d^0(\frac{t}{\epsilon} - \frac{n}{\epsilon})} y(n) + \sum_{i=-N}^{N} (e^{d^0(\frac{t}{\epsilon} - \frac{n}{\epsilon})} - 1) d^{0^{-1}} d_i^0 y(n+i) \\ &+ \sum_{i=-N}^{N} \int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{t}{\epsilon} - \sigma)} (d_{i,\epsilon}(\epsilon\sigma) - d_i^0) d\sigma \cdot y_0(n+i) \\ &+ \sum_{i=-N}^{N} \int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{t}{\epsilon} - \sigma)} c_{i,\epsilon}(\epsilon\sigma) d\sigma \cdot x_0(n+i) \\ &+ \int_{\frac{n}{\epsilon}}^{\frac{t}{\epsilon}} e^{d^0(\frac{t}{\epsilon} - \sigma)} [(d_\epsilon(\epsilon\sigma) - d^0) y_0(\epsilon\sigma) + c_\epsilon(\epsilon\sigma) x_0(\epsilon\sigma) + g(\epsilon\sigma)] d\sigma, \ n \le t < n+1. \end{aligned} \tag{5.4}$$

In view of the continuity of a solution at a point, we arrive at the following difference equation

$$
\begin{aligned}
x(n+1) &= e^{a^0}x(n) + \sum_{i=-N}^{N}(e^{a^0}-1){a^0}^{-1}a_i^0 x(n+i) \\
&+ \sum_{i=-N}^{N}\int_n^{n+1} e^{a^0(n+1-s)}[a_{i,\epsilon}(s)-a_i^0]ds\cdot x_0(n+i) \\
&+ \sum_{i=-N}^{N}\int_n^{n+1} e^{a^0(n+1-s)}b_{i,\epsilon}(s)ds\cdot y(n+i) \\
&+ \int_n^{n+1} e^{a^0(n+1-s)}[(a_\epsilon(s)-a^0)x_0(s)+b_\epsilon(s)y(s)+f(s)]ds, \\
y(n+1) &= e^{\frac{d^0}{\epsilon}}y(n) + \sum_{i=-N}^{N}(e^{\frac{d^0}{\epsilon}}-1){d^0}^{-1}d_i^0 y(n+i) \\
&+ \sum_{i=-N}^{N}\int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon}-\sigma)}(d_{i,\epsilon}(\epsilon\sigma)-d_i^0)d\sigma\cdot y_0(n+i) \\
&+ \sum_{i=-N}^{N}\int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon}-\sigma)}c_{i,\epsilon}(\epsilon\sigma)d\sigma\cdot x_0(n+i) \\
&+ \int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{d^0(\frac{n+1}{\epsilon}-\sigma)}[(d_\epsilon(\epsilon\sigma)-d^0)y_0(\epsilon\sigma)+c_\epsilon(\epsilon\sigma)x_0(\epsilon\sigma)+g(\epsilon\sigma)]d\sigma.
\end{aligned}
\tag{5.5}
$$

Note that the second equation in (5.5) is solved first. From Lemmas 3, 4, 6 and Theorem 2, it has a unique sequence solution $\{y(n,\epsilon)\}\in\mathcal{QP}(\tilde{\omega},\mathbb{Z})$. Similar to the proof in section 4, it can be proved by Lemma 7 that the y defined by (5.4), with values $y(n,\epsilon)$ at $t=n$, is in $\mathcal{GQP}(1,\tilde{\omega})$. Then this y is put into the first equation in (5.5) which is then solved for $\{x(n,\epsilon)\}$ in $\mathcal{QP}(\tilde{\omega},\mathbb{Z})$. Therefore, the inhomogeneous difference equation (5.5) has an sequence solution $(x(n,\epsilon),y(n,\epsilon))\in\mathcal{QP}(\tilde{\omega},\mathbb{Z})$. At this time, it can be proved that the solution $(x(t,\epsilon),y(t,\epsilon))$ defined by (5.4), with values $(x(n,\epsilon),y(n,\epsilon))$ at $t=n$, is the unique solution of Eq.(5.3) in $\mathcal{GQP}(1,\tilde{\omega})$ by Lemmas 2, 3, 4. See section 4 for reference.

Writing $(x,y)=\mathcal{T}(x_0,y_0;f,g,\epsilon)$, then solving (5.1) is equivalent to finding a fixed point of $\mathcal{T}(\cdot,\cdot;f,g,\epsilon)$. The contractibility of $\mathcal{T}$ and (5.2) can be proved in an analogous way to those in [20]. So we omit it. $\square$

6. Almost Periodic Singularly Perturbed Differential Equations

In this section, we consider the following singularly perturbed differential equation.

$$\begin{cases} x'(t) = a(t,\epsilon)x(t) + \sum_{i=-N}^{N} a_i(t,\epsilon)x([t+i]) + b(t,\epsilon)y(t) + \sum_{i=-N}^{N} b_i(t,\epsilon)y([t+i]) + f(t), \\ \epsilon y'(t) = c(t,\epsilon)x(t) + \sum_{i=-N}^{N} c_i(t,\epsilon)x([t+i]) + d(t,\epsilon)y(t) + \sum_{i=-N}^{N} d_i(t,\epsilon)y([t+i]) + g(t), \end{cases} \tag{6.1}$$

where $f(t), g(t), a(t,\epsilon), b(t,\epsilon)$, etc., are almost periodic.

Under (H1), we can show that the spectrums of $a(t,\epsilon)$ and $a_j(t,\epsilon)$, $-N \le j \le N$, are independent of ϵ. Similar results also hold for $b(t,\epsilon), c(t,\epsilon), d(t,\epsilon)$, etc. Without loss of generality, we give a proof for $a(t,\epsilon)$.

Since $a(t,\epsilon)$ is almost periodic in t and continuous in $\epsilon \in [0,\epsilon_0]$ uniformly in $t \in \mathbb{R}$, we have $a^* : \mathbb{R} \to C([0,\epsilon_0],\mathbb{R})$, where $a^*(t) = \{a(t,\epsilon);\ 0 \le \epsilon \le \epsilon_0\}$. From the uniform continuity on ϵ, it follows that for each ϵ', there exists $\sigma(\epsilon')$ such that

$$T(f,\eta,\epsilon) = \{\tau \in \mathbb{R};\ |a(t+\tau,\epsilon) - a(t,\epsilon)| < \eta, \quad \forall t \in \mathbb{R}\}$$

is independent of $\epsilon \in (\epsilon' - \sigma(\epsilon'), \epsilon' + \sigma(\epsilon'))$. Using the compactness of $[0,\epsilon_0]$, we know that $T(f,\eta,\epsilon)$ is independent of $\epsilon \in [0,\epsilon_0]$. It follows that if $\tau \in T(f,\eta,\epsilon)$, we have

$$|a^*(t+\tau) - a^*(t)| = \sup_{0 \le \epsilon \le \epsilon_0} |a(t+\tau,\epsilon) - a(t,\epsilon)| < \eta.$$

Thus, it implies that $a^* : \mathbb{R} \to C([0,\epsilon_0],\mathbb{R})$ is an almost periodic function. a^* has a Fourier expansion

$$a^*(t) \sim \sum_{\nu} a^*_\nu e^{i\lambda_\nu t}, \quad a^*_\nu = a(\lambda_\nu; a^*) \in C([0,\epsilon_0],\mathbb{R}).$$

It follows from the Approximation Theorem (see [13]) that

$$|\tilde{a}^*_k(t) - a^*(t)| < \frac{1}{k}, \quad \forall t \in \mathbb{R},$$

where $\tilde{a}^*_k(t) = \sum_{\nu}^{n_k} \tilde{A}^*_{k,\nu} e^{i\lambda_\nu t}, \tilde{A}^*_{k,\nu} \in C([0,\epsilon_0],\mathbb{R})$. Therefore, $\Lambda_{a(t,\epsilon)} = \Lambda_{a^*} = \{\lambda_1, \lambda_2, \cdots\}$ is independent of ϵ and $|\tilde{a}_k(t,\epsilon) - a(t,\epsilon)| < \frac{1}{k}, \tilde{a}_k(t,\epsilon) = \sum_{\nu=1}^{n_k} \tilde{A}_{k,\nu}(\epsilon) e^{i\lambda_\nu t}$.

In what follows, without loss of generality, we suppose that the spectrums of all coefficients in (6.1) are same, that is,

$$\Lambda_{a(t,\epsilon_1)} = \Lambda_{a(t,\epsilon_2)} = \Lambda_{a_j(t,\epsilon_1)} = \Lambda_{a_j(t,\epsilon_2)} \triangleq \Lambda, \quad 0 \le j \le N, \forall\, 0 \le \epsilon_1, \epsilon_2 < \epsilon_0.$$

$$\Lambda_{b(t,\epsilon)} = \Lambda_{b_j(t,\epsilon)} = \Lambda_{c(t,\epsilon)} = \Lambda_{c_j(t,\epsilon)} = \Lambda_{d(t,\epsilon)} = \Lambda_{d_j(t,\epsilon)} = \Lambda.$$

Otherwise, we could combine the spectrums so that the spectrums coincide. At this time, these modules do not have to the smallest ones. For two number sets $A, B \subset \mathbb{C}$, we define $A+B = \{a+b | \forall a \in A, b \in B\}$. Set $\mathcal{M} = \{\sum_{l=1}^{s} m_l \lambda_l \Big| \forall m_l \in \mathbb{Z}, \forall \lambda_l \in \Lambda, \forall s \in \mathbb{N}\}$. We define a Banach space $\mathcal{AP}$ to be the set of all almost periodic functions, equipped with supremum norm $||\phi|| = \sup_{t\in\mathbb{R}} |\phi(t)|$.

Theorem 5. *If (H1)-(H4) hold, then there exist $\epsilon_1,\ 0 < \epsilon_1 \leq \epsilon_0$, positive functions $\alpha_{i,j}(\epsilon),\ 1 \leq i, j \leq 2$ defined for $0 < \epsilon \leq \epsilon_1$, satisfying*

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \alpha_{1,1}(\epsilon) = U, \qquad \lim_{\epsilon\to 0^+} \alpha_{1,2}(\epsilon) = 2UV(N+1)M, \\
&\lim_{\epsilon\to 0^+} \alpha_{2,1}(\epsilon) = 0, \qquad \lim_{\epsilon\to 0^+} \alpha_{2,2}(\epsilon) = V, \\
&\alpha_{i,j}(\epsilon) \leq 2UV(N+1)M + U + V,\ 1 \leq i, j \leq 2,
\end{aligned}$$

such that for each $(f, g) \in \mathcal{AP},\ 0 < \epsilon \leq \epsilon_1$, with the spectrum relation $\Lambda_f, \Lambda_g \subset \Lambda$, there is a unique solution

$$(x^*(f, g, \epsilon), y^*(f, g, \epsilon) \in \mathcal{AP}$$

of Eq.(6.1) and mod$(x^*(t), y^*(t)) \subset \mathcal{M} + \{2k\pi \big| k \in \mathbb{Z}\}$. The solution satisfies

$$||x^*|| \leq \alpha_{1,1}(\epsilon)||f|| + \alpha_{1,2}(\epsilon)||g||,\ ||y^*|| \leq \alpha_{2,1}(\epsilon)||f|| + \alpha_{2,2}(\epsilon)||g||. \tag{6.2}$$

The map $(f, g) \to (x^*(f, g, \epsilon), y^*(f, g, \epsilon))$ defines a bounded linear operator $\mathcal{L}(\epsilon)$ satisfying $||\mathcal{L}(\epsilon)|| \leq 4UV(N+1)M + 2U + 2V$ and $\epsilon \to \mathcal{L}(\epsilon)$ is continuous for $0 < \epsilon \leq \epsilon_1$.

Proof. The existence and uniqueness of almost periodic solution $(x^*(f, g, \epsilon), y^*(f, g, \epsilon))$, (6.2), and the boundedness and continuity of operator have been shown in [20]. It suffices to study the module containment. Set $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$. From the Approximation Theorem 1, it follows that there exist trigonometric polynomials $\tilde{a}_k(t, \epsilon), \tilde{a}_{j,k}(t, \epsilon), \tilde{b}_k(t, \epsilon),\ \tilde{b}_{j,k}(t, \epsilon), \tilde{c}_k(t, \epsilon), \tilde{c}_{j,k}(t, \epsilon), \tilde{d}_k(t, \epsilon), \tilde{d}_{j,k}(t, \epsilon),\ -N \leq j \leq N$, $\tilde{f}_k(t), \tilde{g}_k(t)$ of the forms

$$\begin{aligned}
\tilde{a}_k(t, \epsilon) &= \sum_{\nu=1}^{n_k} \tilde{A}_{k,\nu}(\epsilon) e^{i\lambda_\nu t}, \quad \tilde{a}_{j,k}(t, \epsilon) = \sum_{\nu=1}^{n_k} \tilde{A}_{j,k,\nu}(\epsilon) e^{i\lambda_\nu t}, \\
\tilde{b}_k(t, \epsilon) &= \sum_{\nu=1}^{n_k} \tilde{B}_{k,\nu}(\epsilon) e^{i\lambda_\nu t}, \quad \tilde{b}_{j,k}(t, \epsilon) = \sum_{\nu=1}^{n_k} \tilde{B}_{j,k,\nu}(\epsilon) e^{i\lambda_\nu t}, \\
\tilde{c}_k(t, \epsilon) &= \sum_{\nu=1}^{n_k} \tilde{C}_{k,\nu}(\epsilon) e^{i\lambda_\nu t}, \quad \tilde{c}_{j,k}(t, \epsilon) = \sum_{\nu=1}^{n_k} \tilde{C}_{j,k,\nu}(\epsilon) e^{i\lambda_\nu t}, \\
\tilde{d}_k(t, \epsilon) &= \sum_{\nu=1}^{n_k} \tilde{D}_{k,\nu}(\epsilon) e^{i\lambda_\nu t}, \quad \tilde{d}_{j,k}(t, \epsilon) = \sum_{\nu=1}^{n_k} \tilde{D}_{j,k,\nu}(\epsilon) e^{i\lambda_\nu t}, \\
\tilde{f}_k(t) &= \sum_{\nu=1}^{n_k} \tilde{F}_{k,\nu} e^{i\lambda_\nu t}, \quad \tilde{g}_k(t) = \sum_{\nu=1}^{n_k} \tilde{G}_{k,\nu} e^{i\lambda_\nu t},
\end{aligned}$$

where $\lambda_\nu \in \Lambda$, such that

$$\begin{aligned}
&|\tilde{a}_k(t,\epsilon) - a(t,\epsilon)| < \frac{1}{k}, \quad |\tilde{a}_{j,k}(t,\epsilon) - a_j(t,\epsilon)| < \frac{1}{k},\\
&|\tilde{b}_k(t,\epsilon) - b(t,\epsilon)| < \frac{1}{k}, \quad |\tilde{b}_{j,k}(t,\epsilon) - b_j(t,\epsilon)| < \frac{1}{k},\\
&|\tilde{c}_k(t,\epsilon) - c(t,\epsilon)| < \frac{1}{k}, \quad |\tilde{c}_{j,k}(t,\epsilon) - c_j(t,\epsilon)| < \frac{1}{k},\\
&|\tilde{d}_k(t,\epsilon) - d(t,\epsilon)| < \frac{1}{k}, \quad |\tilde{d}_{j,k}(t,\epsilon) - d_j(t,\epsilon)| < \frac{1}{k}, \ (-N \le j \le N),\\
&|\tilde{f}_k(t) - f(t)| < \frac{1}{k}, \quad |\tilde{g}_k(t) - g(t)| < \frac{1}{k}, \quad k = 1, 2, \cdots,
\end{aligned}$$

for $t \in \mathbb{R}$ and $0 < \epsilon \le \epsilon_1$. For each k, consider

$$\begin{cases}
x'(t) = \tilde{a}_k(t,\epsilon)x(t) + \displaystyle\sum_{j=-N}^{N} \tilde{a}_{j,k}(t,\epsilon)x([t+j]) \\
\qquad + \tilde{b}_k(t,\epsilon)y(t) + \displaystyle\sum_{j=-N}^{N} \tilde{b}_{j,k}(t,\epsilon)y([t+j]) + \tilde{f}_k(t), \\
\epsilon y'(t) = \tilde{c}_k(t,\epsilon)x(t) + \displaystyle\sum_{j=-N}^{N} \tilde{c}_{j,k}(t,\epsilon)x([t+j]) \\
\qquad + \tilde{d}_k(t,\epsilon)y(t) + \displaystyle\sum_{j=-N}^{N} \tilde{d}_{j,k}(t,\epsilon)y([t+j]) + \tilde{g}_k(t),
\end{cases} \tag{6.3}$$

$n \le t < n+1$. Clearly, $\tilde{a}_k(t,\epsilon), \tilde{a}_{j,k}(t,\epsilon)$, etc., are continuous in ϵ uniformly in $t \in \mathbb{R}$. Let $\tilde{a}_k(t,0) = \tilde{a}_k^0, \tilde{a}_{j,k}(t,0) = \tilde{a}_{j,k}^0$ and $\tilde{d}_k(t,0) = \tilde{d}_k^0, \tilde{d}_{j,k}(t,0) = \tilde{d}_{j,k}^0$,

$$\begin{aligned}
A_{0,k} &= e^{\tilde{a}_k^0} + \tilde{a}_{0,k}^0(e^{\tilde{a}_k^0} - 1)/\tilde{a}_k^0, \quad A_{1,k} = \tilde{a}_{1,k}^0(e^{\tilde{a}_k^0} - 1)/\tilde{a}_k^0 - 1,\\
A_{j,k} &= \tilde{a}_{j,k}^0(e^{\tilde{a}_k^0} - 1)/\tilde{a}_k^0, \quad j = -1, \pm 2, \cdots, \pm N,\\
D_{0,k} &= -\tilde{d}_{0,k}^0/\tilde{d}_k^0, \quad D_{1,k} = -\tilde{d}_{1,k}^0/\tilde{d}_k^0 - 1,\\
D_{j,k} &= -\tilde{d}_{j,k}^0/\tilde{d}_k^0, \quad j = -1, \pm 2, \cdots, \pm N.
\end{aligned}$$

When $k \gg 1$, it follows from (H3) and (H4) that for each fixed k, all roots of algebraic equations

$$\sum_{j=-N}^{N} A_{j,k}\xi^j = 0$$

and

$$\sum_{j=-N}^{N} D_{j,k}\mu^j = 0$$

are not on S^1. Set $\tilde{\omega}_k = (\frac{\lambda_1}{2\pi}, \cdots, \frac{\lambda_{n_k}}{2\pi})$. We establish a lemma for convenience as follows. But, its proof will be put in the finality of this section.

Lemma 8. *Under the assumptions (H1)-(H4), there exist $\epsilon_1 > 0$ and $k_1 > 0$ such that for $k > k_1$ and $0 < \epsilon < \epsilon_1$, system (6.3) possesses a unique solution $(x_k^*(t,\epsilon), y_k^*(t,\epsilon)) \in \mathcal{GQP}(1, \tilde{\omega}_k)$ and*

$$|x_k^*(t,\epsilon)| \le (\alpha_{1,1}(\epsilon)+1)\sup_t |\tilde{f}_k(t)| + (\alpha_{1,2}(\epsilon)+1)\sup_t |\tilde{g}_k(t)|,$$
$$|y_k^*(t,\epsilon)| \le (\alpha_{2,1}(\epsilon)+1)\sup_t |\tilde{f}_k(t)| + (\alpha_{2,2}(\epsilon)+1)\sup_t |\tilde{g}_k(t)|,$$

so that there exists a constant $U_{f,g}$ such that when $0 \le \epsilon \le \epsilon_1$, we have

$$\sup_t |x_k^*(t,\epsilon)| \le U_{f,g}, \quad \sup_t |y_k^*(t,\epsilon)| \le U_{f,g}. \tag{6.4}$$

Here, $\alpha_{i,j}(\epsilon)$ are as in Theorem 4.

We continue the proof of Theorem 5. Let $\tilde{m}_k = (m_1, \cdots, m_{n_k})$. It follows from Lemma 8 that we could write $x_k^*(t,\epsilon)$ and $y_k^*(t,\epsilon)$ as

$$x_k^*(t,\epsilon) = \sum_{\tilde{m}_k} X_{k,\tilde{m}_k}(\{t\},\epsilon)e^{i2\pi<\tilde{m}_k,\tilde{\omega}_k>[t]} \triangleq F_k(t, \tilde{\omega}_k t, \epsilon),$$
$$y_k^*(t,\epsilon) = \sum_{\tilde{m}_k} Y_{k,\tilde{m}_k}(\{t\},\epsilon)e^{i2\pi<\tilde{m}_k,\tilde{\omega}_k>[t]} \triangleq G_k(t, \tilde{\omega}_k t, \epsilon),$$

where

$$F_k(\theta_0, \tilde{\theta}_k, \epsilon) = \sum_{\tilde{m}_k} X_{k,\tilde{m}_k}(\{\theta_0\},\epsilon)e^{-i2\pi<\tilde{m}_k,\tilde{\omega}_k>\{\theta_0\}}e^{i2\pi<\tilde{m}_k,\tilde{\theta}_k>},$$
$$G_k(\theta_0, \tilde{\theta}_k, \epsilon) = \sum_{\tilde{m}_k} Y_{k,\tilde{m}_k}(\{\theta_0\},\epsilon)e^{-i2\pi<\tilde{m}_k,\tilde{\omega}_k>\{\theta_0\}}e^{i2\pi<\tilde{m}_k,\tilde{\theta}_k>}$$

are 1-periodic in $\theta_0, \theta_1, \cdots, \theta_{n_k}$, respectively, and $\tilde{\theta}_k = (\theta_1, \cdots, \theta_{n_k})$. Clearly, $F_k, G_k \in L(\mathbb{T}^{n_k+1})$. Setting $\hat{\theta}_k = (\theta_0, \theta_1, \cdots, \theta_{n_k})$ and $\hat{m}_k = (m_0, m_1, \cdots, m_{n_k})$, we have the Fourier expansion (see [25] for details)

$$F_k(\theta_0, \theta_1, \cdots, \theta_{n_k}, \epsilon) \sim \sum_{\hat{m}_k} F_{k,\hat{m}_k}(\epsilon)e^{i2\pi<\hat{m}_k,\hat{\theta}_k>},$$
$$G_k(\theta_0, \theta_1, \cdots, \theta_{n_k}, \epsilon) \sim \sum_{\hat{m}_k} G_{k,\hat{m}_k}(\epsilon)e^{i2\pi<\hat{m}_k,\hat{\theta}_k>},$$

where the Fourier coefficients

$$F_{k,\hat{m}_k}(\epsilon) = \int_{\mathbb{T}^{n_k+1}} F_k(\hat{\theta}_k,\epsilon)e^{-i2\pi<\hat{m}_k,\hat{\theta}_k>}d\hat{\theta}_k,$$
$$G_{k,\hat{m}_k}(\epsilon) = \int_{\mathbb{T}^{n_k+1}} G_k(\hat{\theta}_k,\epsilon)e^{-i2\pi<\hat{m}_k,\hat{\theta}_k>}d\hat{\theta}_k.$$

We get the Fourier expansions of $x_k^*(t,\epsilon)$ and $y_k^*(t,\epsilon)$ as

$$x_k^*(t,\epsilon) \sim \sum_{\hat{m}_k} F_{k,\hat{m}_k}(\epsilon) e^{i2\pi<\hat{m}_k,\hat{\omega}_k>t},$$
$$y_k^*(t,\epsilon) \sim \sum_{\hat{m}_k} G_{k,\hat{m}_k}(\epsilon) e^{i2\pi<\hat{m}_k,\hat{\omega}_k>t},$$

where $\hat{\omega}_k = (1, \tilde{\omega}_k)$. It follows that

$$\mathrm{mod}(x_k^*(t,\epsilon), y_k^*(t,\epsilon)) \subset \{\sum_{l=1}^{n_k} m_l\lambda_l \big| \lambda_l \in \Lambda \ \ \forall m_l \in \mathbb{Z}\} + \{2j\pi | j \in \mathbb{Z}\},$$

which implies that $\lim_{k\to\infty} \mathrm{mod}(x_k^*(t,\epsilon), y_k^*(t,\epsilon)) \subset \mathcal{M} + \{2k\pi | k \in \mathbb{Z}\}$, here we use the limit of sets. Set $u_k^*(t,\epsilon) = x_k^*(t,\epsilon) - x^*(t,\epsilon)$, $v_k^*(t,\epsilon) = y_k^*(t,\epsilon) - y^*(t,\epsilon)$. Then $u_k^*(t,\epsilon)$, $v_k^*(t,\epsilon)$ satisfy the following systems

$$\begin{aligned}
u'(t) &= a(t,\epsilon)u(t) + \sum_{i=-N}^{N} a_i(t,\epsilon)u([t+i]) + b(t,\epsilon)v(t) + \sum_{i=-N}^{N} b_i(t,\epsilon)v([t+i]) \\
&\quad + \tilde{f}_k(t) - f(t) + (\tilde{a}_k(t,\epsilon) - a(t,\epsilon))x_k^*(t) + \sum_{i=-N}^{N} (\tilde{a}_{i,k}(t,\epsilon) - a_i(t,\epsilon))x_k^*([t+i]) \\
&\quad + (\tilde{b}_k(t,\epsilon) - b(t,\epsilon))y_k^*(t) + \sum_{i=-N}^{N} (\tilde{b}_{i,k}(t,\epsilon) - b_i(t,\epsilon))y_k^*([t+i]), \\
\epsilon v'(t) &= c(t,\epsilon)u(t) + \sum_{i=-N}^{N} c_i(t,\epsilon)u([t+i]) + d(t,\epsilon)v(t) + \sum_{i=-N}^{N} d_i(t,\epsilon)v([t+i]) \\
&\quad + \tilde{g}_k(t) - g(t) + (\tilde{c}_k(t,\epsilon) - c(t,\epsilon))x_k^*(t) + \sum_{i=-N}^{N} (\tilde{c}_{i,k}(t,\epsilon) - c_i(t,\epsilon))x_k^*([t+i]) \\
&\quad + (\tilde{d}_k(t,\epsilon) - d(t,\epsilon))y_k^*(t) + \sum_{i=-N}^{N} (\tilde{d}_{i,k}(t,\epsilon) - d_i(t,\epsilon))y_k^*([t+i]).
\end{aligned}$$

It follows from (6.2) and (6.4) that when $0 \le \epsilon \le \epsilon_1$, we have

$$\sup_t |x_k^*(t,\epsilon) - x^*(t,\epsilon)| \to 0, \quad \sup_t |y_k^*(t,\epsilon) - y^*(t,\epsilon)| \to 0, \quad \text{as } k \to \infty.$$

Thus, we have

$$\lim_{k\to\infty} a(\lambda; x_k^*) = a(\lambda; x^*), \quad \lim_{k\to\infty} a(\lambda; y_k^*) = a(\lambda; y^*), \quad \forall \lambda \in \mathbb{R}.$$

Therefore, we could imply $\mathrm{mod}(x^*, y^*) \subset \mathcal{M} + \{2k\pi | \forall k \in \mathbb{Z}\}$. □

Proof of Lemma 8. It follows from Theorem 4 that there exist $\epsilon_1 > 0$ and $k_1 > 0$ such that for $k > k_1$ and $0 < \epsilon < \epsilon_1$, the system (6.3) possesses a unique solution

$(x_k^*(t,\epsilon), y_k^*(t,\epsilon)) \in \mathcal{GQP}(1,\tilde{\omega}_k)$. It suffices to show (6.4). We give an outline as follows.

(1). If $A_N \neq 0$, then the system

$$\sum_{i=-N}^{N} A_i x(n+i) = h(n) \tag{6.5}$$

is equivalent to

$$u(n+1) = Cu(n) + q(n),$$

where

$$u(n) = \begin{pmatrix} y(n-N) \\ y(n-N+1) \\ \vdots \\ y(n+N-1) \end{pmatrix}, \quad q(n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{A_N} h(n) \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{A_{N-1}}{A_N} & -\frac{A_{N-2}}{A_N} & \cdots & \cdots & -\frac{A_{-N}}{A_N} \end{pmatrix}.$$

From (H3), it follows that

$$u(n+1) = Cu(n)$$

admits an exponential dichotomy.

(2). If $A_{N,k} \neq 0$, then the system

$$\sum_{j=-N}^{N} A_{j,k} x(n+j) = h_k(n) \tag{6.6}$$

is equivalent to

$$u(n+1) = C_k u(n) + q_k(n),$$

where

$$C_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{A_{N-1,k}}{A_{N,k}} & -\frac{A_{N-2,k}}{A_{N,k}} & \cdots & \cdots & -\frac{A_{-N,k}}{A_{N,k}} \end{pmatrix}, \quad q_k(n) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{A_{N,k}} h_k(n) \end{pmatrix}.$$

From the roughness theorem for exponential dichotomies, we know that there exists k_1 (without loss of generalization) such that when $k > k_1$, the system

$$u(n+1) = C_k u(n)$$

also admits an exponential dichotomy. The unique bounded solution $x_k^*(n)$ of system (6.6) can be estimated by the constants appeared in the exponential dichotomy and the constants can be chosen to be independent of the small perturbation (see [16, Prop. 2.10]) so that we can obtain

$$|x_k^*(n)| \leq \tilde{M}_A \sup_{n\in\mathbb{Z}} |h_k(n)|.$$

(3). From the above and Theorem 1, it is easy to see that

$$\dot{x}(t) = \tilde{a}_k^0 x(t) + \sum_{i=-N}^{N} \tilde{a}_{i,k}^0 x([t+i]) + \tilde{f}_k(t) \tag{6.7}$$

possesses a unique solution $\tilde{x}_k^*(t) \in \mathcal{GQP}(1, \tilde{\omega}_k)$ and there exists a constant $\tilde{U}$ independent of k (note that $\tilde{x}_k^*(t)$ can be represented as in (3.5)) such that

$$|\tilde{x}_k^*(t)| \leq \tilde{U} \sup_t |\tilde{f}_k(t)|.$$

(4). Suppose $D_N \neq 0$. Let

$$\begin{aligned}
D_{0,k}(\epsilon) &= e^{\tilde{d}_k^0/\epsilon} + \tilde{d}_{0,k}^0 (e^{\tilde{d}_k^0/\epsilon} - 1)/\tilde{d}_k^0, \\
D_{1,k}(\epsilon) &= \tilde{d}_{1,k}^0 (e^{\tilde{d}_k^0/\epsilon} - 1)/\tilde{d}_k^0 - 1, \\
D_{j,k}(\epsilon) &= \tilde{d}_{j,k}^0 (e^{\tilde{d}_k^0/\epsilon} - 1)/\tilde{d}_k^0, \quad j = -1, \pm 2, \cdots, \pm N, \\
e_k(n,\epsilon) &= -\int_{\frac{n}{\epsilon}}^{\frac{n+1}{\epsilon}} e^{\tilde{d}_k^0 (\frac{n+1}{\epsilon} - \sigma)} \tilde{g}_k(\epsilon\sigma) d\sigma.
\end{aligned}$$

If $k > k_1$ and $0 < \epsilon < \epsilon_1$, then the system

$$\sum_{i=-N}^{N} D_{i,k}(\epsilon) y(n+i) = e_k(n,\epsilon) \tag{6.8}$$

is equivalent to

$$v(n+1) = B_k(\epsilon) v(n) + q_k(\epsilon),$$

where

$$B_k(\epsilon) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \\ -\frac{D_{N-1,k}(\epsilon)}{D_{N,k}(\epsilon)} & -\frac{D_{N-2,k}(\epsilon)}{D_{N,k}(\epsilon)} & \cdots & \cdots & -\frac{D_{-N,k}(\epsilon)}{D_{N,k}(\epsilon)} \end{pmatrix},$$

$$v(n) = \begin{pmatrix} y(n-N) \\ y(n-N+1) \\ \vdots \\ y(n+N-1) \end{pmatrix}, \quad q_k(\epsilon) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{D_{N,k}(\epsilon)} e_k(n,\epsilon) \end{pmatrix}$$

Clearly, $\lim_{k\to\infty,\epsilon\to 0} D_{j,k}(\epsilon) = D_j$ for each fixed j. From (H4) and the roughness theorem for exponential dichotomies, we know that there exist $k_1 > 0$ and $\epsilon_1 > 0$ (without loss of generalization) such that when $k > k_1$ and $0 < \epsilon < \epsilon_1$, the system

$$v(n+1) = B_k(\epsilon)v(n)$$

also admits an exponential dichotomy. The unique bounded solution $y_k^*(n,\epsilon)$ of system (6.8) can be estimated by the constants appeared in the exponential dichotomy and the constants can be chosen to be independent of the small perturbation (see [16, Prop. 2.10]) so that we can obtain

$$|y_k^*(n,\epsilon)| \le \tilde{N}_D \sup_{n\in\mathbb{Z}} |e_k(n,\epsilon)|.$$

(5). From the above and in analogous way to the proof of Theorem 3 (e.g., see [20, Thm. 2.4]), it is easy to prove

$$\epsilon\dot{y}(t) = \tilde{d}_k^0 y(t) + \sum_{i=-N}^{N} \tilde{d}_{i,k} y([t+\epsilon]) + \tilde{g}_k(t)$$

possesses a unique solution $\tilde{y}_k^*(t,\epsilon) \in \mathcal{GQP}(1,\tilde{\omega}_k)$ and there exists a constant $\tilde{V}$ independent of k and ϵ such that

$$|\tilde{y}_k^*(t,\epsilon)| \le \tilde{V} \sup_t |\tilde{g}_k(t)|.$$

The rest is virtually in an analogous way as in the proof of [20, Thm. 3.1]. We omit it. This completes the proof of Lemma 8. □

7. Nonlinear Singularly Perturbed Differential Equations

In this section, we will discuss the systems (1.5). We formulate the following result.

Theorem 6. *Assume that $f(t,\phi,\psi,\epsilon), g(t,\phi,\psi,\epsilon)$ are almost periodic in t uniformly for (ϕ,ψ) with spectrums $\Lambda_f, \Lambda_g \subset \Lambda$. If (H1)-(H5) hold, then there exist $\epsilon_2, \rho_1, 0 < \epsilon_2 \le \epsilon_0$ and $0 < \rho_1 \le \rho_0$, such that for each ϵ satisfying $0 < \epsilon \le \epsilon_2$, (1.5) has a unique almost periodic solution $(x^*(t,\epsilon), y^*(t,\epsilon))$ satisfying $||x|| \le \rho_1, ||y|| \le \rho_1$, $mod((x^*(t), y^*(t))) \subset \mathcal{M} + \{2k\pi |\, k \in \mathbb{Z}\}$, and this solution is continuous in ϵ uniformly in $t \in \mathbb{R}$ and satisfies $||x^*(\epsilon)|| + ||y^*(\epsilon)|| = O(M(\epsilon))$ as $\epsilon \to 0$.*

Proof. The existence of almost periodic solution has been shown in [20]. It suffices to show the module containment. We choose ρ_1 $(\le \rho_0)$ and ϵ_2 $(\le \epsilon_1)$ such that

$$(4UV(N+1)M + 2U + 2V)[4\rho_1\eta(\rho_1,\epsilon_2) + M(\epsilon_2)] < \rho_1,$$
$$2[4UV(N+1)M + 2U + 2V)\eta(\rho_1,\epsilon_2) < \frac{1}{2}.$$

Set $\mathcal{R} = \{(x,y) \in \mathcal{AP} \;\big|\mathrm{mod}(x,y) \subset \mathcal{M} + \{2k\pi\}\}$. Given $(x_0, y_0) \in \mathcal{R}$ satisfying $||x_0|| \le \rho_1, ||y_0|| \le \rho_1$ and $0 < \epsilon \le \epsilon_2$, let (x,y) be the unique solution in $\mathcal{R}$ of

$$\begin{cases} x'(t) = a(t,\epsilon)x(t) + \displaystyle\sum_{i=-N}^{N} a_i(t,\epsilon)x([t+i]) + b(t,\epsilon)y(t) + \sum_{i=-N}^{N} b_i(t,\epsilon)y([t+i]) \\ \qquad\qquad\qquad + f(t, x_{0t}, y_{0t}, \epsilon), \\ \epsilon y'(t) = c(t,\epsilon)x(t) + \displaystyle\sum_{i=-N}^{N} c_i(t,\epsilon)x([t+i]) + d(t,\epsilon)y(t) + \sum_{i=-N}^{N} d_i(t,\epsilon)y([t+i]) \\ \qquad\qquad\qquad + g(t, x_{0t}, y_{0t}, \epsilon). \end{cases}$$

Such an $(x,y) \in \mathcal{R}$ exists by Theorem 5 and the estimate

$$\begin{aligned} &|f(t, x_{0t}, y_{0t}, \epsilon)| \le \eta(\rho_1, \epsilon_2)2[||x_0|| + ||y_0||] + M(\epsilon_2) \\ &\le 4\rho_1\eta(\rho_1, \epsilon_2) + M(\epsilon_2),\ t \in \mathbb{R}, 0 < \epsilon \le \epsilon_2. \end{aligned}$$

In fact, $(x,y) = \mathcal{L}(\epsilon)(f(\cdot, x_0, y_0, \epsilon), g(\cdot, x_0, y_0, \epsilon)) = \mathcal{T}(x_0, y_0, \epsilon)$. The existence of a solution of (1.5) in $\mathcal{R}$ is equivalent to the existence of a fixed point of the mapping $\mathcal{T}$.

We estimate (x,y) using Theorem 5 as

$$||x|| \le (\alpha_{1,1}(\epsilon) + \alpha_{1,2}(\epsilon))[4\rho_1\eta(\rho_1, \epsilon_2) + M(\epsilon_2)] < \rho_1.$$

and similarly for $||y||$. Thus, $\mathcal{T}(\cdot,\cdot,\epsilon)$ maps the closed set $\mathcal{F} = \{(x_0, y_0) \in \mathcal{AP} : \mathrm{mod}(x_0, y_0) \subset \mathcal{M} + \{2k\pi\},\ ||x_0|| \le \rho_1,\ ||y_0|| \le \rho_1\}$ into itself for each ϵ with $0 < \epsilon \le \epsilon_2$.

Setting $(x,y) = \mathcal{T}(x_0, y_0, \epsilon)$ and $(\bar{x}, \bar{y}) = \mathcal{T}(\bar{x}_0, \bar{y}_0, \epsilon)$, it is easily shown that

$$||x - \bar{x}|| \le [4UV(N+1)M + 2U + 2V]\eta(\rho_1, \epsilon_2)[||x_0 - \bar{x}_0|| + ||y_0 - \bar{y}_0||]$$

and similarly for $||y - \bar{y}||$, yielding

$$\begin{aligned} &||x - \bar{x}|| + ||y - \bar{y}|| \\ &\le 2[4UV(N+1)M + 2U + 2V]\eta(\rho_1, \epsilon_2)[||x_0 - \bar{x}_0|| + ||y_0 - \bar{y}_0||] \\ &\le \frac{1}{2}[||x_0 - \bar{x}_0|| + ||y_0 - \bar{y}_0||]. \end{aligned}$$

Hence $\mathcal{T}$ is a uniform contraction. The rest is in an analogous way to those in [20]. So we omit it. □

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