CHAOS AND OPTIMAL CONTROL OF STOCHASTIC LATTICE OF PREY-PREDATOR WITH UNKNOWN PROBABILITIES

Awad El-Gohary and Saba Alwan

Department of Statistics and OR, Faculty of Science, King Saud University, P. O. Box. 2455, Riyadh, 11451, Saudi Arabia
E-mail: elgohary0@yahoo.com,

and

Department of Statistics and OR, Faculty of Science, King Saud University, P. O. Box. 800, Riyadh, 11421, Saudi Arabia
E-mail: alwansaba@yahoo.com

Abstract

In this paper, we will discuss the problem of chaos and control of a stochastic lattice gas model for prey-predator. Such system follows a stochastic dynamics composed of three Markovian sub-processes which are the birth of prey; the death of prey and simultaneous birth of predator, and spontaneous death of predator. The stability analysis and chaos of the stationary states of one-site correlations will be discussed. The optimal control inputs that ensure the asymptotic stability of unstable stationary states are derived. Estimators of the unknown probabilities will be derived. Numerical simulation study will be presented.

Key words: Stochastic Lattice gas Model, Prey-Predator, Optimal Control, Mean field approximation, Estimation.

1 INTRODUCTION

The theory of interacting particle systems began as a branch of probability theory and applied mathematics in the late 1960’s. Many of the original impetus came from the work of Spitzer in the United States and of Dobrushin in the Soviet Union. For example, of their early work, see Spitzer (1969) and Dobrushin (1971a, b). During the decade and a half since then, this area has grown and developed rapidly, establishing unexpected connections with a number of other fields.

In fact, the applications of the stochastic and probability models can extend to describe many models in physical, chemical, ecology and medical fields. A wide variety of
phenomena can be modeled by the mean field theory of interacting particles such as food chain, prey-predator model, Ising and chemostat models. The original motivation for the mean field theory came from applied mathematics, chemistry, physics, biological science, statistical mechanics and many others fields.

The main objective of mean field theory was to describe and analyze stochastic models for the time evolution of systems whose equilibrium measures are the classical Gibbs states. In particular, it was hopped that this would lead to a better understanding of the phenomenon of phase transition. As time passed, it became clear that models with a very similar mathematical structure could be naturally formulated in other contexts - neural networks, tumor growth, spread of infection, and behavioral systems, for example. From a mathematical point of view, interacting particle systems represents a natural departure from the established theory of Markov processes.

The interest in fluctuations of population and in the stochastic methods for their description have grown enormously in the last century. The problems of estimating and controlling stochastic systems are far from solved, and a considerable amount of research is under way. Estimation of the internal states of a stochastic dynamical system is a topic with important applications in different fields such as physics, biology and medicine [4, 16, 17].

A spatial stochastic model to discuss strategies to control the epidemic was introduced by Schinazi [6]. El-Gohary has proposed a stochastic model to study the problem of optimal controlling the epidemic [4, 16]. El-Gohary and Al-Ruziza [7, 18] have suggested a non-linear stochastic model to investigate the optimal control of a non-homogenous prey-predator model. They have derived the feedback control law as non-linear functions of the population densities. El-Gohary, et.al have studied the problems of chaos and optimal control cancer model with complete unknown parameters [17, 19].

In the present paper, we propose a model consisting of a system of two types of interacting particles residing in the sites of a square lattice. One of these types of particles represents a prey and the other represents a predator. Every site of the lattice can be either vacant (0), occupied by a prey \( P_1 \) or occupied by a predator \( P_2 \). The local interactions considered are as follows: predators can be spontaneously annihilated \( P_2 \rightarrow 0 \), prey can be autocatalytically created \( 0 + nP_1 \rightarrow (n+1)P_1 \), and predator can also be autocatalytically created at expense of prey \( P_1 + nP_2 \rightarrow (n+1)P_2 \). The stability analysis and chaos of the stationary states of this model are presented. The problem of the optimal control and estimation of the unknown probabilities will be discussed.

2 STOCHASTIC PROBABILITY MODEL

In this section, we will describe the stochastic rules of the proposed stochastic lattice gas model for prey-predator. The master equation is used to evaluate the time rate of the probability of states of the system at any time. The average of the state function is obtained. The time rate of the sites correlations is derived.

Consider a lattice of \( N \) sites. Every site can be either empty (0) or occupied by a prey
or occupied by a predator \((P_2)\). At any time step a site is randomly chosen. For that site, we suppose that \(n_1\) and \(n_2\) are the numbers of nearest neighbors of that site occupied by prey and predator respectively and \(S\) is the total number of nearest neighbors of this site. For this model the following assumptions are adopted:

1. The site is empty and becomes either occupied by a prey with probability \(p_1 n_1 / S\) or remains empty with probability \(1 - p_1 n_1 / S\).

2. The site is occupied by a prey, it will be either replaced by predator with a probability \(p_2 n_2 / S\) or it remains as a prey with probability \(1 - p_2 n_2 / S\).

3. The site is occupied by a predator, it will be either vacated with a probability \(p_3\) or remains as a predator with probability \(1 - p_3\).

This Markov process contains three probability parameters \(p_1, p_2\) and \(p_3\), they are associated to this process. Sub-process (I) describes the prey birth process, Subprocess (II) describes the death of the prey and simultaneous birth of predator, finally the Subprocess (III) describes the spontaneous death of the predator \([13, 14, 15]\).

The state of the system evolves in time according to stochastic dynamics. Assuming that the state of this system is represented by the random vector \(\vec{X} = (X_1, X_2, \ldots, X_N)\) where \(X_i = 0, 1,\) or \(2\) \((i = 1, 2, \ldots, N)\) according to whether the site \(i\) is empty, occupied by a prey or occupied by a predator respectively. Let \(P(\vec{X}, t)\) be the probability of the state \(\vec{X}\) of the system at time \(t\) and let \(V_i(\vec{X})\) be the transition probability of cyclic permutation of variable \(X_i\). This means that if \(X_i = 0, 1\) or \(2\) then \(V_i(\vec{X})\) is the transition probability to \(X_i = 1, 2\) or \(0\), respectively. The time rate of the probability \(P(\vec{X}, t)\) is governed by the master equation which given by [8]:

\[
\frac{d}{dt} P(\vec{X}, t) = \sum_{i=1}^{N} \left\{ V_i(\vec{X}^i) P(\vec{X}^i, t) - V_i(\vec{X}) P(\vec{X}, t) \right\} 
\]

where the state \(\vec{X}^i\) denotes the state that can be obtained from \(\vec{X}\) by an anticyclic permutation of the variable \(X_i\). According to the local rules of the model considered, we have:

\[
V_i(\vec{X}) = \begin{cases} 
\frac{p_1}{S} \sum_r \delta(X_{i+r}, 1), & \text{if } X_i = 0 \\
\frac{p_2}{S} \sum_r \delta(X_{i+r}, 2), & \text{if } X_i = 1 \\
p_3, & \text{if } X_i = 2 
\end{cases} 
\]

where \(\delta(r, s)\) denotes the kronecker delta and the summation in equation(2) is over the nearest neighbor sites.

The master equation (1) is an equivalent form of the Chapman-Kolmogorov equation for Markov processes, but it is easier to handle and more related to dynamical stochastic models [8].
In the next part of this section, we will obtained the time evolution of one and two site correlations. For this purpose we will define the average of any function $F(\vec{X})$ of the state $\vec{X}$:

$$\langle F(\vec{X}) \rangle = \sum_\vec{X} F(\vec{X}) P(\vec{X}, t).$$  \hspace{1cm} (3)

From equation (2) and the master equation (1) we find that the time rate of average the function $\langle F(\vec{X}) \rangle$ is given by [20]

$$\frac{d}{dt} \langle F(\vec{X}) \rangle = \sum_{i=1}^N \left[ \frac{\partial}{\partial t} F(\vec{X}) \right] V_i(\vec{X}),$$  \hspace{1cm} (4)

where the state $\vec{X}_i$ denotes the state that can be obtained from $\vec{X}$ by a cyclic permutation of the variable $X_i$.

Now, let us introduce the following definitions

$$P_i(\nu_1) = \langle \delta(X_i, \nu_1) \rangle, \quad P_{ij}(\nu_1 \nu_2) = \langle \delta(X_i, \nu_1) \delta(X_j, \nu_2) \rangle,$$

$$P_{ijk}(\nu_1 \nu_2 \nu_3) = \langle \delta(X_i, \nu_1) \delta(X_j, \nu_2) \delta(X_k, \nu_3) \rangle,$$

$$P_{i_1i_2...i_n}(\nu_1, \nu_2, \ldots, \nu_n) = \langle \prod_{s=1}^n \delta(X_{i_s}, \nu_s) \rangle,$$

where the variables $\nu_i$ can take any one of the values 0, 1 or 2.

Using equations (2), (4) and (5) the evolution of these densities which are given by (5) can be derived as follows:

$$\frac{d}{dt} P_i(1) = \frac{p_1}{S} \sum_{r \neq i} P_{i,r+i}(01) - \frac{p_2}{S} \sum_r P_{i,r+i}(12),$$

$$\frac{d}{dt} P_i(2) = \frac{p_2}{S} \sum_r P_{i,r+i}(12) - p_3 P_i(2),$$

$$\frac{d}{dt} P_{ij}(01) = -\frac{p_1}{S} \sum_{r \neq j} \left[ P_{r+i,j}(101) + P_{ij}(01) \right] + \frac{1}{S} \sum_{j \neq i} [p_1 P_{i,j+r}(001) - p_2 P_{i,j+r}(012)] + p_3 P_{ij}(21),$$

$$\frac{d}{dt} P_{ij}(12) = \frac{1}{S} \sum_{r \neq i} [p_2 P_{i+r,j}(012) - p_2 P_{i+r,j}(212) + \frac{p_2}{S} P_{ij,j+r}(112) - (\frac{p_2}{S} + p_3) P_{ij}(12)],$$

$$\frac{d}{dt} P_{ij}(02) = p_3 P_{ij}(22) - \frac{p_1}{S} \sum_{r \neq i} P_{i+r,j}(102) + \frac{p_2}{S} \sum_{j \neq i} P_{ij,j+r}(012) - p_3 P_{ij}(02).$$  \hspace{1cm} (6)

The dynamical system (6) represents a hierarchic system of equations. The time evolution of one-site correlations $P_i(\nu)$ involve the two-site correlations $P_{ij}(\nu_1 \nu_2)$, and the
two-site correlations $P_{ij}(\nu_1\nu_2)$ involve the three-site correlations $P_{ijk}(\nu_1\nu_2\nu_3)$ and so on. In the next section we will use a truncation of mean field approximation of this hierarchic system to obtain an approximate solution for such system.

3 MEAN FIELD THEORY APPROXIMATION

The maximum principle entropy is very difficult to apply here since the solution of the equations of the first moments needs a knowledge about the second moments. Similarly the second moments needs a knowledge about the third order moments and so on. We start this section by discussing a simple mean-field approximation. In present section, we shall use the mean field approximation theory to derive a closed set of equations for one site correlations. In this approximation, the two-site correlations $P_{ij}(\nu_1\nu_2)$ can be expressed in terms of $P_i(\nu_1)$ and $P_j(\nu_2)$ [9]. This assumption lead to: the site mean-field approximation is obtained by neglecting all correlations between the states of the sites in the neighborhood of an updated site [9, 10] that is

$$P_{ij}(\nu_1\nu_2) \simeq P_i(\nu_1)P_j(\nu_2). \tag{7}$$

A truncation of higher order, that is of order $n > 1$, consists in writing any correlations in terms of correlations of order $n$ and less than $n$. Now consider a cluster of $m > n$ sites and denote it by $A$. Let $B$ and $C$ be the sets of points in the core and closure of cluster $A$, respectively. The set $B$ is said to be a core of the set $A$ if $B$ is the closure of its restriction to $B$. The set $C$ is said to be a closure of $A$ if $C$ is the smallest closed extension of $C$. The conditional probability $P(C|B)$ is approximated by the product [10, 11]

$$P(C|B) \simeq \prod_{i \in C} P(i|B). \tag{8}$$

Therefore, the probability of the cluster $A$ is given by

$$P(A) = P(B)P(C|B) = P(B)\prod_{i \in C} \frac{P(i,B)}{P(B)} \tag{9}$$

where $P(i,B)$ is the probability of the cluster of $n$ site formed by site $i$ and the sites of $B$.

Now, to obtain a closed set of equations for one site correlations $P_i(\nu)$, we will use the approximation $P_{ij}(\nu_1\nu_2) \simeq P_i(\nu_1)P_j(\nu_2)$. Thus, for one site correlations we have three correlations $P_i(0), P_i(1)$ and $P_i(2)$. The time evolution of these correlations can be obtained by using (6) in the form:
\[ \frac{d}{dt} P_i(0) = p_3 \sum_r P_i(2) - \frac{p_1 P_i(0)}{S} \sum_r P_{i+r}(1), \]
\[ \frac{d}{dt} P_i(1) = \frac{p_1 P_i(0)}{S} \sum_r P_{i+r}(1) - \frac{p_2}{S} \sum_r P_{i+r}(2), \]
\[ \frac{d}{dt} P_i(2) = \frac{p_2 P_i(1)}{S} \sum_r P_{i+r}(2) - p_3 \sum_r P_i(2). \]

Next, we will only seek for an homogeneous solution. Such this solution can be obtained by applying the conditions:
\[ \sum_r P_{i+r}(1) = SP_i(1), \quad \sum_r P_{i+r}(2) = SP_i(2). \] 

Therefore, the equations (10) reduced to:
\[ \frac{d}{dt} P_i(0) = p_3 P_i(2) - p_1 P_i(0)P_i(1), \]
\[ \frac{d}{dt} P_i(1) = p_1 P_i(0)P_i(1) - p_2 P_i(1)P_i(2), \]
\[ \frac{d}{dt} P_i(2) = p_2 P_i(1)P_i(2) - p_3 P_i(2). \] 

The first success of the model (12) can be shown using the following test. The prey is the only food source available to the predators. In the absence of prey then the grows of the predator population is given by
\[ P_i(2) = P_i^{(0)}(2) e^{-p_3t}, \quad P_i(0) = 1 - P_i^{(0)}(2) e^{-p_3t}. \] 

This means that the predator population decreases exponentially at rate \( p_3 \).

Also in the absence of predator (if \( P_i(2) = 0 \), the prey population has a logistic grow that is:
\[ P_i(1) = \left( P_i^{(0)}(1) \right)^{p_1} e^{p_1t}, \quad P_i(0) = \frac{1}{1 + \left( P_i^{(0)}(1) \right)^{p_1} e^{p_1t}} \] 

where \( P_i^{(0)}(1) \) and \( P_i^{(0)}(2) \) are the initial values of \( P_i(1) \) and \( P_i(2) \), respectively at the initial time.

From equations (14) we find that the density of the prey tends to one as \( t \) tends to infinity. This means that the densities of the vacant and predator sites will be tend to zero as time tends to infinity. Therefore all the sites of the lattice will be occupied by a prey.
4 CHAOS AND LINEAR STABILITY ANALYSIS

In this section, we will discuss the linear stability analysis and chaos of the system (12). The stationary states of this system will be derived. The stability conditions will be discussed. The stationary states of system (12) can be obtained by setting:

\[
\frac{dP_i(0)}{dt} = 0, \quad \frac{dP_i(1)}{dt} = 0, \quad \frac{dP_i(2)}{dt} = 0. \tag{15}
\]

By solving the system (15) one can get the stationary states of the system (12).

Now, one can easily prove that, by recalling the time, the process is found to be invariant under the transformation \( p_i \to \alpha p_i, (i = 1, 2, 3) \) where \( \alpha \) is a positive constant.

Now we will derived all stationary states of the system (12) and study their linear stability.

The stationary states of the system (12) are as follows:

1. The first stationary state is given by

\[
E_1 = (0, 1, 0). \tag{16}
\]

To investigate the local stability of the biologically feasible stationary states we will calculate the Jacobian matrix \( J \) and its eigenvalues at this state.

The Jacobian matrix of the system (15) at the stationary state \( E_1 \) is given by

\[
J_1 = \begin{bmatrix}
-p_1 & 0 & p_3 \\
p_1 & 0 & -p_2 \\
0 & 0 & p_2 - p_3
\end{bmatrix} \tag{17}
\]

The eigenvalues of the Jacobian matrix \( J_1 \) are given by:

\[
\lambda_{11} = 0, \quad \lambda_{12} = -p_1 < 0, \quad \lambda_{13} = p_2 - p_3. \tag{18}
\]

2. The second stationary state is given by

\[
E_2 = \left(\frac{p_2 - p_3}{p_1 + p_2}, \frac{p_3}{p_2}, \frac{p_1(p_2 - p_3)}{p_2(p_1 + p_2)}\right), \tag{19}
\]

\[
J_2 = \begin{bmatrix}
-p_1 p_3 & -p_1 \left(\frac{p_2 - p_3}{p_1 + p_2}\right) & p_3 \\
p_2 & 0 & -p_3 \\
0 & p_1 \left(\frac{p_2 - p_3}{p_1 + p_2}\right) & 0
\end{bmatrix} \tag{20}
\]
The eigenvalues of the Jacobian matrix $J_2$ are given

$$
\lambda_{21} = 0, \quad \lambda_{22} = \frac{p_1 p_3}{2 p_2} \left\{ -1 - \sqrt{1 - \frac{4 p_2}{p_1 p_3} (p_2 - p_3)} \right\} 
\lambda_{23} = \frac{p_1 p_3}{2 p_2^2} \left\{ -1 + \sqrt{1 - \frac{4 p_2}{p_1 p_3} (p_2 - p_3)} \right\}
$$

(21)

The linear stability indicate that, the first stationary state is unstable if $p_2 > p_3$ and the stability decision of this state needs more difficult stability analysis if $p_2 < p_3$. While the linear stability indicate that, the second stationary state is unstable if $p_2 < p_3$ and needs more difficult stability analysis for the stability decision if $p_2 > p_3$. Thus, we can easy conclude that the stochastic lattice gas for prey-predator system is unstable at least in one dimensions.

Figure (1). (a) the density of the empty sites, (b) the density of the prey sites, (c) the density of the predator sites, for the probabilities $p_1 = 0.1, p_2 = 0.6, p_3 = 0.3$ and the initial densities are $P_i(0)|_0 = 0.15, P_i(1)|_0 = 0.25, P_i(2)|_0 = 0.6$.

The study of limit cycles normally includes two aspects: one is the existence, stability and instability, number and relative positions of limit cycles, and the other is the creating and disappearing of limit cycles along with the varying of the parameters in the systems. For the exact number of limit cycles and their relative positions, the known results are not many because determining the number and positions of limit cycles is not easy.
Figure (2). Limit cycles of the empty sites and prey sites, the empty sites and predator sites and the prey and predator sites of the stochastic lattice gas of prey-predator system respectively, for the probabilities \( p_1 = 0.1, p_2 = 0.6, p_3 = 0.3 \) and the initial densities are \( P_1(0)|_0 = 0.15, P_1(1)|_0 = 0.25, P_1(2)|_0 = 0.6 \).

Figure (3). Three different attractors of the stochastic lattice gas of prey-predator system, for the probabilities \( p_1 = 0.08, 0.35, 0.03, p_2 = 0.85, 0.64, 0.9, p_3 = 0.07, 0.01, 0.07 \) and the initial densities are \( P_1(0)|_0 = 0.25, P_1(1)|_0 = 0.6, P_1(2)|_0 = 0.15 \).

5 CONTROL AND ESTIMATIONS OF UNKNOWN PROBABILITIES

In this section we present the dynamic estimators of the unknown probabilities \( p_1, p_2 \) and \( p_3 \) from the conditions of the asymptotic stability of the system (12) about its stationary states \( E_1 \) and \( E_2 \).

Here we consider a system corresponding the system (12) with unknown probabilities \( p_1, p_2 \) and \( p_3 \). Therefore, we assume that, the modified controlled stochastic lattice for prey-predator model with the estimators \( \hat{p}_1(t), \hat{p}_2(t) \) and \( \hat{p}_3(t) \) of the unknown probabilities \( p_1, p_2 \) and \( p_3 \) takes the form:

\[
\begin{align*}
\frac{d}{dt} P_1(0) &= \hat{p}_3(t) P_1(2) - \hat{p}_1(t) P_1(0) P_1(1) + u_0, \\
\frac{d}{dt} P_1(1) &= \hat{p}_1(t) P_1(0) P_1(1) - \hat{p}_2(t) P_1(1) P_1(2) + u_1, \\
\frac{d}{dt} P_1(2) &= \hat{p}_2(t) P_1(1) P_1(2) - \hat{p}_3(t) P_1(2) + u_2.
\end{align*}
\]  

(22)

where \( \hat{p}_i(t) \) are estimators of the unknown probabilities \( p_i, (i = 1, 2, 3) \) and \( u_0, u_1 \) and \( u_2 \) are the control inputs that will be suitably designed to derive the trajectory of the system specified by the steady-states \( E_1 \) and \( E_2 \) to any of these states of the uncontrolled system.

If \( u_j = 0, (j = 0, 1, 2) \) then the system (22) has an unstable special solution:

\[
P_i(0) = \bar{P}_i(0), P_i(1) = \bar{P}_i(1), P_i(2) = \bar{P}_i(2), \hat{p}_s(t) = p_s, (s = 1, 2, 3),
\]  

(23)
where \( \bar{P}_j (j, (j = 0, 1, 2) \) are the steady-states of the uncontrolled system (12) that should be stabilized by finding the control \( u_j, (j = 0, 1, 2) \) that causes the system (22) to follow an stable trajectory.

Therefore, the problem is equivalent to stabilize of the steady-states (23) and determining the update law of the estimators \( \hat{p}_s(t), (s = 1, 2, 3) \) of the system unknown probabilities with the help of the controllers \( u_j, (j = 0, 1, 2) \).

In what follows, we use the Liapunov stability technique to determine both of the controllers and updating rules of the unknown probabilities of the system (22).

Let us consider the Liapunov function

\[
2V(P_i(0), P_i(1), P_i(2), \hat{p}_1, \hat{p}_2, \hat{p}_3, t) = \sum_{j=0}^{2} \left( P_i(j) - \bar{P}_i(j) \right)^2 + \sum_{s=1}^{3} \left( \hat{p}_s - p_s \right)^2
\]

Differentiating the function Liapunov (24) along the trajectories of the system (22), we get

\[
\dot{V} = \begin{cases} \left( P_i(0) - \bar{P}_i(0) \right) \left[ \hat{p}_3(t)P_i(2) - \hat{p}_1(t)P_i(0)P_i(1) + u_0 \right] + \left( P_i(1) - \bar{P}_i(1) \right) \left[ \hat{p}_1(t)P_i(0)P_i(1) - \hat{p}_2(t)P_i(1)P_i(2) + u_1 \right] + \left( P_i(2) - \bar{P}_i(2) \right) \left[ \hat{p}_2(t)P_i(1)P_i(2) - \hat{p}_3(t)P_i(2) + u_2 \right] + \sum_{s=1}^{3} \left( \hat{p}_s(t) - p_s \right) \hat{p}_s(t) . \end{cases}
\]

Now for the suitable choice of both controllers and update law of the unknown probabilities:

\[
u_0 = p_1P_i(0)P_i(1) - p_3P_i(2) - k_0 \left( P_i(0) - \bar{P}_i(0) \right),
\]

\[
u_1 = p_2P_i(1)P_i(2) - p_1P_i(0)P_i(1) - k_1 \left( P_i(1) - \bar{P}_i(1) \right),\]

\[
u_2 = p_3P_i(2) - p_2P_i(1)P_i(2) - k_2 \left( P_i(2) - \bar{P}_i(2) \right).
\]

and

\[
\dot{\hat{p}}_1(t) = \left( P_i(0) - \bar{P}_i(0) \right) P_i(0)P_i(1) - \left( P_i(1) - \bar{P}_i(1) \right) P_i(0)P_i(1) - m_1 \left( \hat{p}_1 - p_1 \right),
\]

\[
\dot{\hat{p}}_2(t) = \left( P_i(1) - \bar{P}_i(1) \right) P_i(1)P_i(2) - \left( P_i(2) - \bar{P}_i(2) \right) P_i(1)P_i(2) - m_2 \left( \hat{p}_2 - p_2 \right),\]

\[
\dot{\hat{p}}_3(t) = \left( P_i(2) - \bar{P}_i(2) \right) P_i(2) - \left( P_i(0) - \bar{P}_i(0) \right) P_i(2) - m_3 \left( \hat{p}_3 - p_3 \right).
\]

where \( k_j, (j = 0, 1, 2) \) and \( m_s, (s = 1, 2, 3) \) are non-negative control gains constants.

The derivative of the Liapunov function (25) takes the form

\[
\dot{V} = - \sum_{j=0}^{2} k_j \left( P_i(j) - \bar{P}_i(j) \right)^2 - \sum_{s=1}^{3} m_s \left( \hat{p}_s - p_s \right)^2
\]
which is a negative definite if \( k_j > 0, m_s > 0 \) and a negative semi-definite if \( k_j = 0, m_s = 0 \).

This implies that, under the action of the controllers (26) and updating rules (27) of the unknown system probabilities, the solution (23) of the systems (22) and (28) is asymptotic stable in the Liapunov sense if \( k_j > 0, m_s > 0 \), and only stable but not necessarily asymptotic stable if \( k_j = 0, m_s = 0 \).

Since \( V \) is radially unbounded, it now follows that all trajectories are bounded. Also, since \( \dot{V} = 0 \) implies that \( P_j = \bar{P}_j \), which gives that \( \dot{P}_j = 0, u_j = 0, (j = 0, 1, 2) \) and \( \hat{p}_s = 0, (s = 1, 2, 3) \), turn on the Stochastic lattice prey-predator system (12) we find that the estimators \( \hat{p}_s \) are constants and hence \( \hat{p}_s = p_s \). It follows that from LaSalle’s theorem [11, 12] that the solution (24) of the closed-loop system (22) and (27) is globally asymptotically stable which completes the proof.

6 THE OPTIMAL CONTROL

The analysis of stability in section (4), and the existence of steady states of the system are unstable, give us an excuse to start studying the problem of the optimal control of the system and to propose a mechanism to control the perfect behavior during a specific time period \( T \). In this section we will discuss the problem of optimal control for the stochastic lattice gas for prey-predator model about its steady states, we will use Pontryagin minimum principle to find the best possible control and to derive the optimal control inputs which investigating the ideal behavior of the system with respect to a proposed criterion. Let us select the following cost function as a selected measure for the optimality

\[
J = \frac{1}{2} \int_0^T \left[ \sum_{j=0}^2 \alpha_j \left( P_i(j) - \bar{P}_i(j) \right)^2 + \beta_j (u_j - \bar{u}_j)^2 \right] dt
\]

subject to:

1. The controlled system of (12), which can be written as in the following form

\[
\begin{align*}
\frac{d}{dt} P_i(0) &= p_3 P_i(2) - p_1 P_i(0) P_i(1) + u_0 \\
\frac{d}{dt} P_i(1) &= p_1 P_i(0) P_i(1) - p_2 P_i(1) P_i(2) + u_1 \\
\frac{d}{dt} P_i(2) &= p_2 P_i(1) P_i(2) - p_3 P_i(2) + u_2 
\end{align*}
\]

2. The initial conditions

\[ P_i(j)|_{t=0} = P_{j0}, \quad j = (0, 1, 2). \]  

3. The terminal conditions

\[ P_i(j)|_{T} = \bar{P}_i(j), \quad j = (0, 1, 2). \]
Awad El-Gohary and Saba Alwan

\[ \alpha_j, \beta_j \] are the controlled inputs which will be determined from the conditions of the optimality of the system (30) about its steady-states \( \bar{P}_i(j) \) with respect to the cost function (29), and \( \alpha_j, \beta_j \) are positive constants. It is clear that the cost function (29) is a positive definite and has a zero solution only at \( P_i(j) = \bar{P}_i(j) \) and \( u_j = \bar{u}_j, i = 0, 1, 2, \)

The first step to apply the procedure of Pontryagin minimum principle, replace the integral of the cost function (29) by an addition state variable \( P^*_i(t) \) such that

\[ \dot{P}^*_i(t) = \frac{1}{2} \left[ \sum_{j=0}^{2} \alpha_j \left( P_i(j) - \bar{P}_i(j) \right)^2 + \beta_j \left( u_j - \bar{u}_j \right)^2 \right]. \] (33)

Next, introduced the so-called co-state variables which related to the state variables in the system (12), that our symbols are \( \lambda^*, \lambda_j, (j = 0, 1, 2) \), thus the Hamiltonian function is given by

\[ H = \lambda^* P^* + \sum_{j=0}^{2} \lambda_j P_j \] (34)

By substituting from (30) and (33) in (34) we will get

\[ H = \frac{\lambda^*}{2} \left[ \sum_{j=0}^{2} \alpha_j \left( P_i(j) - \bar{P}_i(j) \right)^2 + \beta_j \left( u_j - \bar{u}_j \right)^2 \right] + \lambda_0 \left[ p_3 P_i(2) - p_1 P_i(0) P_i(1) + u_0 \right] + \lambda_1 \left[ p_1 P_i(0) P_i(1) - p_2 P_i(1) P_i(2) + u_1 \right] + \lambda_2 \left[ p_2 P_i(1) P_i(2) - p_3 P_i(2) + u_2 \right] \] (35)

The Hamilton equations are

\[ \frac{\partial \lambda^*}{\partial t} = - \frac{\partial H}{\partial P^*} = 0, \quad \frac{\partial \lambda_j}{\partial t} = - \frac{\partial H}{\partial P_j}, \quad (j = 0, 1, 2). \] (36)

From equation (36), it is clear that \( \lambda^* \) is a constant value, we can chose \( \lambda^* = -1 \). From (35) and (36), we get the co-state differential equations as the following forms

\[ \begin{align*}
\dot{\lambda}_0 &= \alpha_0 \left( P_i(0) - \bar{P}_i(0) \right) + p_1 P_i(1) \lambda_0 - p_1 P_i(1) \lambda_1, \\
\dot{\lambda}_1 &= \alpha_1 \left( P_i(1) - \bar{P}_i(1) \right) + p_1 P_i(0) \lambda_0 - \left( p_1 P_i(0) - p_2 P_i(2) \right) \lambda_1 - p_2 P_i(2) \lambda_2, \\
\dot{\lambda}_2 &= \alpha_2 \left( P_i(2) - \bar{P}_i(2) \right) - p_3 \lambda_0 + p_2 P_i(1) \lambda_1 - \left( p_2 P_i(1) - p_3 \right) \lambda_2.
\end{align*} \] (37)

The optimal control inputs are determined by minimize the Hamilton function with respect to \( u_j \), through the following conditions

\[ \frac{\partial H}{\partial u_j} = 0, \quad j = 0, 1, 2 \] (38)
thus

\[ u_j = \bar{u}_j + \frac{\lambda_j}{\beta_j}, \quad (j = 0, 1, 2). \]  

(39)

Substituting (39) in the controlled system (30), in addition, the addition state variable (33) and the co-states equations (37), we get a non-linear system of ordinary differential equations. The numerical solution of this system is shown in Figure (4) below.

Figure (4). In (a),(b) and (c), we display the optimal controlled densities of the empty sites, prey sites and predator sites respectively, for the probabilities \( p_1 = 4, p_2 = 0.45, p_3 = 0.15 \), and the constants \( \alpha_0 = 0.2, \alpha_1 = 1.3, \alpha_2 = 0.5, \beta_0 = 5, \beta_1 = 2, \beta_2 = 1 \) at the initial densities \( P_i(0)|_0 = 0.35, P_i(1)|_0 = 0.4, P_i(2)|_0 = 0.25 \).

Next we are going to look the numerical integration of the nonlinear systems (22) and (27) using runge-Kutta numerical method with step size 0.01.

7 NUMERICAL SOLUTION AND ANALYSIS

Since the controlled system is nonlinear and has no analytical solution, we will discuss its numerical solution for fixed values of the system parameters. This section presents the numerical solution of the controlled nonlinear system of the stochastic lattice gas prey-predator model and estimators of the system unknown probabilities to show how the control for this system is possible. Also, numerical examples for controlled stochastic lattice gas prey-predator model were carried out for various probabilities values and different initial densities.

For illustration purpose, we display graphically the numerical solutions of the stochastic lattice gas prey-predator with unknown system probabilities.

Also the following figures display the stabilized behavior of the stochastic lattice gas prey-predator system densities about the unstable stationary states and the dynamic estimators of the system unknown probabilities.
Figure (5). The densities of the stochastic lattice gas of prey-predator model and the estimators of the unknown probabilities of the system which are (a) the density of the empty sites, (b) the density of the prey sites, (c) the density of the predator sites, (d) the estimator of the unknown probability $p_1$, (e) the estimator of the unknown probability $p_2$ and (f) the estimator of the unknown probability $p_3$, respectively, for the system probabilities, parameters and initial densities are given by

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$k_0$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$\hat{p}_1(0)$</th>
<th>$\hat{p}_2(0)$</th>
<th>$\hat{p}_3(0)$</th>
<th>$P_i(0)\mid_0$</th>
<th>$P_i(1)\mid_0$</th>
<th>$P_i(2)\mid_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.35</td>
<td>0.25</td>
<td>5.5</td>
<td>18.5</td>
<td>20.3</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>0.4</td>
<td>0.25</td>
<td>0.35</td>
<td>0.2</td>
<td>0.35</td>
<td>0.45</td>
</tr>
</tbody>
</table>
Figure (6). The densities of the stochastic lattice gas of prey-predator model and the estimators of the unknown probabilities of the system which are (a) the density of the empty sites, (b) the density of the prey sites, (c) the density of the predator sites, (d) the estimator of the unknown probability $p_1$, (e) the estimator of the unknown probability $p_2$, (f) the estimator of the unknown probability $p_3$, respectively, for the system probabilities, parameters and initial densities are given by

| $p_1$ | $p_2$ | $p_3$ | $m_1$ | $m_2$ | $m_3$ | $k_0$ | $k_1$ | $k_2$ | $\hat{p}_1(0)$ | $\hat{p}_2(0)$ | $\hat{p}_3(0)$ | $P_1(0)|_0$ | $P_1(1)|_0$ | $P_1(2)|_0$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0.3   | 0.4   | 0.3   | 8.5   | 7.5   | 15    | 0.25  | 4.45  | 5.3   | 0.4           | 0.25          | 0.35          | 0.2           | 0.35          | 0.45          |

Note that all densities are exponentially stable. Further also all the densities perturbations go to zero through positive values only.

8 CONCLUSION

The Markov process is adopted to formulate the dynamics of two animal species prey-predator population. The master equation is used to obtain the time evolution of the probability of the state of the system at time $t$. The mean field approximation is used to derive the correlations of one site. The stability analysis and chaos of the stochastic lattice gas of prey-predator are discussed. The control inputs and updating rules of the unknown probabilities are derived.

References


