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Global Attractivity in a Higher Order Difference Equation with Applications

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Abstract. In this paper, we establish a sufficient condition for the global attractivity of solutions of the following higher order difference equation

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k} f(x_{n-k}), \qquad n = 0, 1, \dots, \tag{0.1}$$

where k is a positive integer, $\mu \in (0,1)$ is a real number, and f is a continuous function defined on $[0,B)$ such that $xf(x) : (0,B) \to (0,B)$, where $B \leq \infty$. Applications to some difference equation models derived from mathematical biology are also given.

AMS Subject Classifications: 39A10, 92D25

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1. Introduction

The following difference equation

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]_+, \tag{1.1}$$

where k is a positive integer, $K, q, z \in (0,\infty), \mu \in (0,1)$ and $[x]_+ = \max\{x, 0\}$, has been proposed by the International Whaling Commission as a model that describes the dynamics of baleen whales, see [1], [2], [5] and the references cited therein. The global stability of Eq.(1.1) has been studied in [1] and [2], and a sufficient condition for the positive equilibrium K to be globally asymptotically stable relative to the

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interval $(K-a, K+a) \subset (0, x^*)$ has been established, where $x^* = K\left(\frac{1+q}{q}\right)^{1/z}$ and a is a positive constant with $a \le \min\{K, x^* - K\}$. A question now is how to obtain conditions for the global asymptotic stability of the equilibrium K of Eq.(1.1) relative to the whole interval $(0, x^*)$. In fact, this problem has been posted in [5] as a research project for a quite while. However, to the best of our knowledge, there is no known results in the literature about the problem so far.

Motivated by the above observation, our aim in this paper is to study the global attractivity of the following more general higher order difference equation

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k} f(x_{n-k}), \qquad n = 0, 1, \ldots, \tag{1.2}$$

where k is a positive integer, $\mu \in (0,1)$ is a real number, and f is a continuous function defined on $[0, B)$ such that $xf(x) : (0, B) \to (0, B)$, where $B \le \infty$. Assume that $\bar{x}$ is the only point in $(0, B)$ satisfying $f(x) = 1$. Then $\bar{x}$ is the unique positive equilibrium of Eq.(1.2). With Eq.(1.2) we associate an initial condition of the form

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, B) \qquad \text{with} \quad x_0 > 0. \tag{1.3}$$

Then, by the method of steps, it follows that IVP (1.2) and (1.3) has a unique solution $\{x_n\}$ and $x_n \in (0, B), n = 0, 1, \ldots$.

In the next section, we will establish a sufficient condition for $\bar{x}$ to be a global attractor of all positive solutions of Eq.(1.2). Then, in Section 3, by applying the result obtained in Section 2 and the linearized stability theory, we will obtain a criteria for the globally asymptotic stability of the positive equilibrium K of Eq.(1.1) relative to the whole interval $(0, x^*)$; we will also use another example derived from mathematical biology to illustrate the application of our main result.

2. Global Attractivity of Eq.(1.2)

In this section, we will establish a sufficient condition for the global attractivity of solutions of Eq.(1.2). The following lemma is needed.

Lemma 2.1. *Assume that*

$$(x - \bar{x})(f(x) - 1) < 0, \qquad x \ne \bar{x}. \tag{2.1}$$

Then every positive solution $\{x_n\}$ of Eq.(1.2) satisfies

$$\liminf_{n\to\infty} x_n = l > 0 \quad \text{and} \quad \limsup_{n\to\infty} x_n = L < B.$$

Proof. First we show that

$$\limsup_{n\to\infty} x_n = L < B. \tag{2.2}$$

Suppose it is not true. Then there is a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ such that

$$x_{n_r} = \max_{0 \le n \le n_r} x_n \quad \text{and} \quad \lim_{r\to\infty} x_{n_r} = B. \tag{2.3}$$

From Eq.(1.2) we see that

$$(1-\mu)(x_{n_r} - x_{n_r-1}) + \mu x_{n_r} = \mu x_{n_r-1-k} f(x_{n_r-1-k}).$$

Then, by noting (2.3), it follows that

$$x_{n_r} \leq x_{n_r-1-k} f(x_{n_r-1-k}) \leq x_{n_r} f(x_{n_r-1-k}), \tag{2.4}$$

which implies that $f(x_{n_r-1-k}) \geq 1$. Hence $x_{n_r-1-k} \leq \bar{x}$ and so there is a subsequence $\{x_{n_{r_i}-1-k}\}$ of $\{x_{n_r-1-k}\}$ such that $\lim_{i\to\infty} x_{n_{r_i}-1-k} = b$, where $0 \leq b \leq \bar{x}$. From (2.4) we see that

$$x_{n_{r_i}} \leq x_{n_{r_i}-1-k} f(x_{n_{r_i}-1-k}),$$

which implies that

$$B \leq b f(b).$$

Clearly, this contradicts the hypothesis $xf(x) < B,\ x \in (0, B)$. Hence, (2.2) must hold.

Next, we show that

$$\liminf_{n\to\infty} x_n = l > 0. \tag{2.5}$$

Otherwise, there is a subsequence $\{x_{n_s}\}$ of $\{x_n\}$ such that

$$x_{n_s} = \min_{0\leq n\leq n_s} x_n \text{ and } \lim_{s\to\infty} x_{n_s} = 0. \tag{2.6}$$

From Eq.(1.2) we see that

$$(1-\mu)(x_{n_s} - x_{n_s-1}) + \mu x_{n_s} = \mu x_{n_s-1-k} f(x_{n_s-1-k}).$$

Then, by noting (2.6), it follows that

$$x_{n_s} \geq x_{n_s-1-k} f(x_{n_s-1-k}) \geq x_{n_s} f(x_{n_s-1-k}), \tag{2.7}$$

which implies that $f(x_{n_s-1-k}) \leq 1$. Hence $x_{n_s-1-k} \geq \bar{x}$. Since (2.2) holds, $\{x_n\}$ is bounded and then there is a subsequence $\{x_{n_{s_j}-1-k}\}$ of $\{x_{n_s-1-k}\}$ such that

$$\lim_{j\to\infty} x_{n_{s_j}-1-k} = c,$$

where $\bar{x} \leq c < B$. From (2.7) we see that

$$x_{n_{s_j}} \geq x_{n_{s_j}-1-k} f(x_{n_{s_j}-1-k})$$

which implies that

$$0 \geq c f(c).$$

Clearly, this is a contradiction. Hence, (2.5) must hold. The proof is complete. □

The following theorem is our main result in this section.

Theorem 2.1. *Assume that (2.1) holds and the function $xf(x)$ is L-Lipschitz with*

$$(1-(1-\mu)^{k+1})L < 1. \tag{2.8}$$

Then $\bar{x}$ is a global attractor of solutions of Eq.(1.2) (Relative to the interval $(0, B)$.)

Proof. First we show that every nonoscillatroy (about $\bar{x}$) solution $\{x_n\}$ tends to $\bar{x}$ as $n \to \infty$. Suppose that $x_n - \bar{x}$ is eventually positive. The proof for the case that $x_n - \bar{x}$ is eventually negative is similar and will be omitted. Hence, by noting Lemma 1, there is a positive integer N and a positive constant P such that

$$\bar{x} \le x_n \le P < B \qquad \text{for} \quad n \ \ge N.$$

From Eq.(1.2), we see that

$$x_{n+1+k} - x_{n+k} + \mu(x_{n+k} - x_n) = \mu x_n(f(x_n) - 1).$$

By summing both sides of this equality from N to n, we have

$$x_{n+1+k} - x_{N+k} + \mu \sum_{i=1}^{k}(x_{n+i} - x_{N+i-1}) = \mu \sum_{i=N}^{n} x_i(f(x_i) - 1). \tag{2.9}$$

If $x_n \not\to \bar{x}$ as $n \to \infty$, then there is a convergent subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\lim_{m\to\infty} x_{n_m} > \bar{x}$ and so it follows that

$$\sum_{n=N}^{\infty} x_n(f(x_n) - 1) \le \sum_{m=1}^{\infty} x_{n_m}(f(x_{n_m}) - 1) = -\infty.$$

Hence, the right side of (2.9) is unbounded, while the left side of (2.9) is bounded since $\{x_n\}$ is bounded by Lemma 1. This contradiction implies that $x_n \to \bar{x}$ as $n \to \infty$.

Next, assume that $\{x_n\}$ is a solution of Eq.(1.2) and oscillates about $\bar{x}$, that is, $x_n - \bar{x}$ is not of eventually constant sign. Let $y_n = x_n - \bar{x}$. Then $\{y_n\}$ satisfies the equation

$$y_{n+1} = (1-\mu)y_n + \mu((y_{n-k} + \bar{x})f(y_{n-k} + \bar{x}) - \bar{x}) \tag{2.10}$$

and $\{y_n\}$ oscillates about zero. Let $y_i < y_j$ be two consecutive members of the solution $\{y_n\}$ such that

$$y_i \le 0, \ y_{j+1} \le 0 \quad \text{and} \quad y_n > 0 \quad \text{for} \quad i+1 \le n \le j. \tag{2.11}$$

Let

$$y_r = \max\{y_{i+1}, y_{i+2}, \ldots, y_j\}.$$

We claim that

$$r - (i+1) \le k. \tag{2.12}$$

Suppose, for the sake of contradiction, that $r - (i+1) > k$. Then $y_r \geq y_{r-1-k} > 0$. By noting $y_{r-1-k} + \bar{x} > \bar{x}$ and (2.1), we see that $f(y_{r-1-k} + \bar{x}) < 1$ and so

$$(y_{r-1-k} + \bar{x})f(y_{r-1-k} + \bar{x}) - y_r - \bar{x} < y_{r-1-k} + \bar{x} - y_r - \bar{x} = y_{r-1-k} - y_r \leq 0. \quad (2.13)$$

However, (2.10) yields

$$(y_{r-1-k} + \bar{x})f(y_{r-1-k} + \bar{x}) - y_r - \bar{x} = \frac{1}{\mu}(y_r - y_{r-1}) \geq 0,$$

which contradicts (2.13). Hence (2.12) must hold.

Since $\{x_n\}$ is bounded by Lemma 1, there is a positive constant M such that

$$(y_{n-k} + \bar{x})f(y_{n-k} + \bar{x}) - \bar{x} = x_{n-k}f(x_{n-k}) - \bar{x} \leq M, \qquad n = 0, 1, \ldots.$$

Hence, by noting (2.11), it follows from (2.10) that

$$y_{i+1} \leq \mu M \text{ and } y_{n+1} \leq (1-\mu)y_n + \mu M \quad \text{for} \quad i+1 \leq n \leq j.$$

Then by indiction, it is easy to show that

$$y_r \leq (1-\mu)^{r-i-1} y_{i+1} + (1 - (1-\mu)^{r-i-1})M$$

and so

$$y_r \leq (1-\mu)^{r-i-1}\mu M + (1-(1-\mu)^{r-i-1})M = (1-(1-\mu)^{r-i})M \leq (1-(1-\mu)^{k+1})M.$$

Hence,

$$y_n \leq (1 - (1-\mu)^{k+1})M, \qquad i \leq n \leq j.$$

Since y_i and y_j are arbitrary, we see that there is an integer $n_0 \geq 0$ such that

$$y_n \leq (1 - (1-\mu)^{k+1})M, \qquad n \geq n_0. \quad (2.14)$$

Similarly, it can be shown that

$$y_n \geq -(1 - (1-\mu)^{k+1})M, \qquad n \geq n_0. \quad (2.15)$$

Now, consider any two consecutive members x_i and x_j of the solution with $n_0 + k < i < j$ and with the property (2.11). By a similar argument, we can show that (2.12) holds. Then, by noting (2.14), (2.15) and the Lipschitz property of the function $xf(x)$, we see that

$$|(y_n+\bar{x})f(y_n+\bar{x}) - \bar{x}| = |(y_n+\bar{x})f(y_n+\bar{x}) - \bar{x}f(\bar{x})| \leq L|y_n+\bar{x}-\bar{x}| \leq L(1-(1-\mu)^{k+1})M,$$

$n \geq n_0$, and so it follows from (2.10) that

$$y_{i+1} \leq \mu L(1 - (1-\mu)^{k+1})M$$

and

$$y_{n+1} \leq (1-\mu)y_n + \mu L(1 - (1-\mu)^{k+1})M, \qquad i+1 \leq n \leq j.$$

Then, by induction, we see that

$$\begin{aligned} y_r &\leq (1-\mu)^{r-i-1}y_{i+1} + L(1-(1-\mu)^{k+1})(1-(1-\mu)^{r-i-1})M \\ &\leq (1-\mu)^{r-i-1}\mu L(1-(1-\mu)^{k+1})M + L(1-(1-\mu)^{k+1})(1-(1-\mu)^{r-i-1})M \\ &= L(1-(1-\mu)^{k+1})(1-(1-\mu)^{r-i})M \\ &\leq L(1-(1-\mu)^{k+1})^2 M \end{aligned}$$

which implies that

$$y_n \leq L(1-(1-\mu)^{k+1})^2 M, \qquad i+1 \leq n \leq j.$$

Since y_i and y_j with $j > i > n_0 + k$ are arbitrary, we see that there is an integer $n_1(> n_0 + k)$ such that

$$y_n \leq L(1-(1-\mu)^{k+1})^2 M, \qquad n \geq n_1.$$

Similarly, we may show that

$$y_n \geq -L(1-(1-\mu)^{k+1})^2 M, \qquad n \geq n_1.$$

Finally, by induction, we find that for any positive integer m, there is a positive integer n_m such that

$$|y_n| \leq L^{m-1}(1-(1-\mu)^{k+1})^m M, \qquad n \geq n_m.$$

Clearly, by noting (2.8), it follows that $y_n \to 0$ and so $x_n \to \bar{x}$ as $n \to \infty$. The proof is complete. □

Now, observe that the linearized equation of Eq.(1.2) about the equilibrium $\bar{x}$ is

$$z_{n+1} - (1-\mu)z_n + \mu(1+\bar{x}f'(\bar{x}))z_{n-k} = 0, \qquad n = 0, 1, \ldots. \tag{2.16}$$

A sufficient condition (see [5]) for the zero solution of Eq.(2.16) to be asymptotically stable is

$$1 - \mu + \mu|1 + \bar{x}f'(\bar{x})| < 1,$$

that is,

$$|1 + \bar{x}f'(\bar{x})| < 1. \tag{2.17}$$

Hence, the following result is a direct consequence of Theorem 1 and the linearized theory.

Corollary 2.1. *Assume that (2.1), (2.8) and (2.17) hold. Then the positive equilibrium $\bar{x}$ is globally asymptotically stable (relative to the interval (0, B)).*

3. Applications

In this section, we apply our main result obtained in Section 2 to some difference equation models derived from mathematical biology. First, consider the following model which has been mentioned in Section 1

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]_+, \tag{3.1}$$

where k is a positive integer, $K, q, z \in (0,\infty), \mu \in (0,1)$ and $[x]_+ = \max\{x, 0\}$, and the solutions $\{x_n\}$ of Eq.(3.1) satisfy the initial conditions of the form

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, x^*) \qquad \text{with} \quad x_0 > 0, \tag{3.2}$$

where $x^* = K\left(\frac{1+q}{q}\right)^{1/z}$. The following result provides a sufficient condition for K to be a global attractor of solutions of Eq.(3.1).

Theorem 3.1. *Assume that*

$$q < \frac{(1+z)^{1+1/z}}{z} - 1 \tag{3.3}$$

and

$$(1-(1-\mu)^{k+1})L < 1, \tag{3.4}$$

where $L = \max\{1+q, z(1+q)\}$. *Then the positive equilibrium* K *is a global attractor of solutions of Eq.(3.1) relative to the interval* $(0, x^*)$.

Proof. Let

$$g(x) = x[1 + q(1-(x/K)^z)]_+.$$

It has been shown (see [5]) that $g : (0, x^*) \to (0, x^*)$, and the solutions $\{x_n\}$ of Eq.(3.1) with initial conditions of form (3.2) satisfy $x_n \in (0, x^*),\ n = 0, 1, \ldots$. Hence, with the initial conditions of form (3.2), Eq.(3.1) is equivalent to the equation

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k}\left[1 + q\left(1 - \left(\frac{x_{n-k}}{K}\right)^z\right)\right]. \tag{3.5}$$

Let

$$f(x) = 1 + q(1-(x/K)^z), \qquad x \in (0, x^*).$$

Clearly, K is the only point in $(0, x^*)$ satisfing $f(x) = 1$ and

$$(x-K)(f(x)-1) < 0 \qquad \text{for} \quad x \neq K.$$

Observe that

$$(xf(x))' = 1 + q(1-(1+z)(x/K)^z), \qquad x \in (0, x^*).$$

Clearly

$$\sup\{(xf(x))'\} = 1 + q \text{ and } \inf\{(xf(x))'\} = -z(1 + q).$$

Hence, the function $xf(x)$ is L-Lipschitz with $L = \max\{1 + q, z(1 + q)\}$. By noting all the conditions assumed in Theorem 1 are satisfied, we see that every solution of Eq.(3.1) tends to K as $n \to \infty$. The proof is complete. □

By Corollary 1 and Theorem 2, and by noting (2.17) with $\bar{x} = K$ is equivalent to

$$0 < qz < 2, \tag{3.6}$$

we have the following globally asymptotical stability result for Eq.(3.1) immediately.

Corollary 3.1. *Assume that (3.3), (3.4) and (3.6) hold. Then the positive equilibrium K of Eq.(3.1) is globally asymptotically stable relative to the interval $(0, x^*)$.*

Remark 3.1. It has been shown (see [5]) that if

$$q \leq \min\left\{\frac{2}{z}, \frac{\frac{2a}{K+a}}{(\frac{K+a}{K})^z - 1}\right\} < \frac{(1+z)^{1+1/z}}{z} - 1, \tag{3.7}$$

where a is a positive number satisfing $q \leq \min\{K, x^* - K\}$, then the positive equilibrium K of Eq.(3.1) is globally asymptotically stable relative to the interval $(K - a, K + a) \subset (0, x^*)$.

By comparing this result with ours, we find that we not only obtain a sufficient condition which is very easy to verify for the positive equilibrium K to be globally asymptotically stable relative to the whole interval $(0, x^*)$, but also make the "delay" k and the parameter μ to play roles in the condition.

Next, consider the difference equation

$$x_{n+1} = \alpha x_n + \frac{\beta x_{n-k}}{1 + x_{n-k}^p}, \qquad n = 0, 1, \ldots, \tag{3.8}$$

where $\alpha \in (0, 1), p, \beta \in (0, \infty)$ and k is a positive integer, and the initial conditions of the form

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in [0, \infty) \qquad \text{with} \quad x_0 > 0. \tag{3.9}$$

Clearly, every solution $\{x_n\}$ of Eq.(3.8) with (3.9) exists and is positive forever. When $k = 0$, Eq.(3.8) was proposed by Milton and Belair [7] as a model for the bobwhite quail population of northern Wisconsin, and its local and global stability has been studied in [7] and [3]. Eq.(3.8) is also a discrete analogue of a model that has been used to study blood cell production ([6]). Observe that Eq.(3.8) can be written as

$$x_{n+1} = (1-\mu)x_n + \mu x_{n-k}\frac{\beta/\mu}{1+x_{n-k}^p}, \qquad n = 0, 1, \ldots, \tag{3.10}$$

where $\mu = 1-\alpha$. Then by using Theorem 1, we have the following global attractivity result for Eq.(3.8).

Theorem 3.2. *Assume that*

$$1-\alpha < \beta < \frac{1-\alpha}{1-\alpha^{k+1}}l,$$

where $l = 1$ *if* $p \leq 1$, *and* $l = \min\{1, \frac{4p}{(p-1)^2}\}$ *if* $p > 1$. *Then every solution* $\{x_n\}$*of Eq.(3.8) tends to its positive equilibrium* $\bar{x} = (\frac{\beta+\alpha-1}{1-\alpha})^{1/p}$ *as* $n \to \infty$.

Proof. Let

$$f(x) = \frac{\beta/\mu}{1+x^p}, \qquad x \in (0, \infty).$$

Clearly, $\bar{x} = (\frac{\beta}{\mu} - 1)^{1/p} = (\frac{\beta+\alpha-1}{1-\alpha})^{1/p}$ is the only positive point satisfying $f(x) = 1$, and

$$(x-\bar{x})(f(x)-1) < 0 \qquad \text{for} \quad x \neq \bar{x}.$$

Observe that

$$(xf(x))' = \frac{\beta(1+(1-p)x^p)}{\mu(1+x^p)^2}$$

and

$$(xf(x))'' = \frac{\beta p x^{p-1}((p-1)x^p - (p+1))}{\mu(1+x^p)^3}.$$

Clearly, when $p \leq 1$, $(xf(x))' > 0$ and $(xf(x))'' < 0$, and so it follows that

$$\sup\{|(xf(x))'|\} = \frac{\beta}{\mu}.$$

While for the case that $p > 1$, $\sup\{(xf(x))'\} = \frac{\beta}{\mu}$ and

$$\inf\{(xf(x))'\} = (xf(x))'|_{x=(\frac{p+1}{p-1})^{1/p}} = -\frac{\beta(p-1)^2}{4\mu p},$$

and so it follows that

$$\sup\{|(xf(x))'|\} = \max\left\{\frac{\beta}{\mu}, \frac{\beta(p-1)^2}{4\mu p}\right\}.$$

Hence, the function $xf(x)$ is L-Lipschitz with $L = \frac{\beta}{\mu}$ if $p \leq 1$, and

$$L = \max\left\{\frac{\beta}{\mu}, \frac{\beta(p-1)^2}{4\mu p}\right\}$$

if $p > 1$. Then, it is easy to see that (3.11) implies that (2.8) is satisfied, and so by Theorem 1, every positive solution of Eq.(3.10), that is, every positive solution of Eq.(3.8), tends to its positive equilibrium as $n \to \infty$. The proof is complete. $\square$

Remark 3.2. The result for the case $p \le 1$ in the above theorem has been established in [4]; while for the case $p > 1$, it has been shown in [4] that if

$$1 - \alpha < \beta < (1 - \alpha)\frac{p}{p-1},$$

then every positive solution of Eq.(3.8) converges to its positive equilibrium $\bar{x}$ as $n \to \infty$. Here, we obtain a different condition.

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