MUK PUBLICATIONS
Open Access Publisher

International Jr, of Qualitative Theory of Differential Equations and Applications
Vol. 1 No. 2 (December, 2015)
ISSN: 0973-3590

A Note on Uniqueness of Monotone Mono-Stable Waves for Reaction-Diffusion Equations

Wenzhang Huang*, Matthew Puckett

Department of Mathematical Sciences, University of Alabama in Huntsville, Huntsville, AL 35899, U.S.A.

Abstract. For a monotone reaction-diffusion system, the existence of a monotone mono-stable traveling wave, a traveling wave connecting an unstable equilibrium and a stable equilibrium, has been well understood. However, the uniqueness of a monotone traveling wave systems is in general unclear except for some scalar equations. In this note we prove that, for a class of a monotone reaction-diffusion system, the monotone mono-stable traveling wave solution is indeed unique whenever it exists. Our approach is based on a rigorous analysis of the asymptotical behavior of a monotone traveling wave. It turns out that the information on asymptotical behavior plays essential role in determining the uniqueness of a monotone traveling wave.

AMS Subject Classifications: 34C37, 35K57

Keywords: Reaction-diffusion systems; Monotone mono-stable traveling waves; Eigenvalues and asymptotical behavior; Uniqueness

1. Introduction

Consider the following reaction-diffusion system

$$\frac{\partial u}{\partial t} = \mathbf{d}\Delta u + f(u), \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^n$, $\mathbf{d} = \mathrm{diag}(d_1, \cdots, d_n)$ is an $n \times n$ nonnegative diagonal matrix with at least one $d_j > 0$, $x \in \mathbb{R}^m$ is the spatial variable, $t \geq 0$ is the time, $\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$

E-mail addresses: huangw@email.uah.edu (W. Huang), pucketm@email.uah.edu (M. Puckett)

*Corresponding author

is the Laplace operator, $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. We suppose that the following hold.

[A1] System (1.1) is a monotone system. That is,

$$\frac{\partial f_i(u)}{\partial u_j} \geq 0, \quad i \neq j.$$

[A2] System (1.1) has two constant equilibrium steady states $E_i \in \mathbb{R}^n$, such that $E_1 \ll E_2$, E_1 is unstable, and E_2 is stable.

[A3] Both the matrices $A_i = Df(E_i)$, $i = 1, 2$, are irreducible.

By the assumptions [A1]- [A3] we know that there is a $c^* > 0$, the minimum wave speed, such that for each $c \geq c^*$, there is a monotone traveling wave (called a mono-stable traveling wave) connecting E_1 and E_2. In this paper we are interested in the uniqueness of monotone traveling waves. This is important in applications to practical problems. For instance, when using (1.1) to model many biological or physics problems, it is often the case where the lower equilibrium $E_1 = 0$. If only the nonnegative solutions (the biologically or physically meaningful solutions) are considered, then it can be shown that, for a monotone system, the uniqueness of a monotone traveling wave actually implies the uniqueness of a nonnegative traveling wave (with respect a fixed wave speed). For the simplest equation, i.e, the scalar Fisher equation, the uniqueness of the monotone traveling wave can be proved by analyzing a planar system. Recently, the uniqueness of monotone traveling waves also has been proved for some classes of scalar nonlocal types of reaction-diffusion equations [2, 4] using sophisticated analysis. However, the uniqueness of monotone traveling waves for higher dimensional systems is in general unclear. The purpose of this note is to prove that, for a class of reaction-diffusion systems, the monotone traveling wave solution is indeed unique whenever it exists. That is, we shall prove the following

Theorem 1.1. *Under Assumptions [A1] - [A3], for each $c > 0$, if (1.1) has a monotone traveling wave solution $u(x,t) = U(\nu \cdot x + ct)$ connecting E_1 and E_2, then the function $U(s)$, $s = \nu \cdot x + ct$, is unique up to a time translation, where $\nu \in \mathbb{R}^m$ is a unit vector.*

The extension of Theorem 1.1 to more general systems, such as the system with time delay or nonlocal response, will be addressed in our future research.

A complete proof of Theorem 1.1 will be given in Section 3. We first establish several lemmas in Section 2 that will be used in the proof of Theorem 1.1.

2. Preliminaries

In this paper, a nonnegative vector $h \in \mathbb{R}^n$ is denoted by $h \geq 0$. We use $h > 0$ to indicate that h is nonnegative and nonzero. We say that h is strictly positive, denoted by $h \gg 0$, if all components of h are positive. A real matrix is called positive if all its entries are nonnegative and at least one of its entries is positive. The following lemma is well known for a positive matrix [1].

Lemma 2.1. *Let $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ be a positive and irreducible matrix with $b_{ii} > 0$ for $i = 1, \cdots, n$. If $0 < h \in \mathbb{R}^n$ and h has $k (k < n)$ positive components, then Bh has at least $k + 1$ positive components.*

Definition 2.1. *For a second order system of linear differential equations*

$$c\dot{v} = \mathbf{d}\ddot{v} + Av, \tag{2.1}$$

we say that $\lambda \in \mathbb{C}$ is an eigenvalue and $0 \neq \eta \in \mathbb{C}^n$ is the associated eigenvector corresponding to (2.1) if λ and η satisfy the equation

$$[A + \lambda^2\mathbf{d}]\eta = c\lambda\eta.$$

It is clear that (2.1) has an exponential solution $e^{\lambda t}\eta$ if and only if λ is an eigenvalue and η is the associated eigenvector of (2.1).

Lemma 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be irreducible with the off diagonal entries of A being nonnegative. Then*

- **(i)** *If $\lambda \in \mathbb{R}$ is an eigenvalue and the associated eigenvector η (with respect to (2.1) is nonnegative, then (a1) $\eta \gg 0$ and η is the only associated eigenvector (up to a scalar multiplication); (a2) for $0 \neq \beta \in \mathbb{R}$, $\lambda + i\beta$ is not an eigenvalue.*
- **(ii)** *If in addition A is a stable matrix (all eigenvalues of A have negative real part), then there is a unique $\lambda_* \leq 0$ such that the associated eigenvector η with respect to (2.1) is strictly positive (up to a scalar multiplication). Moreover, λ_* must be negative and is simple.*

Proof. Pick a sufficiently large number α such that $\alpha + a_{ii} > 0$ for $i = 1, \cdots, n$. Then the matrix $A + \alpha I + \lambda^2\mathbf{d}$ is a positive irreducible matrix. Since $[A + \lambda^2\mathbf{d}]\eta = c\lambda\eta$ is equivalent to

$$[A + \alpha I + \lambda^2\mathbf{d}]\eta = (c\lambda + \alpha)\eta, \tag{2.2}$$

the theory for positive matrices implies that $\eta \gg 0$ and the eigenvector is unique. this proves (a1). For (a2), suppose the opposite, i.e. there is a $\beta \neq 0$ and $\xi \in \mathbb{C}$ such that

$$[A + (\lambda + i\beta)^2\mathbf{d}]\xi = c(\lambda + i\beta)\xi.$$

Then

$$[A + \alpha I + (\lambda + i\beta)^2\mathbf{d}]\xi = [c(\lambda + i\beta) + \alpha]\xi.$$

A straightforward computation yields that

$$[A + \alpha I + \lambda^2\mathbf{d}]\xi = [(c\lambda + \alpha)I + \beta^2\mathbf{d} + i(c\beta I - 2\lambda\beta\mathbf{d})]\xi.$$

Let $r(B)$ denote the spectral radius of the matrix B. Then (2.2) yields that $c\lambda+\alpha = r(A+\alpha I+\lambda^2\mathbf{d}) > 0$. Since $\mathbf{d}$ is a nonnegative diagonal matrix, we immediately deduce that

$$[A+\alpha I+\lambda^2\mathbf{d}]\xi^* > (\alpha+c\lambda)\xi^*,$$

where $\xi^* = (|\xi_1|, \cdots, |\xi_n|)$ is a positive vector. The irreducibility therefore implies that

$$r(A+\alpha I+\lambda^2\mathbf{d}) > c\lambda+\alpha,$$

contradicting (2.2).

To prove Part (ii), for real λ, we define the matrix

$$M(\lambda) = A+\lambda^2\mathbf{d}+\alpha I.$$

Then $M(\lambda)$ is a positive and irreducible matrix. Consider the equation

$$M(\lambda)h = (\alpha+c\lambda)h. \tag{2.3}$$

By the theory of positive matrices we know that this equation has a solution $h > 0$ if and only if

$$r(M(\lambda)) = \alpha+c\lambda.$$

It is apparent that $r(M(\lambda))$ is continuous with respect to λ. Let us consider $\lambda \leq 0$ such that the above equality holds. First we note that if $\mu < \lambda \leq 0$, then $M(\mu) \geq M(\lambda)$, which implies

$$r(M(\mu)) \geq r(M(\lambda)).$$

That is, $r(M(\lambda))$ is increasing as $\lambda \leq 0$ decreases. It is obvious that

$$p(\alpha) = \alpha+c\lambda$$

is decreasing as λ decreases. Since A is stable, the dominant eigenvalue λ_0 of A is negative and the corresponding eigenvector ζ is positive. Thus we have

$$M(0)\zeta = [A+\alpha I]\zeta = (\alpha+\lambda_0)\zeta.$$

It follows that

$$r(M(0)) = \alpha+\lambda_0 < \alpha = p(\alpha).$$

On the other hand, we have $r(M(\lambda)) \geq 0$ for all λ and $p(\alpha) < 0$ for $\lambda < -\alpha/c$. Therefore, there is a unique $\lambda_* < 0$ such that

$$r(M(\lambda_*)) = \alpha+c\lambda_*.$$

Consequently, there is a unique strictly positive vector $\eta \in \mathbb{R}^n$ such that

$$M(\lambda_*)\eta = (\alpha+c\lambda_*)\eta.$$

Or equivalently,

$$[A + \lambda_*^2 \mathbf{d}]\eta = c\lambda_* \eta.$$

Finally let us to show that λ_* is simple. If this is not the case, then equation (2.1) must have a solution of the form

$$v(t) = (th_1 + h_2)e^{\lambda_* t}, \quad t \in \mathbb{R},$$

where $h_i \in \mathbb{R}^n$ for $i = 1, 2$, with $h_1 \neq 0$. Upon a substitution of

$$\dot{v}(t) = (\lambda_* th_1 + h_1 + \lambda_* h_2)e^{\lambda_* t}, \quad \ddot{v}(t) = (\lambda_*^2 th_1 + 2\lambda_* h_1 + \lambda_*^2 h_2)e^{\lambda_* t}$$

into (2.1) we obtain the system for h_1 and h_2 as

$$\begin{aligned} [A + \lambda_*^2 \mathbf{d}]h_1 &= c\lambda_* h_1 \\ [A + \lambda_*^2 \mathbf{d}]h_2 &= c\lambda_* h_2 + [cI - 2\lambda_* \mathbf{d}]h_1. \end{aligned} \tag{2.4}$$

The first equation of (2.4) implies that h_1 is a multiple of η. It follows that $h_1 \gg 0$. Moreover, it is known that there is a strictly positive vector $\zeta \in \mathbb{R}^n$ such that

$$\zeta^T [A + \lambda_*^2 \mathbf{d}] = c\lambda_* \zeta^T.$$

where "^{T}" denotes the transpose. Multiplying the second equation of (2.4) from the left by ζ^T we obtain

$$0 = \zeta^T (cI - 2\lambda_* \mathbf{d})h_1.$$

Since $\lambda_* < 0$, $cI - 2\lambda_* \mathbf{d}$ is a strictly positive diagonal matrix. The above equality cannot hold. This leads to a contradiction. $\square$

Throughout the paper, an $n \times n$ matrix function $A(t) = [a_{ij}(t)]$, $t \in \mathbb{R}$, is called an ENN-matrix function (essentially nonnegative matrix function) if $A(t)$ is bounded, continuous, and for all $i \neq j$,

$$a_{ij}(t) \geq 0, \quad t \in \mathbb{R}.$$

Lemma 2.3. *Let $A(t)$ be an ENN-matrix function and $w(t)$ be a nonzero, nonnegative, and bounded solution of the equation*

$$c\dot{w}(t) = \mathbf{d}\ddot{w}(t) + A(t)w(t), \quad t \in \mathbb{R}.$$

Then the following hold.

(1) *There is a real number $b > 0$ such that if $d_j > 0$, then*

$$-bw_j(t) \leq \dot{w}_j(t) \leq bw_j(t), \quad t \in \mathbb{R},$$

where w_j is the jth component of w. Consequently, we have

$$w_j(s) \geq [w(t)e^{-bt}]e^{bs}, \quad s \leq t, \tag{2.5}$$

$$w_j(t) \geq [w(s)e^{bs}]e^{-bt}, \quad t \geq s. \tag{2.6}$$

In particular, $w_j(t_0) > 0$ for some t_0 implies that $w_j(t) > 0$ for all $t \in \mathbb{R}$.

(2) *There is a real number $\delta > 0$ such that if $d_i = 0$, then*

$$w_i(t) \geq [w_i(s)e^{\delta s}]e^{-\delta t} \quad \text{for all} \quad t \geq s.$$

In particular, $w_i(t_0) > 0$ implies that $w_i(t) > 0$ for all $t \geq t_0$.

(3) *If $w \not\equiv 0$ and $A(\infty) = A_2$, then there exists a t^* such that $w(t) \gg 0$ for all $t \geq t^*$.*

(4) *If $A(-\infty) = A_1$ and there exists a t^* such that $w(t) \gg 0$ for all $t \geq t^*$, then $w(t) \gg 0$ for all $t \in \mathbb{R}$.*

(5) *If $A(-\infty) = A_1$ and $A(\infty) = A_2$, then $w \not\equiv 0$ implies $w(t) \gg 0$.*

Proof. Let α be a sufficiently large number such that for $i = 1, \cdots, n$,

$$\alpha + a_{ii}(t) \geq 1, \qquad t \in \mathbb{R}.$$

Then $\alpha I + A(t)$ is a positive matrix for all $t \in \mathbb{R}$. For $d_j > 0$, we rewrite the equation for w_j as

$$d_j \ddot{w}_j(t) - c\dot{w}_j(t) - \alpha w_j(t) = -[\alpha w_j(t) + \sum_{i=1}^{n} a_{ji}(t) w_i(t)]. \tag{2.7}$$

Let

$$\gamma_j^- = \frac{c - \sqrt{c^2 + 4d_j\alpha}}{2d_j} < 0$$

$$\gamma_j^+ = \frac{c + \sqrt{c^2 + 4d_j\alpha}}{2d_j} > 0$$

Then $w_j(t)$ can be expressed by

$$\begin{aligned} w_j(t) &= \frac{1}{d_j(\gamma_j^+ - \gamma_j^-)} \int_{-\infty}^{t} e^{\gamma_j^-(t-s)}[\alpha w_j(s) + \sum_{i=1}^{n} a_{ji}(t) w_i(s)]ds \\ &+ \frac{1}{d_j(\gamma_j^+ - \gamma_j^-)} \int_{t}^{\infty} e^{\gamma_j^+(t-s)}[\alpha w_j(s) + \sum_{i=1}^{n} a_{ji}(t) w_i(s)]ds. \end{aligned} \tag{2.8}$$

Thus we have

$$\begin{aligned} \dot{w}_j(t) &= \frac{\gamma_j^-}{d_j(\gamma_j^+ - \gamma_j^-)} \int_{-\infty}^{t} e^{\gamma_j^-(t-s)}[\alpha w_j(s) + \sum_{i=1}^{n} a_{ji}(t) w_i(s)]ds \\ &+ \frac{\gamma_j^+}{d_j(\gamma_j^+ - \gamma_j^-)} \int_{t}^{\infty} e^{\gamma_j^+(t-s)}[\alpha w_j(s) + \sum_{i=1}^{n} a_{ji}(t) w_i(s)]ds. \end{aligned} \tag{2.9}$$

Let

$$b = \max\{|\gamma_j^-|, \gamma_j^+ : d_j > 0\}.$$

Note that $\alpha w_j(s) + \sum_{i=1}^n a_{ji}(t)w_i(s) \geq 0$. By (2.8) and (2.9) one easily sees

$$-bw_j(t) \leq \dot{w}_j(t) \leq bw_j(t), \quad t \in \mathbb{R}, \quad j = 1, \cdots, n. \tag{2.10}$$

From the last inequalities, we deduce that for any $s < t$,

$$w_j(s) \geq [w_j(t)e^{-bt}]e^{bs}, \qquad w_j(t) \geq [w_j(s)e^{bs}]e^{-bt}.$$

This completes the proof of Part (1).

If $d_i = 0$, then $w_i(t)$ satisfies the inequality

$$c\dot{w}_i(t) + \alpha w_i(t) = \alpha w_i(t) + \sum_{k=1}^n a_{ik}(t)w_k(t) \geq 0.$$

Part (2) of the lemma follows immediately from the above inequality with $\delta = \alpha/c$.

Now we prove Part (3). By (1) and (2), for any i, either $w_i \equiv 0$ or there is a t_i such that $w_i(t) > 0$ for all $t \geq t_i$. Since $w \not\equiv 0$, there is at least one i such that $w_i(t) > 0$ for all $t \geq t_i$. Suppose the statement of Part (3) is false. Then, without loss of generality (otherwise by re-ordering the components of w if necessary), we can suppose that there are $t^* > 0$ and a positive integer $k < n$ such that

$$w_i(t) \quad > 0, \quad t \geq t^*, \quad i = 1, 2, \cdots, k, \tag{2.11}$$

$$w_j(t) \quad \equiv 0, \quad t \in \mathbb{R}, \quad j = k+1, \cdots, n. \tag{2.12}$$

It is clear that the matrix $\alpha I + A(\infty) = \alpha I + A_2$ is positive and irreducible with diagonal entries strictly positive. So is $\alpha I + A(t)$ for all sufficiently large t by the continuity. It therefore follows from (2.11) and Lemma 2.1 that there is a sufficiently large $t_0 > t^*$ such that $[\alpha I + A(t_0)]w(t_0)$ has at least $k+1$ positive components. That is, there is j with $k+1 \leq j \leq n$ such that

$$\alpha w_j(t_0) + \sum_{ji} a_{ji}(t)w_i(t_0) > 0.$$

If $d_j > 0$, then (2.8) implies that

$$w_j(t) > 0, \quad t \in \mathbb{R}.$$

If $d_j = 0$, then

$$\dot{w}_j(t) + \frac{\alpha}{c}w_j(t) = \frac{1}{c}[w_j(t) + \sum_{k=1}^n a_{ji}(t)w_i(t)]$$

implies that

$$w_j(t) = e^{-\frac{\alpha}{c}(t-t_0)}\left[w_j(t_0) + \frac{1}{c}\int_{t_0}^t e^{\frac{\alpha s}{c}}[w_j(s) + \sum_{k=1}^n a_{ji}(s)w_i(s)]ds\right] > 0$$

for all $t > t_0$. In either case we have a contradiction to (2.12).

For the proof of Part (4), recall that there is at least one $d_j \neq 0$. Hence, by the assumption, we have $w_j(t) > 0$ for all $t \in \mathbb{R}$. Arguing in the same way as for the proof of Part (3), one is able to see that $w(t) \gg 0$ for all $t \in \mathbb{R}$.

Finally, it is obvious that Part (5) is a direct consequence of (3) and (4). □

Lemma 2.4. *Let $A(t)$ be an ENN-matrix function and $w(t)$ be a strictly positive and bounded function satisfying*

$$c\dot{w}(t) = \mathbf{d}\ddot{w}(t) + A(t)w(t), \quad t \in \mathbb{R}.$$

(a) *Suppose that $A(t) \to A_2$ and $w(t) \to 0$ as $t \to \infty$. Then there is an $\gamma > 0$ such that*

$$w(t) = \gamma\eta e^{\lambda_* t} + o(e^{\lambda_* t}) \quad \text{as} \quad t \to \infty.$$

where $\lambda_* < 0$ and η is a strictly positive vector satisfying

$$[A_2 + \lambda_*^2\mathbf{d}]\eta = c\lambda_*\eta.$$

[By Assumptions [A1], [A2], and Lemma 2.2, $\lambda_* < 0$ and the strictly positive vector η (up to a scalar multiplication) are uniquely determined by the matrix A_2.]

(b) *Suppose that $A(t) \to A_1$ and $w(t) \to 0$ as $t \to -\infty$. Then there are a $\mu > 0$, a strictly positive vector $\xi \in \mathbb{R}^n$, and an integer $k \geq 0$ such that*

$$w(t) = (-1)^k t^k \xi e^{\mu t} + o(t^k e^{\mu t}) \quad \text{as} \quad t \to -\infty.$$

Proof. First consider the case (a). By Part (1) of Lemma 2.3 we have $w_j(t) \geq w_j(0)e^{-bt}$ for $t \geq 0$ for at least a j. Hence

$$s_* = \inf\{s : \lim_{t\to\infty} |w(t)|e^{-st} = 0\} \geq -b > -\infty.$$

Moreover, $w(t) \to 0$ as $t \to \infty$ implies that $s_* \leq 0$. We write $w(t)$ as a solution of the equation

$$c\dot{w}(t) = \mathbf{d}\ddot{w}(t) + A_2 w(t) + [A(t) - A_2]w(t), \quad t \in \mathbb{R}. \tag{2.13}$$

Now for $T > 0$ and $s > s_*$, the Laplace transform of $w(T + \cdot)$,

$$\hat{w}(T, s) = \int_0^\infty w(T + t)e^{-st}dt,$$

is well defined. Applying the Laplace transform to (2.13) with $s > s_*$ yields

$$-cw(T) + c\hat{w}(T, s) = -\mathbf{d}\dot{w}(T) - s\mathbf{d}w(T) + [s^2\mathbf{d} + A_2]\hat{w}(T, s) + K(T, s), \tag{2.14}$$

where

$$K(T, s) = \int_0^\infty [A(T + t) - A_2]w(T + t)e^{-st}dt.$$

By the same argument used in [3, Lemma 4.2] we can show that there are sequences $\{T_n\}$ and $\{s_n\}$ such that

$$T_n \to \infty, \quad s_n > s_*, \quad s_n \to s_* \quad \text{as} \quad n \to \infty$$

and

$$\frac{w(T_n)}{\|\hat{w}(T_n, s_n)\|} \to 0 \quad \text{as} \quad n \to \infty, \tag{2.15}$$

where $\|x\| = \sum_{k=1}^{n} |x_k|$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Moreover, from (2.15) and Part (1) of Lemma 2.3 it follows that for $d_j \neq 0$,

$$\frac{\dot{w}_j(T_n)}{\|\hat{w}(T_n, s_n)\|} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.16}$$

The assumption

$$\|A(t) - A_2\| \to 0 \quad \text{as} \quad t \to \infty$$

yields that

$$M_n = \sup\{\|A(T_n + t) - A_2\| : \ t \geq 0\} \to 0 \quad \text{as} \quad n \to \infty.$$

By the definition of $\|\cdot\|$ and the positivity of $w(t)$ we obtain that

$$\begin{aligned}
\|K(T_n, s)\| &= \| \textstyle\int_0^\infty [A(T_n + t) - A_2] w(T_n + t) e^{-st} dt\| \\
&\leq M_n \textstyle\int_0^\infty \|w(T_n + t)\| e^{-st} dt \\
&= M_n \| \textstyle\int_0^\infty w(T_n + t) e^{-st} dt\| \\
&= M_n \|\hat{w}(T, s)\|.
\end{aligned} \tag{2.17}$$

It follows that

$$\frac{\|K(T_n, s_n)\|}{\|\hat{w}(T_n, s_n)\|} \leq M_n \to 0 \quad \text{as} \quad n \to \infty. \tag{2.18}$$

It is obvious that $\{\hat{w}(T_n, s_n)/\|\hat{w}(T_n, s_n)\|\}$ has a convergent subsequence. Without loss of generality suppose

$$\lim_{n\to\infty} \frac{\hat{w}(T_n, s_n)}{\|\hat{w}(T_n, s_n)\|} = \tilde{\eta}. \tag{2.19}$$

Then it is clear that $\tilde{\eta} > 0$. Now dividing (2.14) by $\|\hat{w}(T_n, s_n)\|$ and letting $n \to \infty$, with the use of (2.15), (2.16), (2.18) and (2.19), we immediately obtain

$$cs_* \tilde{\eta} = [s_*^2 \mathbf{d} + A_2] \tilde{\eta}.$$

It therefore follows from Part (ii) of Lemma 2.2 that we must have

$$s_* = \lambda_* < 0 \quad \text{and} \quad \tilde{\eta} = \gamma \eta$$

for some $\gamma > 0$. The conclusion of Part (a) follows. The proof of Part (b) is essentially the same as above. Let

$$\mu = \sup\{\tau : \lim_{t\to-\infty} |w(t)|e^{-\tau t} = 0\}.$$

Then $0 \le \mu < \infty$. For for $T < 0$ and $\tau < \mu$, let

$$\tilde{w}(T,\tau) = \int_{-\infty}^{0} w(T+t)e^{-\tau t}dt.$$

Then there are sequences $\{T_n\}$ and $\{\tau_n\}$ such that

$$T_n \to -\infty, \quad \tau_n < \mu, \quad \tau_n \to \mu \quad \text{as} \quad n \to \infty$$

and

$$\begin{aligned} \frac{w(T_n)}{\|\tilde{w}(T_n,\tau_n)\|} &\to 0 \quad \text{as} \quad n \to \infty, \\ \frac{\tilde{w}(T_n,\tau_n)}{\|\tilde{w}(T_n,\tau_n)\|} &\to \xi > 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.20}$$

From (2.20) we deduce that

$$c\mu\xi = [\mu^2\mathbf{d} + A_1]\xi.$$

Thus ξ must be strictly positive. Moreover, it is obvious that $\mu > 0$ because A_1 is unstable. For this case, since μ may not be a simple eigenvalue, we can express $w(t)$ in the form given in Part (b). □

3. Proof of Theorem 1.1

Proof. For a fixed $c > 0$, suppose $u(x,t) = U_1(\nu\cdot x + ct)$ and $v(x,t) = U_2(\nu\cdot x + ct)$ are two monotone traveling wave solutions connecting the equilibria E_1 and E_2. Then

$$c\dot{U}_i(t) = \mathbf{d}\ddot{U}_i(t) + f(U_i(t)), \quad t \in \mathbb{R}, \quad i = 1, 2.$$

To show the uniqueness it suffices to show that U_1 is a translation of U_2. That is, there is a constant a^* such that

$$U_1(t) = U_2(t + a^*), \quad t \in \mathbb{R}.$$

For this purpose let us first establish the

Claim *There are positive numbers γ_i, μ_i and strictly positive vectors $\xi_i \in \mathbb{R}^n$, and integers $k_i \ge 0$, such that for $i = 1, 2$,*

$$U_i(t) = E_2 - \gamma_i e^{\lambda_* t}\eta + o(e^{\lambda_* t}), \quad \text{as} \quad t \to \infty, \tag{3.1}$$

$$U_i(t) = E_1 + (-1)^{k_i} t^{k_i} e^{\mu_i t}\xi_i + o(t^{k_i}e^{\mu_i t}), \quad \text{as} \quad t \to -\infty, \tag{3.2}$$

where $\lambda_* < 0$ and the vector η are defined in Lemma 2.4 (a).

Proof of Claim For $i = 1, 2$, let $V_i(t) = E_2 - U_i(t)$ for $t \in \mathbb{R}$. Then $V_i(t) \to 0$ as $t \to \infty$. A straightforward computation shows that $V_i(t)$ satisfies the equation

$$c\dot{v}(t) = \mathbf{d}\ddot{v}(t) + A^i(t)v(t),$$

where

$$A^i(t) = \int_0^1 Df(E_2 - (1-\theta))V_i(t))d\theta.$$

Thus, by Assumption [A1], $A^i(t)$ is an ENN-matrix that converges to $Df(E_2) = A_2$ as $t \to \infty$. Noting that $V_i(-\infty) = E_2 - E_1 \gg 0$, Lemma 2.3 (1) implies that $V_i(t)$ is strictly positive. It follows from Lemma 2.4 that

$$E_2 - U_i(t) = V_i(t) = \gamma_i e^{\lambda_* t}\eta + o(e^{\lambda_* t}), \quad \text{as} \quad t \to \infty$$

for some $\gamma_i > 0$. This proves (3.1).

To prove (3.2), let $W_i(t) = U_i(t) - E_1$, $i = 1, 2$. Then

$$c\dot{W}_i(t) = \mathbf{d}\ddot{W}_i(t) + B^i(t)W_i(t)$$

with

$$B^i(t) = \int_0^1 Df(E_1 + \theta W_i(t))d\theta.$$

It is clear that $W_i(t)$ is nonnegative. Moreover

$$W_i(\infty) = U_i(\infty) - E_1 = E_2 - E_1 \gg 0$$

implies that there is a $t^* \in \mathbb{R}$ such that $W_i(t) \gg 0$ for all $t \geq t^*$. Consequently we have $W_i(t) \gg 0$ for all $t \in \mathbb{R}$ by Lemma 2.3 (4). It is obvious that $B^i(-\infty) = A_1$. (3.2) therefore follows from Lemma 2.4 (b) .

Now we consider two cases.

Case 1 $\mu_1 \neq \mu_2$, or $\mu_1 = \mu_2$ and $k_1 \neq k_2$.

Without loss of generality, we suppose $\mu_2 < \mu_1$, or $k_2 > k_1$ if $\mu_1 = \mu_2$. We define

$$a^* = \inf\{a \in \mathbb{R} : U_2(t+a) \gg U_1(t),\ t \in \mathbb{R}\}.$$

Let us show that a^* is a real number. First we have

$$U_2(-\infty) = E_1 \ll U_1(0).$$

Hence, by continuity, there is an a_0 such that $U_2(a_0) \ll U_1(0)$. This implies that the set

$$\Omega = \{a \in \mathbb{R} : U_2(t+a) \gg U_1(t),\ t \in \mathbb{R}\}$$

is bounded below. Next, by the assumption and (3.2), we see that there is a $T_1 < 0$ such that

$$U_2(t) \gg U_1(t) \quad \text{for all} \quad t \le T_1. \tag{3.3}$$

Noticing that $\lambda_* < 0$, we can pick a number $a_1 > 0$ such that $\gamma_2 e^{\lambda_* a_1} < \gamma_1$. Then by (3.1) we have

$$U_2(t+a_1) - U_1(t) = [\gamma_1 - \gamma_2 e^{\lambda_* a_1}] e^{\lambda_* t}\eta + o(e^{\lambda_* t}) \quad \text{as} \quad t \to \infty.$$

It follows that there is a $T_2 > 0$ such that

$$U_2(t+a_1) \gg U_1(t), \quad t \ge T_2. \tag{3.4}$$

Let $a_2 = T_2 - T_1 + a_1$. Then, the above inequality and the monotonicity of U_1 and U_2 yield that for all $t \in [T_1, T_2]$, we have

$$U_2(t+a_2) \ge U_2(T_1 + a_2) = U_2(T_2 + a_1) \gg U_1(T_2) \ge U_1(t). \tag{3.5}$$

Since $a_2 > a_1$, from (3.3) and (3.4), it follows that

$$U_2(t+a_2) \gg U_1(t), \quad t \in (-\infty, T_1] \cup [T_2, \infty). \tag{3.6}$$

Thus $a_2 \in \Omega$ by (3.5) and (3.6). Hence Ω is nonempty and bounded below. This implies a^* is a real number. By the definition of a^* and the continuity of $U_i(t)$ for $i = 1, 2$, we conclude that

$$U_2(t+a^*) \ge U_1(t), \quad t \in \mathbb{R}.$$

We claim that $U_2(t+a^*) \equiv U_1(t)$. If this is not the case, let $w(t) = U_2(t+a^*) - U_1(t)$. Then $w(t) \ge 0$ and $w \not\equiv 0$. In addition, we have

$$c\dot{w}(t) = \ddot{w}(t) + A(t)w(t),$$

where

$$A(t) = \int_0^1 Df\big(U_1(t) + \theta[U_2(t+a^*) - U_1(t)]\big) d\theta$$

is an ENN-matrix function and

$$A(t) \to A_1 \quad \text{as} \quad t \to -\infty, \quad \text{and} \quad A(t) \to A_2 \quad as \quad t \to \infty.$$

By Lemma 2.3 (5) and Lemma 2.4 we have $w(t) \gg 0$ and

$$w(t) = \gamma_3 e^{\lambda_* t}\eta + o(e^{\lambda_* t}) \quad \text{as} \quad t \to \infty.$$

That is,

$$\begin{aligned} U_2(t+a^*) &\gg U_1(t), \quad t \in \mathbb{R}, \\ U_2(t+a^*) &= E_2 - \gamma_2 e^{\lambda_* a^*} e^{\lambda_*}\eta + o(e^{\lambda_* t}) \quad \text{as} \quad t \to \infty \end{aligned} \tag{3.7}$$

with

$$\gamma_1 - \gamma_2 e^{\lambda_* a^*} = \gamma_3 > 0.$$

Thus, by using (3.7) and the same discussion as above, we easily see that there is a sufficiently small $\epsilon > 0$ such that

$$U_2(t + a^* - \epsilon) \geq U_1(t), \quad t \in \mathbb{R}.$$

This contradicts the definition of a^*, and hence we must have $U_2(t + a^*) \equiv U_1(t)$. That is, U_1 is a translation of U_2.

Case 2 $\mu_1 = \mu_2 = \mu$ and $k_1 = k_2$.

In this case we must have $\xi_2 = \delta\xi_1$ for some constant $\delta > 0$. Without loss of generality, we can suppose $\delta = 1$, for otherwise we can consider the translation

$$\tilde{U}_2(t) = U_2(t + a) = (-1)^k t^k e^{\mu t}[e^{\mu a}\xi_2] + o(t^k e^{\mu t}) \quad \text{as} \quad t \to -\infty,$$

and choose a such that $e^{\mu a}\delta = 1$. It is obvious that $\tilde{U}_2(t)$ is a monotone traveling wave connecting E_1 and E_2. Now suppose $U_2 \not\equiv U_1$. Then, there is a t_0 and an integer j such that $U_{1,j}(t_0) \neq U_{2,j}(t_0)$, where $U_{i,j}(t)$ is the jth component of $U_i(t)$. For clarity let

$$U_{1,j}(t_0) > U_{2,j}(t_0). \tag{3.8}$$

Let a^* be defined as above. Then, in this case, we must have $a^* > 0$ by (3.8). Also we have

$$U_2(t + a^*) \geq U_1(t), \quad t \in \mathbb{R}.$$

Note that $a^* > 0$ implies that

$$U_2(t + a^*) \gg U_1(t)$$

for all sufficiently negative t. It therefore follows that

$$U_2(t + a^*) \gg U_1(t), \quad t \in \mathbb{R}.$$

By using the same argument as above, we deduce that for some sufficiently small $\epsilon > 0$, we have

$$U_2(t + a^* - \epsilon) \gg U_1(t), \quad t \in \mathbb{R}.$$

This again leads to a contradiction. □

Acknowledgement We thank Prof. X-Q Zhao for calling our attention to the problem on uniqueness of mono-stable monotone traveling wave solutions and to the references [2, 4].

References

[1] A. Berman and R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics 9, SIAM, Philadelphia, 1994.

[2] J. Carr and A. Chmaj, Uniqueness of travelling waves for nonlocal monostable equations. Proc. Amer. Math. Soc. 132 (2004), 2433–2439.

[3] W. Huang, Monotonicity of heteroclinic orbits and spectral properties of variational equations for delay differential equations, J. Diff. Equations, 162 (2000), 91–139.

[4] H. Thieme and X-Q. Zhao, Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models. J. Diff. Equations 195 (2003), 430–470.