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A Novel Numerical Corrective Technique to Mass Spring Systems

Hadi Mohammadi [1], Wankei Wan [1,2,*]

[1]Biomedical engineering graduate program, the University of Western Ontario, London, Ontario, Canada

[1,2]Department of chemical and biochemical engineering, The University of Western Ontario, London, Ontario, Canada

This paper presents a numerical corrective technique that allows control of nonlinearity in a mass-spring system (MSS) independent of its spring constants or system topology. The governing equations of MSS in the form of ordinary differential equations or a regular function companied by any boundary or initial condition as known constraints are employed to modify the results. A least-squares algorithm coupled with the finite difference method is used to discretize the basic residual function implemented in this corrective technique. This numerical solution is applicable to both static and dynamic MSS. This technique is easy to implement and has similar accuracy as the equivalent finite element method (FEM) solution to the same system with the solutions are obtained in a fraction of the CPU time required for FEM. The proposed technique can also be used to smooth solutions from other methods such as finite element method or boundary element method (BEM).

Keywords: *Mass-spring system, finite difference method, least square algorithm, finite element method, soft tissue mechanics*

1. INTRODUCTION

Mass-spring systems (MSS) have been extensively used in mechanical simulation of soft tissues over the past fifteen years [1, 2]. They are also used in modeling of deformable objects for facial animation, animation of artificial animals, cloth modeling and recently in surgical simulation [3-14]. Although, MSS are known as non-time-consuming solvers, it suffers from the lack of accuracy as compared to other numerical techniques such as the finite element method (FEM) [24]. To address this shortcoming, numerical corrective techniques have been developed for the MSS. The approaches used include the inverse dynamics procedure to eliminate super elongation of the springs, the implicit integration method to take large time steps and the heuristic method of handling post-buckling instability for robust modeling [15-18].

Both isotropic and anisotropic elastic materials can be found among the objects to animate. For instance, most soft tissues are strongly anisotropic, due to their fiber structure and/or composite nature. One of the main limitations with the mass-spring systems is that neither isotropic nor anisotropic materials can be generated and controlled easily [19-23].

In this study a new numerical corrective technique to MSS is developed by which the accuracy of MSS results is enhanced by utilizing the material properties and the boundary and/or initial conditions of the system. FORTRAN custom code is used to implement the corrected MSS technique in this study. Application of this new corrective technique is illustrated by three examples. These include the force-displacement of a non-linear spring, a highly nonlinear displacement of a nonlinear spring and a highly non-linear stress-strain relationship of a polymer hydrogel, a biomaterial that mimics cardiovascular tissue properties [25].

correspondence author: *E-mail: wkwan@eng.uwo.ca*

2. METHOD

2.1. Governing Equation for MSS

The internal force of a MSS which is due to the changes of spring length and velocity takes the form of:

$$F_{\text{int}} = [K.(|x| - x_0) + D.\frac{\vec{x} \bullet \vec{v}}{|x|}].\frac{\vec{x}}{|x|} \tag{1}$$

Where $\vec{x}$ and $\vec{v}$ are position and velocity vectors between the two ends of the springs and $|x|$ and x_0 are the current and initial length of the spring, respectively. K and D are linear constants of the MSS elements representing the spring and dashpot constants.

Equation (1) can be represented in the following nonlinear form:

$$F_{\text{int}} = K(\cdot) * x_K + D(\cdot) * x_D \tag{2}$$

$$x_K = (|x| - x_0).\frac{\vec{x}}{|x|} \tag{3}$$

$$x_D = \frac{\vec{x} \bullet \vec{v}}{|x|}.\frac{\vec{x}}{|x|} \tag{4}$$

Equations (3) and (4) are defined as directional vectors and the nonlinear functions $K(\cdot)$ and $D(\cdot)$ represent spring stiffness and damping function. Newton's equation of motion can be rearranged in the following form:

$$F_{\text{int}}(k+1) = \frac{M}{(\Delta t)^2}[\Delta x(k) - \Delta x(k-1)] - F_{ext} \tag{5}$$

Where $\Delta x(k)$ and $\Delta x(k-1)$ are position changes and F_{ext} refers to the external force as boundary conditions. Given the input and output data, nonlinear functions $K(\cdot)$ and $D(\cdot)$ can be approximated using, for instance, ordinary differential equation (ODE) numerical solvers or neural network approaches.

2.2. Corrective Technique to MSS

MSS is governed by a set of ODE as mentioned above. There is also additional information in the form of boundary or initial conditions that can also be formulated. These additional information companied together with the governing equations can be expressed as:

$$R(u_i, \zeta_j, \eta_k) = \sum_i \frac{1}{2}(\bar{u}_i - u_i)^2 + \sum_j \zeta_j F(u) + \sum_k \eta_k T(u) \tag{6}$$

Where R is the residual function to the MSS, j is the number of internal points of the domain to be corrected and ζ_j and η_k are the Lagrange multipliers. The certain quality to be corrected is displacement $\bar{u}_i(x)$ available in i points. The corrected values of the $\bar{u}_i(x)$ are considered as $u_i(x)$. The governing equation of the MSS is $F(u) = 0$ where order of F is either 0 (static) and 2 (time-dependent). Also the additional information from the system such as boundary and initial conditions is $T(u_i) = 0$ and T is a known function of u. Equation 6 is solved for u_i by minimizing the residual function R.

It should be noted i and j have different ranges, i refers to all points of the domain including boundary points and additional constraints, j refers only to the internal points of the domain and k is the number of constraints where $T(u_k)$ is defined. The residual function R should be minimized on the entire domain by taking the first derivative of the function R with respect to u as such:

$$\frac{\partial R}{\partial u} = \sum_i (\bar{u} - u) + \sum_j \zeta_j \frac{\partial F(u)}{\partial u} + \sum_k \eta_k \frac{\partial T(u)}{\partial u} = 0 \tag{7}$$

Also the first derivative of the function R with respect to ζ_j and η_k delivers the following complementary set of equations:

$$F(u) = 0 \tag{8}$$

$$T(u) = 0 \tag{9}$$

The function F can be replaced by the equivalent finite difference forms calculated for the points j. Equation (6) and (7) are now reduced to a set of algebraic equations, with u being the only variable. These sets of equations can be solved by any relevant numerical technique as they are in the form of linear or non-linear algebraic equations. The order of function F which is called r defines the number of connected points, which are to be solved in each step. When order of F is 2 (time-dependent), three consecutive values: u_{i-1}, u_i and u_{i+1} are defined and consequently $j = i - 2$ as $j = i - r$, because equation (8) and (9) can only be written for the internal points of the domain.

Equations (6) can also be written in matrix form, as such:

$$R = \frac{1}{2}(\bar{u} - u)^2 + (Fu)^T \psi \tag{10}$$

where:

$$\psi = \begin{Bmatrix} \zeta \\ \eta \end{Bmatrix} \tag{11}$$

and eventually the matrix form of the error function R takes the form of:

$$\begin{bmatrix} I & B^T \\ B & 0 \end{bmatrix} \begin{Bmatrix} u \\ \zeta \\ \eta \end{Bmatrix} = \begin{Bmatrix} \bar{u} \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

where I is the unit diagonal matrix and B is the matrix resulting from the additional equations (8) and (9), which is of the order $i + j + k$. For ease of calculations, equation (10) can be broken down to two lower systems of the order i and $j + k$. The new partitioning of equation (10) takes the matrix form, as such:

$$Iu + B^T \psi = \bar{u} \tag{13}$$

$$Bu = 0 \tag{14}$$

Equation (13) can be multiplied by B:

$$BIu + BB^T \psi = B\bar{u} \tag{15}$$

Using equation (15) the Lagrange multipliers matrix can be computed as follows:

$$\psi = SB\bar{u} \tag{16}$$

and,

$$u = Y\bar{u} \tag{17}$$

where Y is defined as:

$$Y = I - B^T SB \tag{18}$$

which is considered the corrective matrix where matrix S is defined as:

$$S = (BB^T)^{-1} \tag{19}$$

The order of matrix S is $(j + k) \times (j + k)$ and the correction matrix Y is essentially a square matrix of order

$i \times i$. The correction matrix is unique for a given MSS and once it is solved it can be retained as a modulus for the MSS. It can also be proven that the Lagrange multipliers ζ_j and η_k contributes to correction of the results and therefore, they can be considered as correction factors as well.

Now, the set of equations (6) to (19) can appropriately interpret the MSS and can be solved for u and the multipliers ζ_j and η_k. The solution is unique since it refers only to the corresponding set of governing equations and selected constraints of the MSS.

3. RESULTS

3.1. Example 1 – Time Independent Application (A non-linear MSS)

Part I: The force-displacement of a nonlinear spring ($f = ku^2$, where f, u and k refer to force, displacement and the spring constant respectively) under large deformation is considered. A linear spring is used for uncorrected results and is modified through the proposed technique. For this example, the governing equation is not a differential equation, and however, it is a second order polynomial. The functions F and T, take the form of:

$$F(u) = f - ku^2 = 0 \tag{20}$$

$$T(u) = 0 \tag{21}$$

where k is the spring constant which is set to 1 for this example. The minimized residual function can be written as such:

$$\frac{\partial R}{\partial u} = (\bar{u} - u) + 2ku \tag{22}$$

where the $B_{m\times n}$ matrix is written as ($m = n = 2$):

$$B = \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix} u \tag{23}$$

Given B matrix (equation 23), equation 13 is solved for the corrective matrix Y. The Y matrix is now available for the uncorrected results. The exact solutions to this simple model for the linear and the used nonlinear spring constant are in the form of linear and parabolic functions, respectively. The uncorrected, exact and modified results in company with the Lagrange multipliers are shown in Figure 1.

The trial function as uncorrected results for this simple system is linear. The corrected results are close to the result of the FEM simulation, with a deviation of less then 10%.

Part II: The solution to the Example 1 can be extended to a relatively complicated MSS. A rectangular polymeric plate (5 × 15 cm^2) made of 10% polyvinyl alcohol (PVA) hydrogel is considered. The mechanical

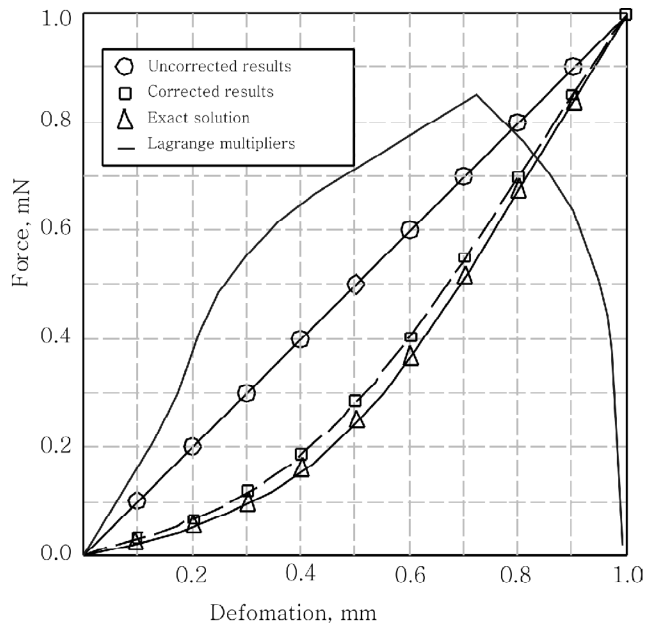

Figure 1: Modeling of a parabolic spring with a linear spring in a MSS - the exact solution and modified results are in the same range with 10% difference

properties of the sample in the form of stress-strain relationship has been determined previously using a uniaxial tensile machine and are expressed as follows [25, 27, 28]:

$$\sigma = -0.05923 + 0.0611e^{2.2347\varepsilon} \tag{24}$$

where s and μ are stress and strain, respectively and the Poisson's ration of the used PVA sample was assumed to be 0.5 (incompressible).

The stress-strain behavior shows a highly non-linear behaviour similar to that of the porcine aortic root [25]. An iterative modified Newton-Raphson method to MSS and a validated finite element solver for soft material considering material nonlinearity has been reported previously [26]. In this finite element code hyperelastic isotropic elements have been employed along with a classical Mooney-Rivlin soft material model. The corrective technique to the results from MSS has also been calculated. The results of uncorrected MSS, modified MSS and nonlinear FEM are shown in Figure 2.

The modified MSS gives results that are close to FEM, with a percentage error of less then 5%. The corrected mass-spring approach is more accurate here than in the solution provided in part 1 as the trial function in this example is already nonlinear. Also in this example, the accuracy approaches the existing finite element solution at 1/20 of the CPU time on a Pentium IV with a CPU speed of 2.4 GHz and 512 MB RAM.

3.2. Example 2 – Time-dependent Application

A vibrating spring with a linear spring constant is considered. The exact results, which are in the form of a

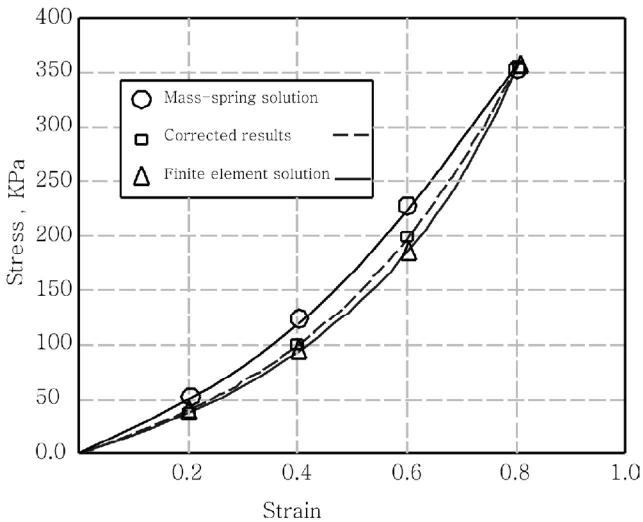

Figure 2: (Right) the nonlinear FEM solution, (Left) the regular MSS solution for a 10%PVA rectangular pate (5 x 15 cm²) for a deformation starting form 15% to 100%

sinusoidal function, are randomly manipulated by a factor of more than 20% to examine the robustness of the proposed numerical technique for a time-dependent example. The equations (8) and (9) take the form of:

$$F(u) = \ddot{u} - ku = 0 \tag{25}$$

$$T(u) = 0 \tag{26}$$

where ü is the second derivative of the u with respect to time and k is the spring constant. The minimized residual function takes the form of:

$$\frac{\partial R}{\partial u} = (\bar{u} - u) + \frac{\partial}{\partial u}(\ddot{u} - ku) = 0 \tag{27}$$

The expansion of the equation (26) together with the implementation of finite difference to the equation (25) gives the following set of equations:

$$(\bar{u}_0 - u_0) + \hat{\zeta}_1 = 0 \tag{28}$$

$$(\bar{u}_1 - u_1) + (-2 + \hat{k})\hat{\zeta}_1 + \hat{\zeta}_2 = 0 \tag{29}$$

$$\dots$$

$$(\bar{u}_j - u_j) + \hat{\zeta}_{j-1} + (-2 + \hat{k})\hat{\zeta}_j + \hat{\zeta}_{j+1} = 0 \tag{30}$$

$$\dots$$

$$(u_n - u_n) + \hat{\zeta}_{n-1} = 0 \tag{31}$$

where $\hat{\zeta}_i = \frac{\hat{\zeta}}{(\Delta t)^2}$ and $\hat{k} = k_i(\Delta t)^2$ and the $B_{m\times n}$ matrix for this case takes the form of (m = 6, n = 4):

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2+k & 1 & 0 & 0 \\ 1 & -2+k & 1 & 0 \\ 0 & 1 & -2+k & 1 \\ 0 & 0 & 1 & -2+k \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{32}$$

The Y matrix is now calculated for the current MSS. The results of this solution and the exact solution for $k=1$ are presented in Figure 3 including the Lagrange multipliers.

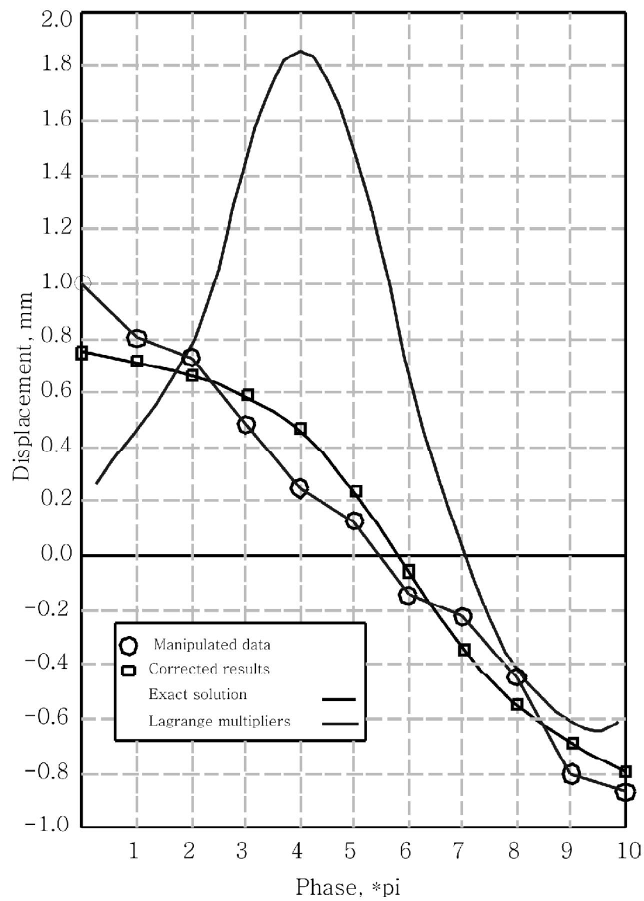

Figure 3: Motion of the vibrating spring for *k=1*, manipulated and modified results

Although the manipulated data are designed to hold more than 20% error, the discrepancy between corrected and exact results are smaller.

The proposed technique can also be implemented on regressing other parameters of a MSS (for instance, for the spring constant used in example 2). For this case, the residual function (R) is the same except a new variable (k) is added up. An additional equation can be obtained where the residual function, R, is minimized with respect to k. This equation takes the form of: $\sum_j u\psi = 0$, which is non-linear and can be solved iteratively using the Newton-Raphson algorithm or similar approaches. The calculated stiffness of the spring was $k = 1.2$ for uncorrected results.

4. CONCLUSION

A corrective numerical technique to modify the MSS results independent of the system topology or spring constants has been developed. Illustrated examples of both static and dynamic systems indicate the successful implementation of this approach to MSS and corrections of more than 80% is achievable. In the dynamic example of the vibrating spring, the corrective technique was able to improve the results to within 10% of the FEM results. In summery, the corrective numerical technique presented here is of general applicability to both static and dynamic application of MSS. Since the proposed technique is computationally efficient and easy to implement, it can also be employed to smooth solutions obtained using other numerical techniques such as finite element, boundary element or finite difference methods to enhance the precision of the solutions.

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REFERENCES

[1] Miller GSP, The motion dynamics of snakes and worms. In SIGGRAPH '88 Conference Proceedings, Addison Wesley, 169–178, 1988.

[2] Terzopoulos D., Platt J., Barr A. and Fleischer K., Elastically deformable models, *Comput Graph*, **21**(4): 205–214, 1987.

[3] Lee Y, Terzopoulos D and Waltes K, Realistic modeling for facial animation, In: Proceedings of the 22nd annual conference on computer graphics and interactive techniques, 55–62, 1995.

[4] Platt S. and Badler N., Animating facial expressions. *Comput Graph*, **15**(3): 245–252, 1981.

[5] Terzopoulos D. and Waters K., Physically-based facial modeling, analysis and animation, *J. Visualizat Comput Animat*, 1:73–80, 1990.

[6] Miller GSP, The motion dynamics of snakes and worms. Comput Graph, **22**(4): 169–177, 1988.

[7] Tu X and Terzopoulos D, Artifical fishes: physics, locomotion, perception, behaviour, In: Proceedings of the 21st annual conference on computer graphics and interactive techniques, 43–50, 1994.

[8] Breen DE, House DH and Wozny MJ, Predicting the drape of woven cloth using interacting particles, In: Proceedings of ACM SIGGRAPH, 365–372, 1994.

[9] Eberhardt B, Weber A and Strasser W, A fast, flexible particle-system model for cloth draping, *IEEE Comput Graph Appl.*, 16(5): 52–59, 1996.

[10] Carignan M, Yang Y, Thalmann MN and Thalmann D, Dressing animated synthetic actors with complex deformable clothes, Computer Graphics (Proceedings of SIGGRAPH), 99–104, 1992.

[11] Choi K and Ko H, Stable but responsive cloth, *ACM Trans Graph*, **21**(3): 604–611, 2002.

[12] Vassilev T, Spanlang B and Chrysanthou Y, Fast cloth animation on walking avatars, In: Computer Graphics Forum (Proceedings of Eurographics'01), **20**(3): 260–267, 2001.

[13] Volino P, Courchesne M, Thalmann NM, Versatile and efficient techniques for simulating cloth and other deformable objects, In: Computer Graphics (Proceedings of SIGGRAPH), 137–144, 1995.

[14] Brown J, Montgomery K, Latombe JC and Stephanides M, A microsurgery simulation system, In: Proceedings of the 4th international conference on medical image computing and computer-assisted intervention, 137–144, 2001.

[15] Gelder AV, Approximate simulation of elastic membranes by triangulated spring meshes. J Graph Tools, **3**(2): 21–42, 1998.

[16] Baraff D and Witkin A, Large steps in cloth simulation, In: Proceedings of ACM SIGGRAPH, 43–54, 1998.

[17] Choi K and Ko H, Stable but responsive cloth, *ACM Trans Graph*, **21**(3): 604–611, 2002.

[18] Luciani A, Jimenez S, J. Florens L, Cadoz C, and Raoult O, Computational physics: A modeller simulator for animated physical objects, In Proceedings of Eurographics '91, 425–436, 1991.

[19] Bhat KS, Twigg CD, Hodgins JK, Khosla PK, Popovic Z and Seitz SM, Estimation cloth simulation parameters from video, In: Proceedings of ACM SIGGRAPH/Eurographics symposium on computer animation, 37–51, 2003.

[20] Gelder AV, Approximate simulation of elastic membranes by triangulated spring meshes. *J Graph Tools*, **3**(2): 21–42, 1998.

[21] Jojic N and Huang TS, Estimating cloth draping parameters from range data, In: International workshop on synthetic-natural hybrid coding and 3-D imaging, 73–76, 1997.

[22] Maciel A, Boulic R and Thalmann D, Deformable tissue parameterized by properties of real biological tissue, In: Proceedings of the international symposium on surgery simulation and soft tissue modeling, 74–87, 2003.

[23] Terzopoulos D, Platt J, Barr A and Fleischer K, Heating and melting deformable models (from goop to glop), In: Graphics Interface'89, 219–226, 1989.

[24] Provot X, Deformation constraints in a mass-spring model to describe rigid cloth behaviour, In: Proceedings of Graphics Interface, 147–154, 1995.

[25] Millon LE, Mohammadi H, Wan WK. Anisotropic polyvinyl alcohol for cardiovascular applications, Journal of biomedical material research B; Applied biomaterials, 2006; 79B: 305-311.

[26] Mohammadi H, Millon LE, Wan WK. Simulation and design improvement of a bacterial Cellulose–polyvinyl alcohol nanocomposite mechanical aortic heart valve prosthesis. To be submitted.

[27] B Calvo, E Peña, MA Martinez, M Doblaré, Computational modeling of ligaments at non-physiological situations, *Int. Journal for Comp Visison & Biomechanics*, in press.

[28] M. Boukerrou, C. Rubod, M. Cosson, M. Brieu, M. Vermandel, P. Dubois, A 3-D mechanical model for the pelvic surgery, Proceedings of the Int Symposium on Computational Modelling of Objects Represented in Images: Fundamentals, Methods and Applications, JMRS Tavares & RM Natal Jorge (Eds), Coimbra, Portugal, 20-21 2007, 107-110.