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Synchronization Design of Coupled Chaotic Systems using Fuzzy Approach

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Abstract: *In this paper, we focus on the synchronization design of coupled chaotic systems with various coupling congurations. To handle the highly coupled nonlinear system, the T-S fuzzy model approach is adopted to model the overall system. First, the fuzzy representation for general coupled systems is shown. Then, we discuss how to design the inner coupling matrix for a given configuration matrix. Based on the T-S fuzzy model approach, we derive the synchronization conditions in terms of linear matrix inequalities (LMIs) according to Lyapunov stability theory. Considering the well-known Lorenz attractors as an example for coupled networks, we find that the LMIs are feasible for the fully connected network, a regular network, or a small-world network. Finally, nice synchronization behavior is demonstrated via numerical simulation.*

Keywords: *Synchronization, T-S fuzzy model, linear matrix inequalities, complex network.*

I. INTRODUTION

Synchronization is an interesting phenomenon, which occurs in many situations such as fire-flies light up synchronously in the dark; the claps of audiences are spontaneously synchronized in a concert hall; and so on. The synchronization is the process where two or many systems interact with each other and come to move together. The technique due to synchronization finds various applications. There are many benets of having synchronization phenomena in engineering applications such as secure communications [1]-[2] and harmonic oscillation generation [3]. In secure communication, the synchronization indicates the capability of recovering the transmitted message from the masking signals. Here, our question is how to effectively synchronize coupled chaotic systems for dierent type of coupling configuration.

In 1998, Watts and Strogatz introduced networks of coupled dynamical systems, namely small-world networks, to model biological oscillators, Josephson junction arrays, excitable media, neural networks, spatial games, genetic control networks and many other self-organizing systems [4]. Compared to the regular networks and the random networks, the small-world networks display their own importance. Small-world networks have intermediate connectivity properties but exhibit a high degree of clustering as in regular networks and small average distance between vertices as in random networks [4]-[11]. The well-known small-world phenomenon is six degrees of separation which means that something relates one person to another person by at most six mediums [4]. In terms of the coupled complex systems, the research issues in [12]-[15] focus on Turing pattern. In [16], the stability criterion is depicted numerically in a set of coupled Rossler-like oscillations. The threshold of the coupling strength of a scale-free dynamical network is shown in [17]. For any given coupling strength, it is shown in [18] that the small-world networks will synchronize when the numbers of nodes increase large enough. In [19], it indicates that the synchronizability can be determined by an associated feedback system, where the sensitive edge and the robust edge are also introduced.

Chaotic signals are typically broadband, noise-like, and dicult to predict. This property leads to some interesting communications applications. For example, they can be used in various contexts for masking information-bearing waveforms. The chaotic signal masking technique introduced in [20] appears to be a useful approach to secure communications. The embedded message can be retrieved from the information-bearing chaotic signals if the synchronization between transmitter and receivers are achieved. However, we notice that the treatment of the nonlinearities for complex coupled chaotic systems is always a challenge issue. Nowadays, most researches on synchronization simplify the nonlinear systems into linear ones based on approximated linearization at the equilibrium points [17]-[19]. These methods may lead to ambiguous results if the concerning signals are not close to the equilibrium points. The T-S fuzzy approach [21]-[23] gives a nice solution for synchronization design of coupled chaotic systems. From the investigation of many well-known continuous-time and discrete-time chaotic systems, we find that an exact model without approximation error can be constructed for each well-known chaotic system [1]-[2]. Moreover, accompanied by linear matrix inequality (LMI) technique [24], the analysis and design for the synchronization can be transformed into an easily solvable problem.

The rest of the paper is organized as follows. The general complex network model will be described in the next section. In Section III, the proposed T-S fuzzy model method is illustrated, where the way to design the synchronization and the stability analysis are given. Numerical simulation results are shown in Section IV. Finally, some conclusions with summary are made in the Section V.

II. GENERAL COMPLEX NETWORK MODELS

We are now discussing the synchronization of coupled chaotic system. At first, suppose that there are N independent nonlinear dynamic systems which are with the same form but exist individually. The N independent nonlinear systems are listed as the following equations:

$$\dot{x}_i(t) = f(x_i(t)),\ i = 1, 2, ..., N \tag{1}$$

where $x_i \in \mathbb{R}^n$ Rn denotes that state variables for ith system. Different coupling configuration can affect the synchronization behavior of coupled dynamical systems. The coupling configuration can be described by graph theory. Informally, a graph is simply a collection of nodes (or called vertices) and connections (or called edges) together with a rule about how the nodes are linked to one another with the connections. From the small world theory [4]-[11] and the studies of coupled complex systems [17]-[18], we know that the coupled systems will be synchronous if appropriate connections among the nodes exist. In this work, the coupling connection between from node i to node j will be denoted by c_{ij}. If there is no connection between these two nodes, then $c_{ij} = 0$, or else $c_{ij} = 1$. Hence, the structure topology of the coupled systems can be described the matrix $C = (c_{ij})_{N\times N}$, which is called the coupling conguration matrix. Each diagonal entry of C is dened by

$$c_{ii}(t) = -\sum_{j=1,\ j\neq i}^{N} c_{ij}(t),\ i = 1, 2, ..., N \tag{2}$$

which means that the matrix C is with zero row sum, i.e.

$$\sum_{j=1}^{N} c_{ij}(t) = 0,\ i = 1, 2, ..., N \tag{3}$$

A matrix satises (3) is called a diffusively coupled matrix [18]. In each node, the coupling signals fed to and from other nodes will be denoted by a matrix C, which will be called the inner coupled matrix. We assume the inner coupled matrix H is constant in this work. When the N independent nodes in (1) are coupled with each other, the coupled network is written as the following form:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij}(t) H x_j(t) \tag{4}$$

The synchronization of the coupled dynamical network means that the state variables will be all the same; that is, $x_1(t) = x_2(t) = \cdots = x_N(t) = x_0(t)$ as $t \to \infty$, where $x_0(t)$ is the solution of (1), i.e., $\dot{x}_0(t) = f(x_0(t))$. To further development, we let the synchronization error be dened as follows:

$$e_i(t) = x_i(t) - x_0(t),\ i = 1, 2, \ldots, N \tag{5}$$

After substituting (4) into (5), and using the property (3), we obtain the error system for synchronization as follows:

$$\dot{e}_i(t) = f(x_i(t)) - f(x_0(t)) + \sum_{j=1}^{N} c_{ij}(t) H x_j(t) - \sum_{j=1}^{N} c_{ij}(t) H x_0(t)$$

$$= f(x_i(t)) - f(x_0(t)) + \sum_{j=1}^{N} c_{ij}(t) H\, e_j(t),\ i = 1, 2, ..., N \tag{6}$$

III. SYNCHRONIZATION ON T-S FUZZY MODEL

As mentioned in introduction, most well-known chaotic systems can be exactly represented by Takagi-Sugeno fuzzy linear model. In light of this, IF-THEN rules will be employed here to deal with the synchronization problem for the nonlinear systems (1). The exact Takagi-Sugeno fuzzy model for the Nth independent nonlinear dynamic systems (1) has the following fuzzy rule representation :

Plant Rule k :

IF $z_{i1}(t)$ is F_{1k} and $\cdots$ $z_{iq}(t)$ is F_{qk}

$$\text{THEN}\ \ \dot{x}_i(t) = A_k x_i(t) \tag{7}$$

where $i = 1, 2, \ldots, N$ and $z_{i1}(t) \sim z_{iq}(t)$ are the premise variables which would consist of the states of each system; F_{jk} are the fuzzy sets; $k = 1, 2, ...r$ is the number of IF-THEN rules; A_k is system matrices of appropriate dimensions. Hence we let each system have the same membership function F_{jk}. To obtain the inferred output of the fuzzy rules, we use the singleton fuzzier, product fuzzy inference, and weighted average defuzzier. Therefore, we have

$$\dot{x}_i(t) = \sum_{k=1}^{r} h_k(x_i(t)) A_k x_i(t) \tag{8}$$

where $i = 1, 2, \ldots, N$. Form the fuzzication, inference, and defuzzication procedure, we have a properties $\mu_k(x_i(t)) = \prod_{j=1}^{q} F_{jk}(x_i(t))$, and $h_k(x_i(t)) = \mu_k(x_i(t)) \,/\, \sum_{k=1}^{r} \mu_k(x_i(t))$; and $\sum_{k=1}^{r} h_k(x_i(t)) = 1$ for all t, where $h_k(x_i(t)) \geqslant 0$ are normalized weights. Applying T-S fuzzy model approach to (6) the error system can be rewritten as follows:

$$\dot{e}_i(t) = \sum_{k=1}^{r} h_k(x_i(t)) A_k x_i(t) - \sum_{k=1}^{r} h_k(x_0(t)) A_k x_0(t)$$

$$+\sum_{j=1}^{N} c_{ij} H e_j(t),\ i=1,2,\ldots,N \tag{9}$$

Let $\bar{C}=C\otimes H$, where $\otimes$ is the Kronecker product. Also we dene

$$\bar{e}(t)=\begin{bmatrix} e_1(t)\\ \vdots \\ e_N(t)\end{bmatrix},\ \bar{x}(t)=\begin{bmatrix} x_1(t)\\ \vdots \\ x_N(t)\end{bmatrix},\ and\ \bar{x}_0(t)=\begin{bmatrix} x_0(t)\\ \vdots \\ x_0(t)\end{bmatrix}\in R^N.$$

Consequently, the augmented error system is obtained as follows:

$$\dot{\bar{e}}(t)=diag\left\{\sum_{k=1}^{r} h_k(x_1(t))A_k,\ldots,\sum_{k=1}^{r} h_k(x_N(t))A_k\right\}\bar{x}(t)$$

$$-diag\left\{\sum_{k=1}^{r} h_k(x_0(t))A_k,\ldots,\sum_{k=1}^{r} h_k(x_0(t))A_k\right\}\bar{x}_0(t)$$

$$+\bar{C}\bar{e}(t), \tag{10}$$

After adding and subtracting the term below to (10)

$$diag\left\{\sum_{k=1}^{r} h_k(x_0(t))A_k,\ldots,\sum_{k=1}^{r} h_k(x_0(t))A_k\right\}\bar{x}(t),$$

We can obtain the following equation:

$$\dot{\bar{e}}(t)=\sum_{k=1}^{r} h_k(x_0(t))[I_N\otimes A_k+\bar{C}]\bar{e}(t)$$

$$+diag\{\Xi_1,\ldots,\Xi_N\}\bar{x}(t), \tag{11}$$

where

$$\Xi_i=\sum_{k=1}^{r} h_k(x_i(t))A_k-\sum_{k=1}^{r} h_k(x_0(t))A_k,\ i=1,2,\ldots,N.$$

However, the high dimension induced by Kronecker product will make the solution H for synchronization becomes hard to find. To cope with this problem, we transpose (9) and obtain the following equation

$$\dot{e}_i^T(t)=x_i^T(t)\sum_{k=1}^{r} h_k(x_0(t))A_k^T-x_0^T(t)\sum_{k=1}^{r} h_k(x_0(t))A_k^T$$

$$+x_i^T(t)\sum_{k=1}^{r} h_k(x_i(t))A_k^T-x_i^T(t)\sum_{k=1}^{r} h_k(x_0(t))A_k^T$$

$$+\sum_{j=1}^{N} e_j^T(t)H^T c_{ij}. \tag{12}$$

Which leads to the following overall error system

$$\frac{d}{dt}\begin{bmatrix} e_1^T(t)\\ \vdots \\ e_N^T(t)\end{bmatrix}=\begin{bmatrix} e_1^T(t)\\ \vdots \\ e_N^T(t)\end{bmatrix}\sum_{k=1}^{r} h_k(x_0(t))A_k^T$$

$$+\begin{bmatrix} e_1^T(t)\\ \vdots \\ e_N^T(t)\end{bmatrix}\bar{\Delta}(x(t))+C\begin{bmatrix} e_1^T(t)\\ \vdots \\ e_N^T(t)\end{bmatrix}H^T \tag{13}$$

where we dene $e_i^T(t)\bar{\Delta}(x(t))=e_i^T(t)\bar{\Delta}(x_i(t))$ and

$$e_i^T(t)\bar{\Delta}_i(x_i(t))=x_i^T(t)\sum_{k=1}^{r} h_k(x_i(t))A_k^T-x_i^T(t)\sum_{k=1}^{r} h_k(x_0(t))A_k^T$$

It is obvious that, the high dimension structure like $C\otimes H$ has vanished. We know that if C satises (3) and C is a symmetric matrix, the eigenvalues of C are real and can be expressed as follows:

$$0=\lambda_1>\lambda_2\geq\lambda_3\geq\cdots\geq\lambda_N$$

Besides, C is orthogonally diagonalizable. That is, there exists the following transformation:

$$D=\begin{bmatrix} \lambda_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \lambda_N\end{bmatrix}=S^T C S.$$

After dening the transformation $E\triangleq S^T\left[e_1(t)\cdots e_N(t)\right]^T$, (13) can be simplied to:

$$\dot{E}=\sum_{k=1}^{r} h_k(x_0(t))EA_k^T+E\bar{\Delta}(x(t))+DEH^T.$$

Obviously, $\lim_{t\to\infty}\|E\|=0$ is the goal of synchronization. Before addressing the synchronization theorem, we need the following assumption:

Assumption: The uncertainty $\bar{\Delta}_i(x_i(t))$ satises the following bounding fashion inequalities

$$\bar{\Delta}_i(x_i(t))\bar{\Delta}_i^T(x_i(t))\leq WW^T,\ i=1,2,\ldots,N, \tag{14}$$

with a constant matrix W.

Theorem 1: The chaotic synchronous system (4) which satises (3) is asymptotically stable if there exist a positive denite matrix P and a symmetric matrix H satisfy the following inequalities

$$\Omega P+P\Omega^T<0 \tag{15}$$

$$WP+PW^T>0 \tag{16}$$

where

$$\Omega=A_k^T+W+\lambda_i H^T,\ i=2,3,\ldots,N.$$

Proof : Dene a Lyapunov function candidate $V=E_iPE_i^T$, where E_i is ith row of E. After taking time derivative, we obtain

$$\begin{aligned}\dot{V}&=\dot{E}_iPE_i^T+E_iP\dot{E}_i^T\\ &=E_i((A_k^T+\bar{\Delta}(x(t))+\lambda_iH^T)P\end{aligned}$$

$$+P(A_k^T+\bar{\Delta}(x(t))+\lambda_i H^T)^T)E_i^T$$

From (14), it follows that

$$\bar{\Delta}(t)\bar{\Delta}^T(t)\le WW^T,$$

and

$$\left|\bar{\Delta}^T(t)P\right|\le\left|W^TP\right|.$$

Hence,

$$\begin{aligned}E_i\bar{\Delta}(t)PE_i+E_iP\bar{\Delta}(t)^TE_i^T &\le \left|E_iW\,PE_i\right|+\left|E_iPW^T\,E_i^T\right|\\ &= \left|E_iW\,PE_i+E_iPW^TE_i^T\right|\\ &= E_i(W\,P+PW^T)E_i^T\end{aligned}$$

If the inequality (16) is sustained. Therefore, by (17), it follows that:

$$\dot{V}\le E_i((A_k^T+W+\lambda_iH^T)P+P(A_k^T+W+\lambda_iH^T)^T)E_i^T$$

Hence, if (15) is feasible, it results in $\dot{V}<0$. Hence, $\bar{e}(t)\to 0$ as $t\to\infty$ by the Lyapunov theorem.

Let $PH = M$, and $M > 0$. Then the inequality (15) can be transformed to following LMI form:

$$A_k^TP+PA_k+WP+PW^T+\lambda_N(M^T+M)<0 \tag{18}$$

Hence, we can obtain P and M from (18) to get H. If no constraints are applied to the inner coupling matrix H, the existence of the feasible solutions is a trivial problem. But, a higher rank of the matrix H will make the real implementation become harder. In light of this observation, we now consider the case where $H = KL$ with a column vector K and a row vector L. Then, two cases are investigated in the following subsections.

(A) Known K but with Unknown L

In this case, inequality (18) leads to

$$A_k^TP + PA_k + WP + PW^T + \lambda_NL^TK^TP + \lambda_NPKL < 0 \tag{19}$$

After pre-multiply and post-multiply $X = P^{-1}$ on the both side of (19), we obtain

$$XA_k^T + A_kX + XW + W^TX + \lambda_NXL^TK^T + \lambda_NKLX < 0 \tag{20}$$

Let $XL^T = Y$. Then (20) can be written as the following LMIs:

$$XA_k^T + A_kX + XW + W^TX + \lambda_NY\,K^T + \lambda_NKY^T < 0 \tag{21}$$

Hence, we can solve P and Y from (21) to get L.

(B) Known L but with Unknown K

In this case, we let $K^TP = Z$. Then (19) can be written as the follows LMIs:

$$A_k^T\,P + PA_k + WP + PW^T + \lambda_NL^TZ + \lambda_NZ^TL < 0 \tag{22}$$

Hence, we can solve P and Z from (22) to get K.

IV. SIMULATIONS RESULTS

To verify the eectiveness of the proposed theoretical derivation, we apply above method to Lorenz attractor, which is a well known chaotic system. Every Lorenz attractor will be coupled by other Lorenz attractors. Let $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t)]^T$. The Lorenz attractor of ith node is shown as

$$\dot{x}_i(t)=\begin{bmatrix}a(x_{i2}(t)-x_{i1}(t))\\ cx_{i1}(t)-x_{i1}(t)x_{i3}(t)-x_{i2}(t)\\ x_{i1}(t)x_{i2}(t)-bx_{i3}(t)\end{bmatrix}$$

$$=\begin{bmatrix}-a & a & 0\\ c & -1 & -x_{i1}(t)\\ 0 & x_{i1}(t) & -b\end{bmatrix}x_i(t) \tag{23}$$

By setting the parameters $a = 10$, $b = \frac{8}{3}$, $c = 28$, it will show chaotic phenomenon. According to T-S fuzzy modeling approach, the premise variable of the fuzzy system is $z_1 = x_{i1}(t)$ and the fuzzy sets are

$$F_{i1}=\frac{x_{i1}(t)-d_{i1}}{D_{i1}-d_{i1}},\; F_{i2}=\frac{D_{i1}-x_{i1}(t)}{D_{i1}-d_{i1}}.$$

Indeed, $D_{i1} = D$, and $d_{i1} = d$, $i = 1, 2, \ldots N$, for a common region $[d, D] = [50, 50]$ in which x_i(t) lies.

At rst, we consider the small-world network with 10 nodes to discuss for simulation. The model of small-world network and its coupling conguration matrix are shown in Fig. 1 and Table 1, respectively. From the denition in [4] and [10], we can obtain the average path length $L = 1.644$ and the clustering coecient $C = 0.55$ which show the short average length and the high clustering coefficient.

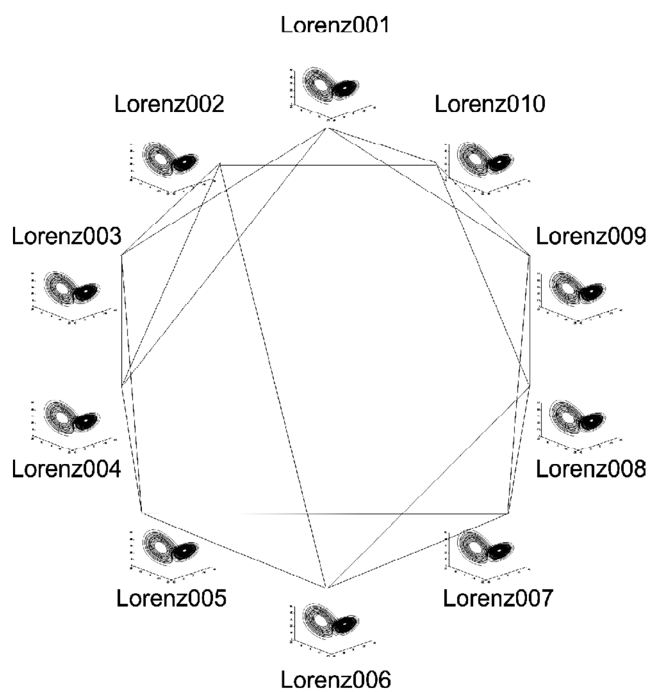

Figure 1: The Small World Network

Table 1
Small world Coupling Conguration Matrix

c_{ij}	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	-4	0	1	1	0	0	0	0	1	1
2	0	-4	1	1	0	1	0	0	0	1
3	1	1	-4	1	1	0	0	0	0	0
4	1	1	1	-4	1	0	0	0	0	0
5	0	0	1	1	-4	1	1	0	0	0
6	0	1	0	0	1	-4	1	1	0	0
7	0	0	0	1	0	1	-4	1	1	0
8	0	0	0	0	0	1	1	-4	1	1
9	1	0	0	0	0	0	1	1	-4	1
10	1	1	0	0	0	0	0	1	1	-4

The inner coupling matrix H of Table 1 obtained via LMI toolbox is given below:

$$\begin{bmatrix} 2.5966 & 0.2927 & 0 \\ 0.2927 & 12.984 & 0 \\ 0 & 0 & 13.581 \end{bmatrix}$$

In simulation, all initial values are set to be dierent as shown as follows:

$$x_1(t_0) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, x_2(t_0) = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}, x_3(t_0) = \begin{bmatrix} 13 \\ 15 \\ 17 \end{bmatrix}, x_4(t_0) = \begin{bmatrix} 19 \\ 21 \\ 23 \end{bmatrix},$$

$$x_5(t_0) = \begin{bmatrix} 25 \\ 27 \\ 29 \end{bmatrix}, x_6(t_0) = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, x_7(t_0) = \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix}, x_8(t_0) = \begin{bmatrix} 14 \\ 16 \\ 18 \end{bmatrix},$$

$$x_9(t_0) = \begin{bmatrix} 20 \\ 22 \\ 24 \end{bmatrix}, and\ x_{10}(t_0) = \begin{bmatrix} 26 \\ 28 \\ 30 \end{bmatrix} \tag{24}$$

The response of each error system and its phase portrait are shown in Fig. 2 and Fig. 3-4 respectively. All systems achieve synchronization rapidly. For the sake of space consideration. In the following, we will use the regular network with 4 nodes to discuss the case where the inner coupling matrix H is restricted to be with rank 1.

First, the coupling conguration matrix of the regular network is shown in Table 2. The initial values which are set to be much dierent from each other are given below (under the assumption that plant is still chaotic system):

$$x_1(t_0) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, x_2(t_0) = \begin{bmatrix} 7 \\ 9 \\ 11 \end{bmatrix}, x_3(t_0) = \begin{bmatrix} 13 \\ 15 \\ 17 \end{bmatrix}, and\ x_4(t_0) = \begin{bmatrix} 19 \\ 21 \\ 23 \end{bmatrix} \tag{25}$$

Case 1: Known *K* but with Unknown *L*

Here, we let $K = [1\ 0\ 0]^T$ and obtain the following L by solving LMI (20):

$$L = [8.8086\ \ 7.3397\ \ 0.60156]$$

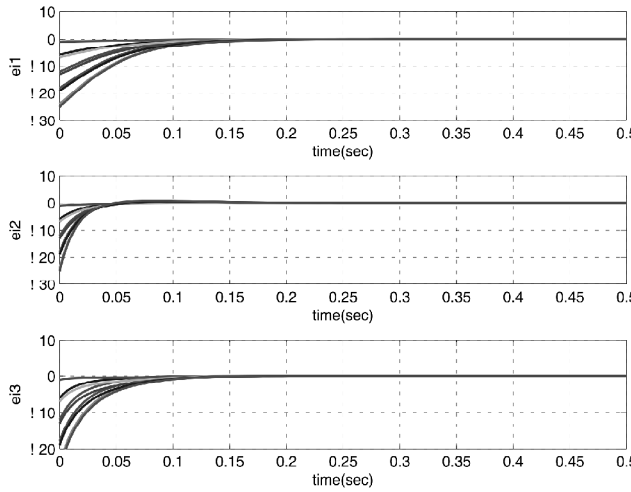

Figure 2: Synchronization Errors between Node 1 and node i, i = 2, 3, 4..., 10

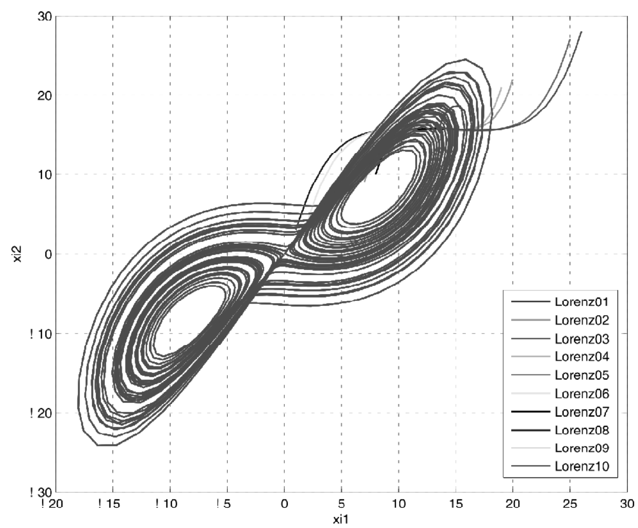

Figure 3: Phase Portrait between Each Node State 1 and state 2, i = 2, 3, 4..., 10

Table 2
Coupling Conguration Matrix of the Regular Network

c_{ij}	*1*	*2*	*3*	*4*
1	-2	1	0	1
2	1	-2	1	0
3	0	1	-2	1
4	1	0	1	-2

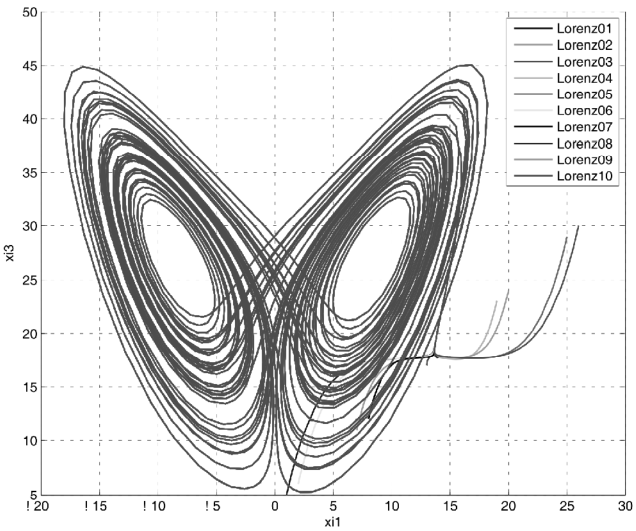

Figure 4: Phase Portrait between Each Node State 1 and State 3, $i = 2, 3, 4..., 10$

The synchronization errors of state variables between node 1 and node 2-4 are shown in Fig. 5.

Case 2: Known L but with Unknown K

Here, we let $L = [1\ 0\ 0]$ and obtain the following K by solving LMI (21):

$$K = \begin{bmatrix} 61.651 \\ 15.918 \\ 0 \end{bmatrix}$$

The synchronization errors of state variables between node 1 and node 2-4 are shown in Fig. 6.

Again, the synchronization is achieved rapidly.

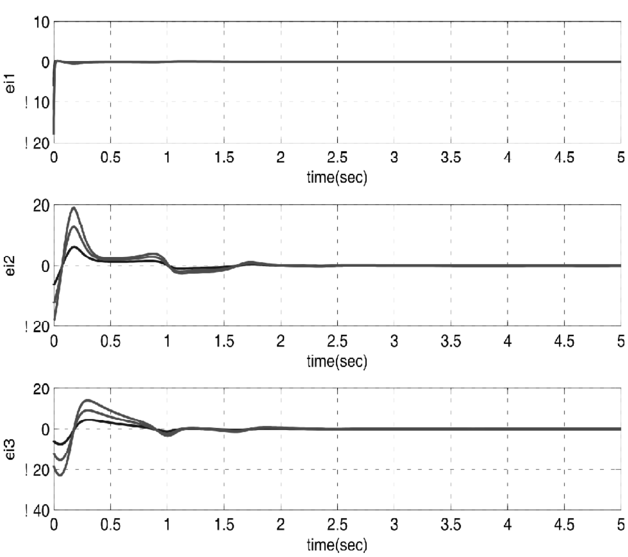

Figure 5: Synchronization Errors between Node 1 and Node 2-4 when we Design L

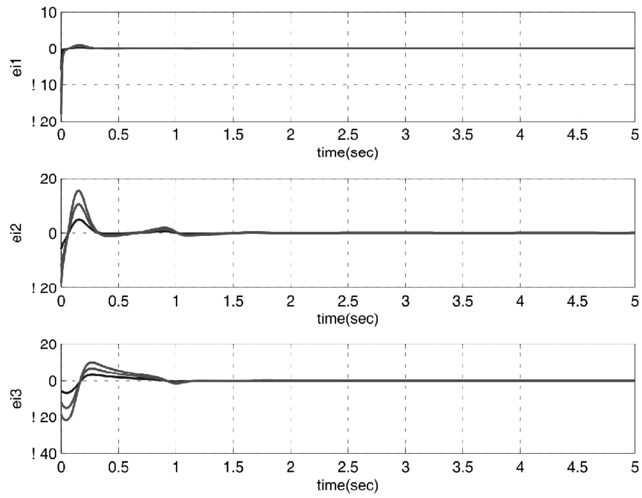

Figure 6: Synchronization Errors between Node 1 and Node 2-4 when we Design K

V. CONCLUSION

A framework of inner coupling matrix design for synchronization of complex dynamical networks has been proposed in this work. Based on the Lyapunov stability theory, some synchronization conditions have been established to ensure the complex dynamical networks achieve synchronization. Moreover, the derived results are transformed to the LMIs problem, so that suitable inner coupling matrices can be easily obtained by using the Matlab LMI toolbox. Lorenz chaotic attractors are adopted to verify the theoretic results. Numerical simulations of synchronization are shown to be consistent with theoretical statements.

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