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Exponential Decay in Integrodifferential Equations with Nonlocal Conditions

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Abstract. We study the existence, uniqueness, and exponential decay of solutions for a semi-linear integrodifferential equation with a nonlocal initial condition

$$
\begin{aligned}
u'(t) &= Au(t) + \int_0^t F(t-s)Au(s)ds + f(t,u(t)), \quad t \geq 0, \\
u(0) &= \int_0^\infty g(s)u(s)ds + u_0,
\end{aligned}
$$

in a Banach space X, with A the generator of a strongly continuous semigroup. The nonlocal condition can be applied in physics with better effect than the "classical" Cauchy problem $u(0) = u_0$ since more measurements at $t \geq 0$ are allowed. The variation of constants formula for solutions via a resolvent operator and the iteration techniques are used in the study.

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1. Introduction

In this paper, we will study the existence, uniqueness, and exponential decay of solutions for a semi-linear integrodifferential equation with nonlocal initial condition

$$u'(t) = Au(t) + \int_0^t F(t-s)Au(s)ds + f(t, u(t)), \quad t \geq 0, \tag{1.1}$$

$$u(0) = \int_0^\infty g(s)u(s)ds + u_0, \tag{1.2}$$

in a Banach space X, with A the generator of a strongly continuous semigroup and $F(t)$ a bounded operator for $t \geq 0$. g is a real-valued function.

The nonlocal condition can be applied in physics with better effect than the "classical" Cauchy problem $u(0) = u_0$ since more measurements at $t \geq 0$ are allowed. See, e.g., Byszewski [1], Byszewski and Lakshmikantham [2], Deng [3], Friedman [4], Jackson [6], Liang, Liu and Xiao [7, 8], Liang and Xiao [9], Lin and Liu [10], and references therein for other comments and motivations.

The exponential decay for (1.1) with local conditions, i.e., $g = 0$ in (1.2), is studied in Grimmer [5]; the exponential decay for parabolic equations (no integral term in (1.1)) with nonlocal conditions is studied in Deng [3] and Friedman [4]; the existence of solutions for an equation with nonlocal conditions that is slightly different from (1.1) is studied in Lin and Liu [10]. Here, we generalize the ideas in [3, 4, 5, 10] and extend their studies to derive the existence and exponential decay for (1.1)-(1.2) with nonlocal conditions. Also note that in Byszewski [1] and Lin and Liu [10], fixed point arguments are used so that the existences of solutions are proven only for finite intervals. Here, we will see that with the iteration techniques, studied, e.g., in Deng [3] and Friedman [4], we can show the existence of solutions on $\mathbb{R}^+$ for (1.1)-(1.2).

In the above mentioned papers [1, 2, 3, 4, 6, 10], the following idea is used: If a property holds for a local problem (i.e., $g = 0$ in (1.2)) and if g is very small in some sense, then that property also hold for the nonlocal problem. We will see that this idea can also be used here to study the existence, uniqueness, and exponential decay of solutions for (1.1)-(1.2). That is, we will put some smallness conditions on g to study the nonlocal problem (1.1)-(1.2).

As in Lin and Liu [10], the resolvent operator will play an important role, so we list the following hypotheses and definitions. We use $B(X)$ to denote the space of all linear and bounded operators on X, and use Y to denote the Banach space formed from $D(A)$ (the domain of A) endowed with the graph norm.

(H1) *A generates a strongly continuous semigroup in Banach space X.*

(H2) $F(t) \in B(X)$, $t \geq 0$. *For $x \in X, F'(t)x$ is continuous for $t \geq 0$,*

Definition 1.1. [5, 11] *$R(\cdot)$ is a resolvent operator of Eq.(1.1) if $R(t) \in B(X)$ for $t \geq 0$ and satisfies*

1. *$R(0) = I$ (the identity operator on X).*
2. *For all $u \in X$, $R(t)u$ is continuous for $t \geq 0$.*

3. $R(t) \in B(Y)$, $t \geq 0$. *For* $y \in Y$, $R(\cdot)y \in C([0,\infty),Y) \cap C^1([0,\infty),X)$ *and*

$$\begin{aligned}\frac{d}{dt}R(t)y &= AR(t)y + \int_0^t F(t-s)AR(s)yds \\ &= R(t)Ay + \int_0^t R(t-s)F(s)Ayds, \quad t \geq 0. \end{aligned} \tag{1.3}$$

The existence and uniqueness of resolvent operators are guaranteed by the following result.

Theorem 1.1. [5, 11] *Let Assumptions (H1)–(H2) be satisfied. Then Eq.(1.1) has a unique resolvent operator R.*

Definition 1.2. *A mild solution of Eq.(1.1)-(1.2) is a function* $u(\cdot) \in C([0,\infty),X)$ *which satisfies*

$$u(t) = R(t)\Big[\int_0^\infty g(s)u(s)ds + u_0\Big] + \int_0^t R(t-s)f(s,u(s))ds, \quad t \geq 0.$$

A classical solution of Eq.(1.1)-(1.2) is a function $u(\cdot) \in C([0,\infty),Y) \cap C^1([0,\infty),X)$ *which satisfies Eq.(1.1)-(1.2) for* $t \geq 0$.

In order to study the exponential decay of (1.1)-(1.2), we require that the resolvent operator R decays exponentially. (Because in a special case when $f = g = 0$, solutions are given by $u(t) = R(t)u(0)$.) Thus we make the following hypothesis.

(H3) *There are* M, $\alpha > 0$ *such that* $\|R(t)\| \leq Me^{-\alpha t}$, $t \geq 0$.

Since the exponential decay of the resolvent operator R for Eq.(1.1) is studied in Grimmer [5], this hypothesis makes sense.

We close this section by showing the variation of constants formula, which generalizes some results in literatures when $f(t,u) = f(t)$ and $g = 0$. We need other hypotheses on f and g.

(H4) $f \in C^1([0,\infty) \times X, X)$. *There is a constant* $p > 0$ *such that* $\|f(t,u_1) - f(t,u_2)\| \leq p\|u_1 - u_2\|$, $\gamma \equiv \alpha - Mp > 0$, *and* $\int_0^\infty |g(s)|Me^{-\gamma s}ds < 1$.

(H5) $\int_0^\infty g(s)u(s)ds \in D(A)$ *if* $u(\cdot)$ *is a classical solution of (1.1).*

Remark 1.1. Note that A is a closed operator, so that if $g(s) = 0$ for large s, or if $Au(\cdot)$ is bounded on $\mathbb{R}^+$ and $g \in L^1(\mathbb{R}^+)$, then (H5) is true, as $\int_0^\infty g(s)Au(s)ds$ would exist.

Theorem 1.2. *Let (H1)-(H2) be satisfied and let* $R(\cdot)$ *be the resolvent operator of Eq.(1.1). Assume that* $u_0 \in D(A)$. *If* $f \in C([0,\infty) \times X, X)$ *and if u is a classical solution of (1.1)-(1.2), then u is a mild solution of (1.1)-(1.2).*

On the other hand, if $f \in C^1([0,\infty) \times X, X)$ *and if u is a mild solution of (1.1)-(1.2), then* $u \in C^1([0,\infty),X)$. *If, in addition,* $u(0) \in D(A)$, *then u is a classical solution of (1.1)-(1.2).*

Next, if (H1)-(H4) are satisfied, then for each $u_0 \in X$, the mild solution is uniquely determined.

Proof. If $f \in C([0,\infty) \times X, X)$ and if u is a classical solution of (1.1)-(1.2), then Fubini's Theorem can be used to show that u is a mild solution, see [5, 10] for details. Next, if $f \in C^1([0,\infty) \times X, X)$ and if u is a mild solution, then we can show that $u(\cdot) \in C^1$, and then verify that u is a classical solution of (1.1)-(1.2) if $u(0) \in D(A)$. Details will be omitted here since they are similar to those in [10].

For the uniqueness, we let u_1 and u_2 be two mild solutions of (1.1)-(1.2) for a given $u_0 \in X$ and let $w = u_1 - u_2$. Then

$$\begin{aligned} w(t) &= R(t)w(0) + \int_0^t R(t-s)[f(s,u_1(s)) - f(s,u_2(s))]ds, \\ w(0) &= \int_0^\infty g(s)w(s)ds. \end{aligned} \tag{1.4}$$

Therefore, from (H3)-(H4), we have

$$\begin{aligned} \|w(t)\| &\le \|R(t)w(0)\| + \int_0^t \|R(t-s)[f(s,u_1(s)) - f(s,u_2(s))]\|ds \\ &\le \|R(t)w(0)\| + \int_0^t \|R(t-s)pw(s)\|ds \\ &\le \|w(0)\|Me^{-\alpha t} + \int_0^t Me^{-\alpha(t-s)}p\|w(s)\|ds. \end{aligned} \tag{1.5}$$

Then from Gronwall's inequality and (H4), we get

$$\begin{aligned} \|w(t)\| &\le \|w(0)\|Me^{-\alpha t + \int_0^t Mpds} \\ &\le \|w(0)\|Me^{-(\alpha - Mp)t} = \|w(0)\|Me^{-\gamma t}, \quad t \ge 0. \end{aligned} \tag{1.6}$$

Next, from (H4), we have $\rho \equiv \int_0^\infty |g(s)|Me^{-\gamma s}ds < 1$. Now, from (1.4) and (1.6), we obtain

$$\begin{aligned} \|w(0)\| &\le \int_0^\infty |g(s)|\|w(s)\|ds \\ &\le \int_0^\infty |g(s)|\|w(0)\|Me^{-\gamma s}ds \\ &= \rho\|w(0)\|. \end{aligned} \tag{1.7}$$

This implies $w(0) = 0$ since $\rho < 1$, and therefore, from (1.6), we obtain $w(t) = 0$, $t \ge 0$. □

2. Existence and decay

In this section, we will obtain the existence, uniqueness and exponential decay of mild and classical solutions for (1.1)-(1.2), by using the variation of constants formula and the iteration techniques.

Theorem 2.1. *Let Assumptions (H1)–(H5) be satisfied. Then for every $u_0 \in D(A)$, (1.1)-(1.2) has a unique mild solution, and the mild solution decays exponentially.*

Proof. Define $u_1(\cdot) \equiv 0$ and for $k \geq 2$ define u_k to be the classical solutions of

$$u_k'(t) = Au_k(t) + \int_0^t F(t-s)Au_k(s)ds + f(t, u_k(t)), \quad t \geq 0, \tag{2.1}$$

$$u_k(0) = \int_0^\infty g(s)u_{k-1}(s)ds + u_0. \tag{2.2}$$

The existence and uniqueness of u_k are guaranteed by [5, 11] and Theorem 1.2 since (H5) implies $u_k(0) \in D(A)$. And we have for $k \geq 2$,

$$u_k(t) = R(t)\Big[\int_0^\infty g(s)u_{k-1}(s)ds + u_0\Big] + \int_0^t R(t-s)f(s, u_k(s))ds, \quad t \geq 0. \tag{2.3}$$

Next, define $w_k(t) = u_{k+1}(t) - u_k(t)$, $t \geq 0$, $k \geq 1$. Then

$$w_k(t) = R(t)\Big[\int_0^\infty g(s)w_{k-1}(s)ds\Big] + \int_0^t R(t-s)\Big[f(s, u_{k+1}(s)) - f(s, u_k(s))\Big]ds, \tag{2.4}$$

$$w_k(0) = \int_0^\infty g(s)w_{k-1}(s)ds, \ k \geq 2,$$

$$w_1(0) = u_2(0) = u_0.$$

From (2.4) and similar to (1.5)-(1.6), we have for $k \geq 1$,

$$\|w_k(t)\| \leq \|w_k(0)\|Me^{-\gamma t}, \quad t \geq 0. \tag{2.5}$$

Thus, $\|w_1(t)\| \leq \|w_1(0)\|Me^{-\gamma t} = \|u_0\|Me^{-\gamma t}$. So that from (2.4),

$$\begin{aligned} \|w_2(0)\| &\leq \int_0^\infty \|g(s)w_1(s)\|ds \\ &\leq \int_0^\infty |g(s)|\|u_0\|Me^{-\gamma s}ds \\ &= \|u_0\|\int_0^\infty |g(s)|Me^{-\gamma s}ds \\ &\equiv \|u_0\|\rho. \end{aligned} \tag{2.6}$$

Therefore from (2.5), $\|w_2(t)\| \leq \|u_0\|\rho Me^{-\gamma t}$. And hence

$$\begin{aligned} \|w_3(0)\| &\leq \int_0^\infty \|g(s)w_2(s)\|ds \\ &\leq \|u_0\|\rho\int_0^\infty |g(s)|Me^{-\gamma s}ds \\ &= \|u_0\|\rho^2. \end{aligned} \tag{2.7}$$

Thus

$$\|w_3(t)\| \quad \leq \quad \|u_0\|\rho^2 M e^{-\gamma t}, \quad t \geq 0. \tag{2.8}$$

By induction, we have

$$\|w_k(t)\| \quad \leq \quad \|u_0\|\rho^{k-1} M e^{-\gamma t}, \;\; t \geq 0, \;\; k \geq 1. \tag{2.9}$$

Since $\rho < 1$, this implies that the series $\sum w_k(t)$ is convergent. Hence the sequence $\{u_k(t)\}$ is convergent. And we also have

$$\begin{aligned} \|u_k(t)\| &\leq \|u_k(t) - u_{k-1}(t)\| + \|u_{k-1}(t) - u_{k-2}(t)\| + ... + \|u_2(t) - u_1(t)\| \\ &= \|w_{k-1}(t)\| + \|w_{k-2}(t)\| + ... + \|w_1(t)\| \\ &\leq \|u_0\| M e^{-\gamma t}\Big[\rho^{k-2} + \rho^{k-2} + ... + 1\Big] \\ &\leq \frac{\|u_0\|M}{1-\rho} e^{-\gamma t}, \quad t \geq 0. \end{aligned} \tag{2.10}$$

Now, one can easily verify, by the Cauchy sequence and the diagonal sequence techniques, that there exists $u \in C([0,\infty), X)$ such that

$$\|u(t)\| \quad \leq \quad \frac{\|u_0\|M}{1-\rho} e^{-\gamma t}, \quad t \geq 0, \tag{2.11}$$

and

$$\lim_{k\to\infty} \sup_{t\geq 0} \|u_k(t) - u(t)\| = 0. \tag{2.12}$$

Next, since

$$u_k(t) = R(t)\Big[\int_0^\infty g(s)u_{k-1}(s)ds + u_0\Big] + \int_0^t R(t-s)f(s, u_k(s))ds, \tag{2.13}$$

one easily verifies that

$$u(t) = R(t)\Big[\int_0^\infty g(s)u(s)ds + u_0\Big] + \int_0^t R(t-s)f(s, u(s))ds. \tag{2.14}$$

Therefore, Theorem 1.2 shows that u is the unique mild solution of (1.1)-(1.2). u decays exponentially according to (2.11). □

Results for classical solutions can be obtained by using Theorem 1.2.

Theorem 2.2. *Let Assumptions (H1)–(H5) be satisfied and $u_0 \in D(A)$. Let u be the unique mild solution of (1.1)-(1.2) determined by Theorem 2.1. If $u(0) \in D(A)$, then u is the classical solution of (1.1)-(1.2) and u decays exponentially.*

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