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Complete $\mathcal{D}$-stability Intervals of Matrices—Generalization of the Stability Feeler

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***Abstract**: In this paper we study the robust $\mathcal{D}$-stability of single-parameter polynomially-dependent matrices. $\mathcal{D}$-stability of a matrix means that all the eigenvalues are in a prescribed open region, which is symmetric with respect to the real axis in the complex plane. We propose a method based on generalization of the stability feeler. The proposed method enables one to derive complete $\mathcal{D}$-stability intervals for a class of single-parameter polynomially-dependent matrices. This method does not require that a nominal matrix is stable. Numerical example shows that both Hurwitz stability intervals and Schur stability intervals of single-parameter polynomially-dependent matrices can be obtained by using the proposed method.*

***Keywords**: robust control, uncertain systems, stability of linear systems, system matrices, parameter space, stability feeler.*

1. INTRODUCTION

In many engineering applications it is required that uncertain systems are robustly stable. In this paper we study the robust $\mathcal{D}$-stability of parameter-dependent real matrices. $\mathcal{D}$-stability of a matrix means that all the eigenvalues are in a prescribed open region $\mathcal{D}$, which is symmetric with respect to the real axis in the complex plane. $\mathcal{D}$-stability contains various types of stability. For example, $\mathcal{D}$-stability corresponds to Hurwitz stability if $\mathcal{D}$ is the open left half of the complex plane. When $\mathcal{D}$ is taken as a unit circle, $\mathcal{D}$-stability corresponds to Schur stability.

Various robust stability conditions of systems for a given domain in the parameter space have been given [3], [6], [1], [7], [14]. Bialas gave a necessary and sufficient condition for stability of convex combinations of stable matrices [7]. Lyapunov-type necessary and sufficient conditions for Hurwitz stability of single-parameter polynomially-dependent matrices in the case that the parameter belongs to a compact interval are given in [14].

Not only robust stability conditions for a given domain, but also computation methods for stability domain have been given. A formula for computation of the real stability radius has been given by Qiu *et al.* [11]. Bounds on the system uncertainty that guarantee that the perturbed system remains stable are given in [10], [15], [18], [4], [5], [12] and [16]. The results in [10], [15], [18], [4] and [5] are all based on Lyapunov stability theory. The result in [12] is based on guardian maps. The result in [16] is based on generalized Lyapunov theory. Fu and Barmish [8] gave a method to synthesize the maximal Hurwitz stability interval for a convex hull of two matrices around a nominally stable matrix, based on a result in [7]. Saydy *et al.* [13] presented closed-form expression for the maximal interval of $\mathcal{D}$-stability of single-parameter polynomially-dependent matrices around a nominally stable matrix, using guardian maps. Results in [17] are also based on guardian maps. A method to find complete Hurwitz stability domain for multi-parameter offinely-dependent matrices is given in [17]. This method does not require that a nominal matrix is stable. However, methods to derive complete $\mathcal{D}$-stability domain for single-parameter polynomially-dependent matrices have not been proposed yet.

In this paper, we propose a method based on generalization of the stability feeler [9]. The stability feeler is a tool for robust stability analysis of uncertain characteristic polynomials. By using this method, we can obtain complete $\mathcal{D}$-stability intervals for a class of single-parameter polynomially-dependent matrices. This method does not require that a nominal matrix is stable. Comparison of [14], [11], [12], [16], [8], [13] and [17] with this paper is shown in Table 1. We also show some numerical examples.

The notations used in this paper are as follows: $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the set of real numbers, n-dimensional real vectors and n-by-n real matrices, respectively. The superscript T stands for matrix transposition.

2. PRELIMINARIES

2.1. Stability Feeler

We propose a method to derive complete intervals of parameter r such that single-parameter polynomially-dependent matrices given by

$$A(r) := \sum_{i=0}^{m} r^i A_i,\ r \in \mathbb{R},\ \ A_i \in \mathbb{R}^{n\times n} \tag{1}$$

Table 1
Comparison of the References with this Paper

References	*Stability domains*	*Stability types*	*Methods*
Tsiotras[14]	Given domain	Hurwitz stability	Lyapunov
Qiu[11]	Stability radius	$\mathcal{D}$-stability	—
Rern[12]	Bounds	Hurwitz stability	Guardian maps
Yedavalli[16]	Bounds	D-stability	Lyapunov
Fu[8]	Maximal interval	Hurwitz stability	Result in [7]
Saydy[13]	Maximal interval	D-stability	Guardian maps
Zhang[17]	Complete intervals	Hurwitz stability	Guardian maps
This paper	Complete intervals	D-stability	Stability feeler

are $\mathcal{D}$-stable based on generalization of the stability feeler [9].

The stability feeler is a tool to derive complete intervals of parameter r that keep $\mathcal{D}$-stability of a characteristic polynomial of the form

$$p_0(s) + rp_1(s),\ r \in \mathbb{R}, \tag{2}$$

where $p_0(s) = \sum_{i=0}^{n} p_{0,i}s^i$ and $p_1(s) = \sum_{i=0}^{n_1} p_{1,i}s^i$ are fixed real polynomials with degree n and $n_1(< n)$, respectively. The stability feeler approach needs the following two results:

Lemma 1. *Let $\boldsymbol{q} := [q_0\ q_1 \cdots q_n]^T$ be a coefficient vector of $q(s) = \sum_{i=0}^{n} q_i s^i$. Then, $x \in \mathbb{R}$ is a zero of $q(s)$ if and only if*

$$\boldsymbol{e}_x^T \boldsymbol{q} = 0 \tag{3}$$

is satisfied, where $\boldsymbol{e}_x := [1\ x\ x^2 \cdots x^n]^T$.

Lemma 2. *[2] Let $\boldsymbol{q} := [q_0\ q_1 \cdots q_n]^T$ be a coefficient vector of $q(s) = \sum_{i=0}^{n} q_i s^i$. Then, $x + jy,\ j = \sqrt{-1},\ x, y \in \mathbb{R}$, $y \neq 0$ is a zero of $q(s)$ if and only if*

$$E_{x+jy}\boldsymbol{q} = \boldsymbol{0} \tag{4}$$

is satisfied, where

$$E_{x+jy} := \begin{bmatrix} \boldsymbol{h}_{x+jy}^1 \\ \boldsymbol{h}_{x+jy}^2 \end{bmatrix} \in \mathbb{R}^{2\times(n+1)}, \tag{5}$$

$$\boldsymbol{h}_{x+jy}^1 := [h_0\ h_1\ h_2\ \cdots\ h_n], \tag{6}$$

$$\boldsymbol{h}_{x+jy}^2 := [0\ h_0\ h_1\ \cdots\ h_{n-1}], \tag{7}$$

$$h_i := 2xh_{i-1} - (x^2 + y^2)h_{i-2},\ \ i = 2, \ldots, n, \tag{8}$$

$$h_0 := 1,\ \ h_1 := 2x. \tag{9}$$

Let $\partial\mathcal{D}_r$ and $\partial\mathcal{D}_c$ be the sets of real numbers and complex conjugates that constitute the boundary of the region $\mathcal{D}$, respectively. From the above two lemmas, the sets of parameter r such that (2) has a zero on $\partial\mathcal{D}_r$ and $\partial\mathcal{D}_c$ are given by

$$\mathcal{R}_r := \{r \mid \boldsymbol{e}_x^T(\boldsymbol{p}_0 + r\boldsymbol{p}_1) = 0,\ x \in \partial\mathcal{D}_r,\ r \in \mathbb{R}\}, \tag{10}$$

$$\mathcal{R}_c := \{r \mid E_{x+jy}(\boldsymbol{p}_0 + r\boldsymbol{p}_1) = \boldsymbol{0},\ x + jy \in \partial\mathcal{D}_c,\ x, y, r \in \mathbb{R}\}, \tag{11}$$

respectively, where $\boldsymbol{p}_0 := [p_{0,0} \cdots p_{0,n}]^T$, $\boldsymbol{p}_1 := [p_{1,0} \cdots p_{1,n1}\ 0 \cdots 0]^T$. Then, the following lemma is satisfied [9].

Lemma 3. *Assume that $\mathcal{R}_r$ and $\mathcal{R}_c$ are sets consisting of finite real numbers and let $r_1 \le r_2 \le \cdots \le r_k$ be all the real numbers in $\mathcal{R}_r \cup \mathcal{R}_c$. Define $r_0 := -\infty$, $r_{k+1} := +\infty$ and*

$$\mathcal{I} := \{i \in \{0, 1, \cdots, k\} \mid \text{There exists } r \in (r_i, r_{i+1}) \text{ such that } p_0(s) + rp_1(s) \text{ is } \mathcal{D}\text{-stable}\}. \tag{12}$$

Then, $p_0(s) + rp_1(s)$ is $\mathcal{D}$-stable if and only if

$$r \in \bigcup_{i\in\mathcal{I}} (r_i, r_{i+1}). \tag{13}$$

Therefore, complete intervals of parameter r such that (2) is $\mathcal{D}$-stable can be derived by checking $\mathcal{D}$-stability of a single polynomial in $\{p_0(s) + rp_1(s),\ r \in (r_i, r_{i+1})\}$, $i = 0, \cdots, k$, respectively.

3. MAIN RESULT

We now generalize the stability feeler to derive the complete $\mathcal{D}$-stability intervals of $A(r)$ given by (1). The eigenvalues of $A(r)$ are equivalent to the zeros of the following characteristic polynomial:

$$\sum_{i=0}^{mn} r^i [1\ s\ s^2 \cdots s^n]\boldsymbol{p}_i := \det\left(sI - \sum_{i=0}^{m} r^i A_i\right), \tag{14}$$

where $\boldsymbol{p}_i \in \mathbb{R}^{n+1}$, $i = 0, \cdots, mn$ are constant vectors and $I \in \mathbb{R}^{n\times n}$ is the identity matrix. The coefficient vector of the above characteristic polynomial is given by $\sum_{i=0}^{mn} r^i \boldsymbol{p}_i$. Therefore, from Lemma 1 and 2, the sets of parameter r such that $A(r)$ has an eigenvalue on $\partial\mathcal{D}_r$ and $\partial\mathcal{D}_c$ are given by

$$\mathcal{R}_{r,A} := \left\{ r \,\middle|\, \boldsymbol{e}_x^T \left(\sum_{i=0}^{mn} r^i \boldsymbol{p}_i \right) = 0,\ \ x \in \partial\mathcal{D}_r,\ r \in \mathbb{R} \right\} \tag{15}$$

$$\mathcal{R}_{c,A} := \left\{ r \,\middle|\, E_{x+jy} \left(\sum_{i=0}^{mn} r^i \boldsymbol{p}_i \right) = 1,\ \ x + jy \in \partial\mathcal{D}_c,\ x, y, r \in \mathbb{R} \right\} \tag{16}$$

respectively. Because of continuity of the eigenvalues of $A(r)$ with respect to parameter r, the following main theorem is satisfied.

Theorem 1. *Assume that $\mathcal{R}_{r,A}$ and $\mathcal{R}_{c,A}$ are sets consisting of finite real numbers and let $r_1 \le r_2 \le \cdots \le r_k$ be all the real numbers in $\mathcal{R}_{r,A} \cup \mathcal{R}_{c,A}$. Define $r_0 := -\infty$ $r_{k+1} := +\infty$ and*

$$\mathcal{I}_A := \{i \in \{0, 1, \cdots, k\} \mid \text{There exists } r \in (r_i, r_{i+1}) \text{ such that } A(r) \text{ is } \mathcal{D}\text{-stable}\}. \tag{17}$$

Then, $A(r)$ is $\mathcal{D}$-stable if and only if

$$r \in \bigcup_{i \in \mathcal{I}_A} (r_i, r_{i+1}). \tag{18}$$

Therefore, the complete intervals of parameter r such that $A(r)$ is $\mathcal{D}$-stable can be derived by checking $\mathcal{D}$-stability of a single matrix in $\{A(r),\ r \in (r_i,\ r_{i+1})\}$, $i = 0, \cdots, k$, respectively if we can derive $\mathcal{R}_{r,A}$ and $\mathcal{R}_{c,A}$.

This result has been obtained by generalization of the stability feeler because $\mathcal{R}_{r,A}$ and $\mathcal{R}_{c,A}$ coincide with $\mathcal{R}_r$ and $\mathcal{R}_c$, respectively in the case of $\boldsymbol{p}_i = \boldsymbol{0}$, $i \geq 2$.

4. EXAMPLES

In this section we show examples, which show that complete $\mathcal{D}$-stability intervals are derived by the proposed method. Sets $\mathcal{R}_{r,A}$ and $\mathcal{R}_{c,A}$ of the examples are all derived by the "solve" command in MATLAB.

4.1. Hurwitz Stability

The following three numerical examples show that complete Hurwitz stability intervals of matrices can be obtained by the proposed method. These examples are from [17].

Example 1. *Consider the matrix*

$$A(r) = A_0 + rA_1, \tag{19}$$

where

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{20}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{21}$$

We obtain the complete intervals of parameter r such that $A(r)$ is Hurwitz; all the eigenvalues of $A(r)$ are in

$$\mathcal{D} = \{x + jy \mid x < 0,\ x, y \in \mathbb{R}\}. \tag{22}$$

In this case, $\partial\mathcal{D}_r$ and $\partial\mathcal{D}_c$ are given by

$$\partial\mathcal{D}_r = \{0\}, \tag{23}$$

$$\partial\mathcal{D}_c = \{\ jy \mid y \neq 0,\ y \in \mathbb{R}\}, \tag{24}$$

respectively. From (14), $\boldsymbol{p}_i$, $i = 0, 1, 2$ are defined as

$$\boldsymbol{p}_0 := [1\ \ 2\ \ \ 1]^T, \tag{25}$$

$$\boldsymbol{p}_1 := \boldsymbol{0}, \tag{26}$$

$$\boldsymbol{p}_2 := \boldsymbol{0}, \tag{27}$$

respectively. Hence, from (15) and (16), one can easily see

$$\mathcal{R}_{r,A} = \phi, \tag{28}$$

$$\mathcal{R}_{c,A} = \phi, \tag{29}$$

where ϕ denotes the empty set. Therefore, one obtains $r_0 = -\infty$, $r_1 = +\infty$. It is also easily seen *that $A(0)$ is Hurwitz by the direct calculation of the eigenvalues. Hence, the stability intervals of parameter r can be concluded to be*

$$(-\infty, +\infty) \tag{30}$$

by Theorem 1.

Example 2. *Consider the matrix*

$$A(r) = A_0 + rA_1, \tag{31}$$

where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \tag{32}$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{33}$$

We obtain the complete intervals of parameter r such that $A(r)$ is Hurwitz. From (14), $\boldsymbol{p}_i$, i = 0, 1, 2 are defined as

$$\boldsymbol{p}_0 := [2\ \ 3\ \ 1]^T \tag{34}$$

$$\boldsymbol{p}_1 := [1\ \ 0\ \ 0]^T, \tag{35}$$

$$\boldsymbol{p}_2 := [-1\ \ 0\ \ 0]^T, \tag{36}$$

respectively. From (15) and (16), one obtains

$$\mathcal{R}_{r,A} = \{-1, 2\}, \tag{37}$$

$$\mathcal{R}_{c,A} = \phi. \tag{38}$$

Hence, one obtains $r_0 = -\infty$, $r_1 = -1$, $r_2 = 2$, $r_3 = +\infty$. It is easily seen that $A(r_1 - 1)$ and $A(r_2 + 1)$ are not Hurwitz and $A\left(\frac{r_1+r_2}{2}\right)$ is Hurwitz. Therefore, the stability intervals of parameter r can be concluded to be

$$(-1, 2) \tag{39}$$

by Theorem 1.

Example 3. Consider the matrix

$$A(r) = A_0 + rA_1, \tag{40}$$

where

$$A_0 = \begin{bmatrix} -10.64 & 3.395 & 8.841 & 4.558 & -10.25 \\ -11.28 & -0.1536 & 14.67 & 9.852 & -13.53 \\ 0.7320 & 3.811 & -0.6047 & 2.408 & -10.44 \\ -12.14 & 4.938 & 9.649 & 1.152 & -6.297 \\ -11.66 & 6.451 & 11.70 & 9.453 & -17.28 \end{bmatrix} \tag{41}$$

$$A_1 = \begin{bmatrix} -110.9 & -247.0 & 162.4 & -57.61 & 194.2 \\ 241.82 & 731.3 & -446.6 & 87.68 & -511.8 \\ 366.8 & 987.5 & -617.4 & 181.9 & -777.1 \\ 385.3 & 1118.5 & -666.7 & 137.4 & -809.4 \\ 100.8 & 237.1 & -142.4 & 57.89 & -234.3 \end{bmatrix} \tag{42}$$

From (14), $\boldsymbol{p}_i$, $i = 0, \cdots, 5$ are defined as

$$\boldsymbol{p}_0 := [5653\ \ 3540\ \ 1352\ \ 280.9\ \ 27.52\ \ 1.000]^T, \tag{43}$$

$$\boldsymbol{p}_1 := [-1.190 \times 10^5 - 6.615 \times 10^5 \ \ -1.764 \times 10^5 \ \ -1.145 \times 10^4 \ \ 93.90 \ \ 0]^T, \tag{44}$$

$$\boldsymbol{p}_2 := [-9.599 \times 10^6 \ \ -1.759 \times 10^7 \ -1.910 \times 10^6 \ \ 2.793 \times 10^4 \ \ 0 \ \ 0]^T, \tag{45}$$

$$\boldsymbol{p}_3 := [-3.131\times10^8 \ \ -6.574\times10^7 \ \ 1.446\times10^6 \ \ 0 \ 0 \ 0]^T, \tag{46}$$

$$\boldsymbol{p}_4 := [-6.869 \times 10^8 \ \ 2.738 \times 10^7 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T, \tag{47}$$

$$\boldsymbol{p}_5 := [1.810 \times 10^8 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T, \tag{48}$$

respectively. Hence, from (15) and (16), one obtains

$$\mathcal{R}_{r,A} = \{-0.382, 0.0159, 4.21\}, \tag{49}$$

$$\mathcal{R}_{c,A} = \{-1.60, -0.0463, 0.00241\}. \tag{50}$$

Therefore, one obtains $r_0 = -\infty$, $r_1 = -1.60$, $r_2 = -0.382$, $r_3 = -0.0463$, $r_4 = 0.00241$, $r_5 = 0.0159$, $r_6 = 4.21$, $r_7 = +\infty$. It is shown that $A(r)$ is Hurwitz for $r = \frac{r_3+r_4}{2}$, $r = r_6 + 1$ and $A(r)$ is not Hurwitz for $r = r_1 - 1$, $r = \frac{r_1+r_2}{2}$, $r = \frac{r_2+r_3}{2}$, $r = \frac{r_4+r_5}{2}$, $r = \frac{r_5+r_6}{2}$ from the direct calculation of eigenvalues. Therefore, the stability intervals of parameter r can be concluded to be

$$(-0.0463, \ 0.00241) \cup (4.21, \ +\infty) \tag{51}$$

by Theorem 1.

We claim that the obtained intervals are complete stability intervals because these are almost the same as the complete stability intervals shown in [17].

4.2. Schur Stability

Now we show an example for Schur stability. In this case, stability region $\mathcal{D}$ is the interior of the unit circle. The following example shows stability analysis of a discrete-time system by the proposed method.

Example 4. Consider the matrix

$$A(r) = A_0 + rA_1 + r^2A_2, \tag{52}$$

where

$$A_0 = \begin{bmatrix} 0.2895 & -1.2919 & 0.4978 & -0.2463 \\ 1.4789 & -0.0729 & 1.4885 & 0.6630 \\ 1.1380 & -0.3306 & -0.5465 & -0.8542 \\ -0.6841 & -0.8436 & -0.8468 & -1.2013 \end{bmatrix} \tag{53}$$

$$A_1 = \begin{bmatrix} 0.9863 & 0.0215 & -1.1859 & -1.2173 \\ -0.5186 & -1.0039 & -1.0559 & -0.0412 \\ 0.3274 & -0.9471 & 1.4725 & -1.1283 \\ 0.2341 & -0.3744 & 0.0557 & -1.3493 \end{bmatrix} \tag{54}$$

$$A_2 = \begin{bmatrix} -0.2611 & -1.1678 & -1.3194 & 0.8057 \\ 0.9535 & -0.4606 & 0.9312 & 0.2316 \\ 0.1286 & -0.2624 & 0.0112 & -0.9898 \\ 0.6565 & -1.2132 & -0.6451 & 1.3396 \end{bmatrix} \tag{55}$$

We obtain the complete intervals of parameter r such that $A(r)$ is Schur stable; all the eigenvalues of $A(r)$ are in

$$\mathcal{D} = \{x + jy \mid |x + jy| < 1, x, y \in \mathbb{R}\}. \tag{56}$$

In this case, $\partial\mathcal{D}_r$ and $\partial\mathcal{D}_c$ are given by

$$\partial\mathcal{D}_r = \{-1, 1\}, \tag{57}$$

$$\partial\mathcal{D}_c = \{x + jy \mid |x + jy| = 1, y \neq 0, x, y \in \mathbb{R}\}, \tag{58}$$

respectively. From (14), $\boldsymbol{p}_i$, $i = 0, 1, \cdots, 8$ are definedas

$$\boldsymbol{p}_0 := [1.3601 \ 2.8352 \ \ 1.7605 \ \ 1.5312 \ \ 1.0000]^T, \tag{59}$$

$$\boldsymbol{p}_1 := [-2.0142 \ \ -6.7561 \ \ -1.2606 \ \ -0.1056 \ \ 0]^T, \tag{60}$$

$$\boldsymbol{p}_2 := [-3.1604 \ \ 7.6615 \ \ 2.8649 \ \ -0.6291 \ 0]^T, \tag{61}$$

$$\boldsymbol{p}_3 := [18.1577 \ \ -7.8512 \ \ 2.1904 \ \ 0 \ \ 0]^T, \tag{62}$$

$$\boldsymbol{p}_4 := [9.4088 \ \ -2.4744 \ \ -0.1986 \ \ 0 \ \ 0]^T, \tag{63}$$

$$\boldsymbol{p}_5 := [0.6822 \ \ -4.1540 \ \ 0 \ \ 0 \ \ 0]^T, \tag{64}$$

$$\boldsymbol{p}_6 := [-2.2548 \ \ -3.7253 \ \ 0 \ \ 0 \ \ 0]^T, \tag{65}$$

$$\boldsymbol{p}_7 := [3.0073 \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T, \tag{66}$$

$$\boldsymbol{p}_8 := \mathbf{0}, \tag{67}$$

respectively. From (15) and (16), one obtains

$$\mathcal{R}_{r,A} = \{-1.2262, 0.0769\}, \tag{68}$$

$$\mathcal{R}_{c,A} = \{0.2544, 0.2608, 0.5328\}. \tag{69}$$

Hence, one obtains $r_0 = -\infty$, $r_1 = -1.2262$, $r_2 = 0.0769$, $r_3 = 0.2544$, $r_4 = 0.2608$, $r_5 = 0.5328$, $r_6 = +\infty$. It is shown that $A(r)$ is Schur stable for $r = \frac{r_3+r_4}{2}$ and $A(r)$ is not stable for $r = r_1 - 1$, $r = \frac{r_1+r_2}{2}$, $r = \frac{r_2+r_3}{2}$, $r = \frac{r_4+r_5}{2}$, $r = r_5 + 1$ from the direct calculation of eigenvalues. Therefore, we conclude that the Schur stability interval is

$$(0.2544, \ 0.2608) \tag{70}$$

by Theorem 1.

5. CONCLUSION

In this paper we study the robust $\mathcal{D}$-stability of parameter-dependent real matrices. We propose a method based on generalization of the stability feeler [9]. By this method, we can obtain complete $\mathcal{D}$-stability intervals for a class of single-parameter polynomially-dependent matrices. This method does not require that a nominal matrix is stable.

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