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Global Attractivity for a Nonlinear Differential Equation with Unbounded Delay

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Abstract. In this paper, we obtain sufficient conditions for every solution of the nonlinear differential equation with unbounded delay $x'(t) = -p(t)f(x(g(t)))$ to tend to zero as $t \to \infty$, without requiring the nondecreasing assumption of $f(x)$.

AMS Subject Classifications: 34K20, 34K25

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1. Introduction

We consider the nonlinear differential equation with unbounded delay

$$x'(t) = -p(t)f(x(g(t))), \quad t \geq 0, \tag{1.1}$$

where $p : [0, \infty) \to [0, \infty)$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $xf(x) > 0$ if $x \neq 0$, and $g : [0, \infty) \to \mathbb{R}$ is a nondecreasing continuous function such that $g(t) \leq t$ for $t \geq 0$ and $\lim_{t\to\infty} g(t) = \infty$. We see that $g(t) = t - 1$ and $g(t) = t/2$ are typical examples of $g(t)$.

By a solution of (1.1), we mean that a continuous function $x(t)$ which is defined for $t \geq g(0)$ and satisfies (1.1) for $t \geq 0$. We note that if $|f(x)| \leq |x|$ for $x \in \mathbb{R}$ and $\int_{g(t)}^{t} p(s)ds$ is finite for all large $t > 0$, then solutions of (1.1) exist for $t \geq 0$, by an argument similar to [14].

Global asymptotic behavior of solutions of scalar delay differential equations containing (1.1) has been studied by many authors, see [1–6, 8–17] and the references

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cited therein. In [8], the author et al. discussed global attractivity for (1.1) under the following strict nonlinearity of $f(x)$:

$$|f(x)| < |x| \quad \text{if } x \neq 0. \tag{1.2}$$

Their result is stated as follows:

Theorem A. *Let $f(x)$ be a nondecreasing function satisfying* (1.2). *Suppose that*

$$\int_{g(t)}^{t} p(s)ds \leq \frac{3}{2} \quad \textit{for all large } t > 0 \tag{1.3}$$

and

$$\int_{0}^{\infty} p(s)ds = \infty. \tag{1.4}$$

Then every solution of (1.1) *tends to zero as $t \to \infty$.*

Theorem A is obtained under the 3/2 stability condition (see, e.g., [2, 3, 11, 13–16]). An example which shows that the condition (1.3) is the best possible is also given in [8]. Unfortunately, however, in case $f(x) = x(4 - 3\sin^2 x)/5$, Theorem A cannot be applied to (1.1), because the monotonicity of $f(x)$ is not satisfied. The purpose of this paper is to remove this restriction in Theorem A. This research is inspired by our recent paper on global attractivity for nonlinear delay difference equations in [7].

2. Main Result

The following theorem is our main result.

Theorem 2.1. *If* (1.2), (1.3) *and* (1.4) *hold, then every solution of* (1.1) *tends to zero as $t \to \infty$.*

To prove Theorem 2.1, we give some remarks. First, in view of the assumption of $g(t)$, there exists a sufficiently large $T > 0$ such that $g(t) \geq 0$ for $t \geq T$. Note that

$$g(g(t)) \leq g(t) \leq t \quad \text{for } t \geq T.$$

Let $g^{-1}(t) = \sup\{s : g(s) = t\}$ for $t \geq 0$. Then $g^{-1}(t)$ is a piecewise continuous function satisfying $t \leq g^{-1}(t)$ for $t \geq T$.

Next, we notice that if solutions of (1.1) are nonoscillatory, then the following result holds.

Lemma 2.1. *If* (1.4) *holds, then every nonoscillatory solution of* (1.1) *tends to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Then there exists $t_1 \geq g^{-1}(T)$ such that $x(t)$ has a constant sign for $t \geq g(t_1)$. Assume that $x(t) \geq 0$ for $t \geq g(t_1)$. (In case $x(t) \leq 0$, the proof is similar.) Then we have $x'(t) \leq 0$ for $t \geq t_1$. Hence,

$x(t)$ is nonincreasing on $[t_1, \infty)$ and there exists $\alpha \geq 0$ such that $\lim_{t\to\infty} x(t) = \alpha$. If $\alpha \neq 0$, there exists $t_2 = t_2(\alpha) \geq g^{-1}(t_1)$ such that

$$\frac{\alpha}{2} \leq x(t) \leq \frac{3}{2}\alpha \quad \text{for } t \geq g(t_2),$$

which implies

$$x'(t) \leq -p(t)\beta \quad \text{for } t \geq t_2, \tag{2.1}$$

where $\beta = \min_{\alpha/2\leq x\leq 3\alpha/2} f(x) > 0$. Thus, integrating (2.1) from t_2 to t and using (1.4), we obtain

$$x(t) - x(t_2) \leq -\beta \int_{t_2}^{t} p(s)ds \to -\infty \quad \text{as } t \to \infty.$$

This contradicts the fact that the left-hand side tends to a finite limit as $t \to \infty$, and so $\alpha = 0$. The proof is complete. □

Proof of Theorem 2.1. By virtue of Lemma 2.1, we have only to consider the case where solutions of (1.1) are oscillatory.

Let $x(t)$ be an oscillatory solution of (1.1). Then, by (1.3), there exists a sufficiently large $t^* \geq g^{-1}(g^{-1}(T))$ such that $x(t^*) = 0$ and

$$\int_{g(t)}^{t} p(s)ds \leq \frac{3}{2} \quad \text{for } t \geq g(g(t^*)). \tag{2.2}$$

Let M be a positive constant such that

$$\max_{g(g(t^*))\leq t\leq t^*} |x(t)| \leq M.$$

We define

$$\tilde{f}(x) = \max\left\{\sup_{0\leq u\leq x} f(u),\ \sup_{0\leq u\leq x}(-f(-u))\right\} \quad \text{for } x \geq 0.$$

Then, from the nondecreasing property of $\tilde{f}(x)$, it turns out that

$$|f(x(s))| \leq \tilde{f}(|x(s)|) \leq \tilde{f}(M) \quad \text{for } s \in [g(g(t^*)), t^*].$$

Hence, we get

$$|x'(t)| = p(t)|f(x(g(t)))| \leq \tilde{f}(M)\, p(t) \quad \text{for } t \in [t^*, g^{-1}(t^*)]. \tag{2.3}$$

Also since

$$\begin{aligned} |x(g(t))| &\leq \int_{g(t)}^{t^*} p(\xi)|f(x(g(\xi)))|d\xi \\ &\leq \tilde{f}(M)\int_{g(t)}^{t^*} p(\xi)d\xi \quad \text{for } t \in [t^*, g^{-1}(t^*)], \end{aligned}$$

we have, together with (1.2),

$$|x'(t)| \le p(t)|x(g(t))| \le \tilde{f}(M)\, p(t) \int_{g(t)}^{t^*} p(\xi) d\xi \quad \text{for } t \in [t^*, g^{-1}(t^*)]. \tag{2.4}$$

Thus, by (2.3) and (2.4), we obtain

$$\begin{aligned}
|x(t)| &\le \min\left\{ \tilde{f}(M) \int_{t^*}^{t} p(s)ds,\ \tilde{f}(M) \int_{t^*}^{t} p(s) \int_{g(s)}^{t^*} p(\xi) d\xi ds \right\} \\
&\le \tilde{f}(M) \int_{t^*}^{t} p(s) \min\left\{ 1, \int_{g(s)}^{t^*} p(\xi) d\xi \right\} ds \quad \text{for } t \in [t^*, g^{-1}(t^*)].
\end{aligned}$$

Now we will show that

$$|x(t)| \le \tilde{f}(M) \quad \text{for } t \in [t^*, g^{-1}(t^*)]. \tag{2.5}$$

We consider two cases:
Case (I). $\lambda = \int_{g(t^*)}^{t^*} p(s)ds \le 1$. Then for $t \in [t^*, g^{-1}(t^*)]$ we see

$$\begin{aligned}
|x(t)| &\le \tilde{f}(M) \int_{t^*}^{g^{-1}(t^*)} p(s) \int_{g(s)}^{t^*} p(\xi) d\xi ds \\
&= \tilde{f}(M) \int_{g(t^*)}^{t^*} p(\xi) \int_{t^*}^{g^{-1}(\xi)} p(s) ds d\xi \\
&= \tilde{f}(M) \left\{ \int_{g(t^*)}^{t^*} p(\xi) \int_{\xi}^{g^{-1}(\xi)} p(s) ds d\xi - \int_{g(t^*)}^{t^*} p(\xi) \int_{\xi}^{t^*} p(s) ds d\xi \right\} \\
&\le \tilde{f}(M) \left\{ \frac{3}{2} \int_{g(t^*)}^{t^*} p(\xi) d\xi + \left[\frac{1}{2} \left(\int_{\xi}^{t^*} p(s) ds \right)^2 \right]_{g(t^*)}^{t^*} \right\} \\
&= \tilde{f}(M) \left(\frac{3}{2}\lambda - \frac{1}{2}\lambda^2 \right) \\
&\le \tilde{f}(M).
\end{aligned}$$

Case (II). $1 < \int_{g(t^*)}^{t^*} p(s)ds \le 3/2$. Then there exists $s_0 \in (t^*, g^{-1}(t^*))$ such that $\int_{g(s_0)}^{t^*} p(\xi)d\xi = 1$, and we have for $t \in [t^*, g^{-1}(t^*)]$,

$$\begin{aligned}
|x(t)| &\le \tilde{f}(M) \left\{ \int_{t^*}^{s_0} p(s) \cdot 1 ds + \int_{s_0}^{g^{-1}(t^*)} p(s) \int_{g(s)}^{t^*} p(\xi) d\xi ds \right\} \\
&= \tilde{f}(M) \left\{ \int_{g(s_0)}^{t^*} p(\xi) d\xi \int_{t^*}^{s_0} p(s) ds + \int_{g(s_0)}^{t^*} p(\xi) \int_{s_0}^{g^{-1}(\xi)} p(s) ds d\xi \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{f}(M) \int_{g(s_0)}^{t^*} p(\xi) \int_{t^*}^{g^{-1}(\xi)} p(s) ds d\xi \\
&= \tilde{f}(M) \left\{ \int_{g(s_0)}^{t^*} p(\xi) \int_{\xi}^{g^{-1}(\xi)} p(s) ds d\xi - \int_{g(s_0)}^{t^*} p(\xi) \int_{\xi}^{t^*} p(s) ds d\xi \right\} \\
&\le \tilde{f}(M) \left\{ \frac{3}{2} \int_{g(s_0)}^{t^*} p(\xi) d\xi + \left[\frac{1}{2} \left(\int_{\xi}^{t^*} p(s) ds \right)^2 \right]_{g(s_0)}^{t^*} \right\} \\
&= \tilde{f}(M).
\end{aligned}
$$

Furthermore, using (2.5), we claim that

$$
|x(t)| \le \tilde{f}(M) \quad \text{for } t \ge t^*. \tag{2.6}
$$

Suppose, for the sake of contradiction, that

$$
|x(t)| > \tilde{f}(M) \quad \text{for some } t \ge g^{-1}(t^*).
$$

Then there exist $T_1 = \sup\{t > g^{-1}(t^*) : |x(s)| \le \tilde{f}(M) \text{ for } s \in [g^{-1}(t^*), t]\}$ and $T_2 = \sup\{t^* \le t < T_1 : x(t) = 0\}$. In case $T_1 < g^{-1}(T_2)$, noting that $\max_{g(g(T_2)) \le t \le T_2} |x(t)| \le M$, we get

$$
|x(t)| \le \tilde{f}(M) \quad \text{for } t \in [T_2, g^{-1}(T_2)],
$$

which contradicts the definition of T_1. In case $T_1 \ge g^{-1}(T_2)$, it follows from the choice of T_1 and T_2 that

$$
x(t) \ge 0 \quad \text{or} \quad x(t) \le 0 \quad \text{for } t \in [T_2, T_1].
$$

Assume first that $x(t) \ge 0$ for $t \in [T_2, T_1]$. (In case $x(t) \le 0$, the proof is similar.) Then we have

$$
x'(t) \le 0 \quad \text{for } t \in [T_1, g^{-1}(T_1)],
$$

and so

$$
x(t) \le x(T_1) = \tilde{f}(M) \quad \text{for } t \in [T_1, g^{-1}(T_1)].
$$

This contradicts the definition of T_1. Consequently, we obtain (2.6).

From the argument above, we can establish an increasing sequence $\{t_n^*\}$ with $t_1^* = t^*$ such that $\lim_{n\to\infty} t_n^* = \infty$ and $x(t_n^*) = 0$, and a sequence $\{X_n\}$ with $X_1 = M$, $X_{n+1} = \tilde{f}(X_n)$ such that

$$
\max_{g(g(t_n^*)) \le t \le t_n^*} |x(t)| \le X_n \quad \text{and} \quad \sup_{t \ge t_n^*} |x(t)| \le X_{n+1}. \tag{2.7}
$$

Here, in view of (1.2) and the definition of $\tilde{f}(x)$, we obtain $\tilde{f}(0) = 0$ and

$$
0 < \tilde{f}(x) < x \quad \text{for } x > 0,
$$

which imply that X_n tends to zero as $n \to \infty$. Finally, by (2.7), we conclude that $x(t)$ tends to zero as $t \to \infty$. This completes the proof. □

Remark 2.1. *In our proof of Theorem 2.1, the nondecreasing assumption of* $g(t)$ *cannot be omitted, because we need to use it in leading* (2.3) *and* (2.4)*, and need to exchange the order of integration on the double integral in leading* (2.5)*.*

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