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THE CLASSICAL SOLUTION OF THE MIXED PROBLEM FOR THE SECOND-ORDER HYPERBOLIC EQUATION WITH HIGH ORDER DERIVATIVES IN THE BOUNDARY CONDITIONS

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ABSTRACT. The second order one-dimensional hyperbolic equation is considered. The classical solution of the mixed problem with the high-order derivatives is constructed. Necessary and sufficient conditions for unique solution existence in specific class of functions are proven.

1. Introduction

Hyperbolic equations have a great sense for all areas of human beings. Thus, a lot of scientists are trying to analyze this types of equations. There are several methods described in literature like using Fourier's method, and most of these methods are reduced to the functional series. A lot of authors didn't even proof that these series are convergent and use a formal integration and differentiation. As for numeric methods: many algorithms use initial and boundary functions to construct a numeric approximation of analytic solution. But they are missing such an important thing as matching conditions.

In this paper with the help of the method of characteristics [1] is shown that under some conditions smoothness of the solution can be corrupted disregard on smoothness of the given functions.

This method also shows good results in tricky cases like Klein-Gordon-Fock equation [2], where solution cannot be expressed in general form and can only be presented as integral equations.

2. Statement of the problem

2.1. Canonical form of the equation. The second-order hyperbolic equation with partial derivatives

$$(\partial_{x_0} - a^{(1)}\partial_{x_1})(\partial_{x_0} - a^{(2)}\partial_{x_1})u = p(\mathbf{x}) \tag{2.1}$$

is considered in the area $Q = \{(x_0, x_1)|x_0 \in [0, +\infty), x_1 \in [0, l]\}$, where $a^{(1)} < a^{(2)} < 0$, $\mathbf{x} = (x_0, x_1)$. The case $a^{(1)} < 0 < a^{(2)}$ was studied in [3]. Additionally the equation (2.1) is a generalization of the wave equation which was considered in [4].

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Initial conditions

$$\begin{aligned}
&u(0,x_1)=\varphi(x_1),\\
&\partial_{x_0}u(0,x_1)=\psi'(x_1),\\
&x_1\in[0,l],
\end{aligned} \tag{2.2}$$

and boundary conditions

$$\begin{aligned}
&\frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}(x_0,0)=\widetilde{\mu^{(0)}}(x_0),x_0\in[0,+\infty),\\
&\frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}(x_0,l)=\widetilde{\mu^{(l)}}(x_0),x_0\in[-l/a^{(1)},+\infty),
\end{aligned} \tag{2.3}$$

where $\boldsymbol{\alpha}=(\alpha_0,\alpha_1),|\boldsymbol{\alpha}|=\alpha_0+\alpha_1$ and $\alpha_i\in\mathbb{N}\bigcup 0,i=0,1$ are joined to the equation (2.1).

According to the algorithm described in [1], equation (2.1) can be converted to the second canonical form

$$\partial_{\xi_0\xi_1}v=h(\boldsymbol{\xi}) \tag{2.4}$$

using transformation $x_1+a^{(1)}x_0=\xi_0, x_1+a^{(2)}x_0=\xi_1$.

3. The general solution of the hyperbolic equation

3.1. The general solution of a homogeneous equation. Homogeneous equation (2.1), namely equation with $p(\boldsymbol{x})\equiv 0$, is considered further. The second canonical form of the equation (2.1) is

$$\partial_{\xi_0\xi_1}v=0. \tag{3.1}$$

The initial conditions in terms of variables $\boldsymbol{\xi}$, where $\boldsymbol{\xi}=(\xi_0,\xi_1)$, are presented as

$$\begin{aligned}
&v(\xi_1,\xi_1)=\varphi(\xi_1),\\
&a^{(1)}\partial_{\xi_0}v(\xi_1,\xi_1)+a^{(2)}\partial_{\xi_1}v(\xi_1,\xi_1)=\psi'(\xi_1),\\
&\xi_1\in[0,l].
\end{aligned} \tag{3.2}$$

The first condition from (2.3) is transformed to

$$\frac{\partial^{|\boldsymbol{\alpha}|}v}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}\left(\frac{a^{(1)}}{a^{(2)}}\xi_1,\xi_1\right)=\widetilde{\mu^{(0)}}\left(\frac{\xi_1}{a^{(2)}}\right),\xi_1\in(-\infty,0], \tag{3.3}$$

and the second condition from (2.3) is transformed to

$$\frac{\partial^{|\boldsymbol{\alpha}|}v}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}\left(\xi_0,l+\frac{a^{(2)}}{a^{(1)}}(\xi_0-l)\right)=\widetilde{\mu^{(l)}}\left(\frac{\xi_0-l}{a^{(1)}}\right),\xi_0\in[-l,+\infty). \tag{3.4}$$

The general solution of the equation (3.1), as was shown in [3], is

$$v(\boldsymbol{\xi})=f(\xi_0)+g(\xi_1). \tag{3.5}$$

Using reverse algorithm to the process described in [3], general solution of the homogeneous equation (2.1) is presented like

$$u(\mathbf{x})=p(x_1+a^{(1)}x_0)+g(x_1+a^{(2)}x_0). \tag{3.6}$$

3.2. The general solution of a non-homogeneous equation. Since the equation (2.1) is linear, then the general solution can be represented as the following sum

$$u(\mathbf{x}) = U(\mathbf{x}) + \overline{u}(\mathbf{x}), \tag{3.7}$$

where $U(\mathbf{x})$ is the general solution of the homogeneous equation and $\overline{u}(\mathbf{x})$ is some particular solution of the non-homogeneous equation.

Theorem 3.1. *Expression*

$$\overline{u}(\boldsymbol{x}) = \int\limits_0^{x_0} w(x_0 - \tau, \tau, x_1) d\tau \tag{3.8}$$

is a particular solution of the non-homogeneous equation with homogeneous initial conditions $\overline{u}(x_1, x_1) = 0, \partial_{x_0}\overline{u}(x_1, x_1) = 0,$ *where the function* $w(x_0, \tau, x_1)$ *is*

$$w(x_0, \tau, x_1) = \left(\frac{1}{a^{(2)} - a^{(1)}} \int\limits_{x_1 + a^{(1)}x_0}^{x_1 + a^{(2)}x_0} p(\tau, y) dy \right). \tag{3.9}$$

In the formula (3.9) the function p is the same as in the equation (2.1).

Proof. Proof of this theorem is provided in the [1]. □

The general solution of the homogeneous equation is $U(\mathbf{x}) = u(\mathbf{x}) - \overline{u}(\mathbf{x})$ according to the equation (3.7). This expression can be used for modifications of the boundary conditions. Values of the solution on bounds of the area Q are $\frac{\partial^{|\boldsymbol{\alpha}|}\overline{u}}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}(x_0, i) = \overline{\mu^{(i)}}(x_0), i = 0, l.$

Thus, for the homogeneous equation the boundary conditions (2.3) are

$$\begin{aligned} &\frac{\partial^{|\boldsymbol{\alpha}|}U}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}(x_0, 0) = \widetilde{\mu^{(0)}}(x_0) - \overline{\mu^{(0)}}(x_0) = \mu^{(0)}(x_0), x_0 \in [0, +\infty), \\ &\frac{\partial^{|\boldsymbol{\alpha}|}U}{\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}}(x_0, l) = \widetilde{\mu^{(l)}}(x_0) - \overline{\mu^{(l)}}(x_0) = \mu^{(l)}(x_0), x_0 \in [-l/a^{(1)}, +\infty). \end{aligned} \tag{3.10}$$

Previous reasoning leads to the following statement of the mixed problem for the homogeneous hyperbolic equation

$$(\partial_{x_0} - a^{(1)}\partial_{x_1})(\partial_{x_0} - a^{(2)}\partial_{x_1})U = 0 \tag{3.11}$$

with initial conditions

$$\begin{aligned} &U(0, x_1) = \varphi(x_1), \\ &\partial_{x_0}U(0, x_1) = \psi'(x_1), \\ &x_1 \in [0, l], \end{aligned} \tag{3.12}$$

and the boundary conditions (3.10) are connected to the equation (3.11).

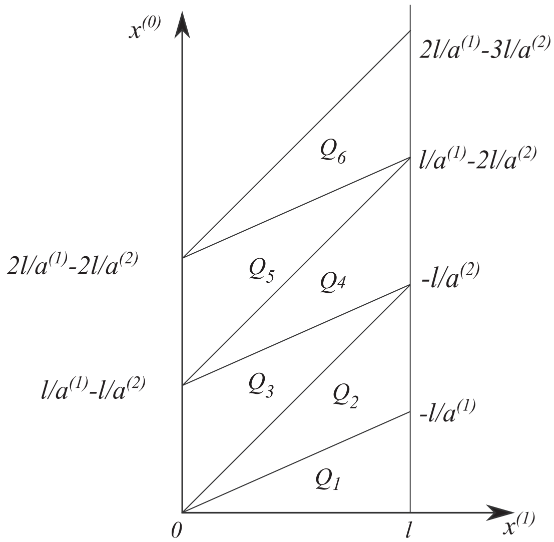

FIGURE 1. The Q area

3.3. Analysis of the Q area. The area $Q = \{(x_0, x_1)|x_0 \in [0, +\infty), x_1 \in [0, l]\}$ is shown on the figure 1.

Transformation $x_1 + a^{(1)}x_0 = \xi_0, x_1 + a^{(2)}x_0 = \xi_1$ will convert each of the domain Q_k to the area Ω_k. Besides the Q area is converted to Ω and $\Omega = \bigcup\limits_{k=1}^{\infty} \Omega_k$, moreover $\Omega_k \bigcap \Omega_h = \emptyset, k \neq h$. Functions $f_m(\xi_0)$ and $g_m(\xi_1)$ are defined in each domain Ω_k and the function $f_m : \Omega_{2m} \bigcup \Omega_{2m+1} \to \mathbb{R}, m \in \mathbb{N} \bigcup 0$, and the function $g_m : \Omega_{2m+1} \bigcup \Omega_{2m+2} \to \mathbb{R}, m \in \mathbb{N} \bigcup 0$.

Let's consider an area of definition of the function f. The result of simple calculation is $f_m(\xi_0) : \mathbb{R} \ni [ml(1-\frac{a^{(1)}}{a^{(2)}}), (m-1)l(1-\frac{a^{(1)}}{a^{(2)}})] \to \mathbb{R}, m = 1, 2, ...$ Two facts should be marked about the functions f_m :

(1) the expression $1 - \frac{a^{(1)}}{a^{(2)}} < 0$,
(2) $f_0(\xi_0) : \mathbb{R} \ni [0, l] \to \mathbb{R}$.

Also the notation $\xi_0^{(m)} = ml(1 - \frac{a^{(1)}}{a^{(2)}})$ is introduced. This implies to the fact $\xi_0 \in (-\infty, l]$. Under the function f is meant $f(\xi_0) = \{f_m(\xi_0), \xi_0 \in [ml(1 - \frac{a^{(1)}}{a^{(2)}}), (m-1)l(1 - \frac{a^{(1)}}{a^{(2)}})] \bigcup \in [0, l]\}$.

Lemma 3.2. *$f(\xi_0) \in C^{(i)}((-\infty, l])$ if and only if $f_m(\xi_0) \in C^{(i)}$ on corresponding areas of definition, as well as the conditions*

$$\frac{d^j f_m}{d\xi_0^j}(ml(1 - \frac{a^{(1)}}{a^{(2)}})) = \frac{d^j f_{m+1}}{d\xi_0^j}(ml(1 - \frac{a^{(1)}}{a^{(2)}}))\ \forall j \leq i, \forall m \geq 0 \qquad (3.13)$$

are fulfilled.

The function $g_m(\xi_1) : \mathbb{R} \ni [ml(\frac{a^{(2)}}{a^{(1)}} - 1), (m-1)l(\frac{a^{(2)}}{a^{(1)}} - 1)] \to \mathbb{R}, m = 1, 2, ...$ The next two facts should be marked about the functions g_m

(1) the expression $\frac{a^{(2)}}{a^{(1)}} - 1 < 0$,
(2) $g_0(\xi_1) : \mathbb{R} \ni [0, l] \to \mathbb{R}$.

Introducing notation $\xi_1^{(m)} = ml(\frac{a^{(2)}}{a^{(1)}} - 1)$ leads to $\xi_1 \in (-\infty, l]$. Also the function g denotes $g(\xi_0) = \{g_m(\xi_0), \xi_1 \in [ml(\frac{a^{(2)}}{a^{(1)}} - 1), (m-1)l(\frac{a^{(2)}}{a^{(1)}} - 1)] \bigcup \in [0, l]\}$.

Lemma 3.3. *$g(\xi_1) \in C^{(i)}((-\infty, l])$ if and only if $g_m(\xi_1) \in C^{(i)}$ on corresponding areas of definition, as well as the conditions*

$$\frac{d^j g_m}{d\xi_1^j}(ml(\frac{a^{(2)}}{a^{(1)}} - 1) = \frac{d^j g_{m+1}}{d\xi_1^j}(ml(\frac{a^{(2)}}{a^{(1)}} - 1))\ \forall j \le i, \forall m \ge 0 \tag{3.14}$$

are fulfilled.

4. Solutions of the problems in sub-domains

4.1. The solution of the problem with Cauchy conditions. The solution of Cauchy problem can be obtained from the general solution (3.5) of the equation (3.1) with the help of the initial conditions (3.6).

A system

$$\begin{cases} f_0(\xi_1) + g_0(\xi_1) = \varphi(\xi_1), \\ a^{(1)} f_0'(\xi_1) + a^{(2)} g_0'(\xi_1) = \psi'(\xi_1) \end{cases} \tag{4.1}$$

is obtained by calculating values of the solution (3.6) and its first derivative at the point (ξ_1, ξ_1).

A solution of this system is a pair of uniquely defined functions (according to the condition $a^{(1)} < a^{(2)}$)

$$\begin{cases} f_0(z) = \frac{1}{a^{(2)} - a^{(1)}}(a^{(2)}\varphi(z) - \psi(z)) - \frac{C}{a^{(2)} - a^{(1)}}, \\ g_0(z) = \frac{1}{a^{(2)} - a^{(1)}}(-a^{(1)}\varphi(z) + \psi(z)) + \frac{C}{a^{(2)} - a^{(1)}}. \end{cases} \tag{4.2}$$

Here $\psi(\xi_1) = \int\limits_0^{\xi_1} \psi'(y)dy$, C is an arbitrary constant.

Functions f_0, g_0 from the system (4.2) belongs to the class $C^{(i)}[0, l]$ only if $\varphi(z) \in C^{(i)}[0, l]$, and function $\psi'(z) \in C^{(i-1)}[0, l]$.

Solution of Cauchy problem is received by substitution functions from the (4.2) to the (3.6)

$$v(\boldsymbol{\xi}) = \frac{1}{a^{(2)} - a^{(1)}}(a^{(2)}\varphi(\xi_0) - \psi(\xi_0)) + \frac{1}{a^{(2)} - a^{(1)}}(-a^{(1)}\varphi(\xi_1) + \psi(\xi_1)). \tag{4.3}$$

4.2. Boundary conditions with high-order derivatives. The function f_0 is defined in the area Ω_1 from Cauchy conditions, and function g_0 is defined in the area $\Omega_1 \bigcup \Omega_2$. To find the function f_1 in the area Ω_2 condition (3.4) specified on right boundary should be used.

The right boundary condition is considered further

$$\frac{\partial^{|\boldsymbol{\alpha}|} v}{\partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1}}(\xi_0, l + \frac{a^{(2)}}{a^{(1)}}(\xi_0 - l)) = \mu^{(l)}\left(\frac{\xi_0 - l}{a^{(1)}}\right). \tag{4.4}$$

The solution $v(\boldsymbol{\xi}) = f_m(\xi_0) + g_{m-1}(\xi_1)$ is substituted in (4.4). Obviously smoothness of the function g_{m-1} defined on the previous step should not be less than an order of the derivative in the condition (4.4).

Since the solution of the problem should be in the class $C^{(2)}(\Omega)$ then consistency conditions are to be valid as for functions themselves so for the first-order and the second-order derivatives of these functions. In this paper we have specific interest in the case $|\boldsymbol{\alpha}| \geq 2$. The case when $|\boldsymbol{\alpha}| = 1$ was discussed in [5] and the case $|\boldsymbol{\alpha}| = 0$ was studied in [6].

Boundary conditions with $|\boldsymbol{\alpha}| \geq 2$ are concerned further.

$$\begin{gathered}
\frac{\partial^{|\boldsymbol{\alpha}|} v}{\partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1}} \left(\xi_1 \frac{a^{(2)}}{a^{(1)}}, \xi_1 \right) = \mu^{(0)} \left(\frac{\xi_1}{a^{(2)}} \right) \\
\frac{\partial^{|\boldsymbol{\alpha}|} v}{\partial_{x_0}^{\alpha_0} \partial_{x_1}^{\alpha_1}} \left(\xi_0, l + \frac{a^{(2)}}{a^{(1)}} (\xi_0 - l) \right) = \mu^{(l)} \left(\frac{\xi_0 - l}{a^{(1)}} \right)
\end{gathered} \tag{4.5}$$

Rewriting the left boundary condition in comfortable form

$$\frac{d^{|\boldsymbol{\alpha}|} g_m}{d\xi_1^{|\boldsymbol{\alpha}|}} = \frac{1}{(a^{(2)})^{\alpha_0}} \left(\mu^{(0)} \left(\frac{\xi_1}{a^{(2)}} \right) - (a^{(1)})^{\alpha_0} \frac{d^{|\boldsymbol{\alpha}|} f_m}{d\xi_0^{|\boldsymbol{\alpha}|}} \left(\frac{a^{(1)}}{a^{(2)}} \xi_1 \right) \right) \tag{4.6}$$

and defining the right part of it as $G^{(m)}(\xi_1)$ helps to write the solution of (4.6) as

$$g_m(\xi_0) = \sum_{k=0}^{|\boldsymbol{\alpha}|-1} C_{(g,k)}^{(m)} \xi_1^k + \int\limits_{\xi_1^{(m-1)}}^{\xi_1} G^{(m)}(y) \frac{(\xi_1 - y)^{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}| - 1)!} dy. \tag{4.7}$$

A system of matching conditions for the functions g_m and g_{m-1} and theirs derivatives up to the $|\boldsymbol{\alpha}| - 1$ order at the point $\xi_1 = \xi_1^{(m)}$ should be constructed to determine arbitrary constants $C_{(g,k)}^{(m)}, k = \overline{0, |\boldsymbol{\alpha}| - 1}$. This system of equations is lower triangular and that's why it have a unique solution.

$$\left\{ \begin{aligned}
& \sum_{k=0}^{|\boldsymbol{\alpha}|-1-i} C_{(g,k)}^{(m)} (\xi_1^{(m-1)})^k = \sum_{k=0}^{|\boldsymbol{\alpha}|-1-i} C_{(g,k)}^{(m-1)} (\xi_1^{(m-1)})^k + \\
& + \int\limits_{\xi_1^{(m-2)}}^{\xi_1^{(m-1)}} G^{(m-1)}(y) \tfrac{(\xi_1 - y)^{|\boldsymbol{\alpha}|-1-i}}{(|\boldsymbol{\alpha}|-1-i)!} dy, i = \overline{1, |\boldsymbol{\alpha}| - 1}, \\
& C_{(g,|\boldsymbol{\alpha}|-1)}^{(m)} = C_{(g,|\boldsymbol{\alpha}|-1)}^{(m-1)} + \int\limits_{\xi_1^{(m-2)}}^{\xi_1^{(m-1)}} G^{(m-1)}(y) dy.
\end{aligned} \right. \tag{4.8}$$

A solution of the system (4.8) can be found by sequential substitution of the values starting from the last equation.

Consistency conditions for the $|\boldsymbol{\alpha}|$-order derivatives of the functions g_m and g_{m-1} at the point $\xi_1 = \xi_1^{(m)}$ provides a condition for $C^{|\boldsymbol{\alpha}|}$ smoothness of the solution.

$$\begin{aligned}
&\frac{d^{|\boldsymbol{\alpha}|}g_m}{d\xi_1^{|\boldsymbol{\alpha}|}}(\xi_1^{(m-1)}) - \frac{d^{|\boldsymbol{\alpha}|}g_{m-1}}{d\xi_1^{|\boldsymbol{\alpha}|}}(\xi_1^{(m-1)}) = \\
&= \frac{d^{|\boldsymbol{\alpha}|}f_m}{d\xi_0^{|\boldsymbol{\alpha}|}}(\xi_0^{(m-1)}) - \frac{d^{|\boldsymbol{\alpha}|}f_{m-1}}{d\xi_0^{|\boldsymbol{\alpha}|}}(\xi_0^{(m-1)}) = \delta_{(m,|\boldsymbol{\alpha}|)}^{(|\boldsymbol{\alpha}|)}, \\
&m = 2, ...
\end{aligned} \tag{4.9}$$

The same results are obtained for the function f. To have an easy form the right boundary condition is rewritten in the next way

$$\frac{d^{|\boldsymbol{\alpha}|}f_m}{d\xi_0^{|\boldsymbol{\alpha}|}} = \frac{1}{(a^{(1)})^{\alpha_0}}\left(\mu^{(l)}\left(\frac{\xi_0 - l}{a^{(1)}}\right) - (a^{(2)})^{\alpha_0}\frac{d^{|\boldsymbol{\alpha}|}g_{m-1}}{d\xi_1^{|\boldsymbol{\alpha}|}}\left(l + \frac{a^{(1)}}{a^{(2)}}(\xi_0 - l)\right)\right). \tag{4.10}$$

The general solution of the equation (4.10) is

$$f_m(\xi_0) = \sum_{k=0}^{|\boldsymbol{\alpha}|-1} C_{(f,k)}^{(m)}\xi_0^k + \int\limits_{\xi_0^{(m-1)}}^{\xi_0} F^{(m-1)}(y)\frac{(\xi_0 - y)^{|\boldsymbol{\alpha}|-1}}{(|\boldsymbol{\alpha}|-1)!}dy, \tag{4.11}$$

where $F^{(m-1)}(\xi_0)$ denotes the right part of the equation (4.10).

A system of matching conditions for the functions f_m and f_{m-1} and theirs derivatives up to the $|\boldsymbol{\alpha}| - 1$ order at the point $\xi_0 = \xi_0^{(m)}$ should be constructed to determine arbitrary constants $C_{(f,k)}^{(m)}$, $k = \overline{0, |\boldsymbol{\alpha}| - 1}$. This system of equations will be lower triangular and that's why it will have a unique solution.

$$\begin{cases}
\sum\limits_{k=0}^{|\boldsymbol{\alpha}|-1-i} C_{(f,k)}^{(m)}(\xi_0^{(m-1)})^k = \sum\limits_{k=0}^{|\boldsymbol{\alpha}|-1-i} C_{(f,k)}^{(m-1)}(\xi_0^{(m-1)})^k + \\
+ \int\limits_{\xi_0^{(m-2)}}^{\xi_0^{(m-1)}} F^{(m-2)}(y)\frac{(\xi_0-y)^{|\boldsymbol{\alpha}|-1-i}}{(|\boldsymbol{\alpha}|-1-i)!}dy, i = \overline{1, |\boldsymbol{\alpha}| - 1}, \\
C_{(f,|\boldsymbol{\alpha}|-1)}^{(m)} = C_{(f,|\boldsymbol{\alpha}|-1)}^{(m-1)} + \int\limits_{\xi_0^{(m-2)}}^{\xi_0^{(m-1)}} F^{(m-2)}(y)dy.
\end{cases} \tag{4.12}$$

A solution of the system (4.12) is found by sequential substitution of values starting from the last equation.

Consistency condition for the $|\boldsymbol{\alpha}|$-order derivatives of the functions f_m and f_{m-1} at the point $\xi_0 = \xi_0^{(m)}$ gives a condition on $C^{|\boldsymbol{\alpha}|}$ smoothness of the solution.

$$\begin{aligned}
&\frac{d^{|\boldsymbol{\alpha}|}f_m}{d\xi_1^{|\boldsymbol{\alpha}|}}(\xi_0^{(m-1)}) - \frac{d^{|\boldsymbol{\alpha}|}f_{m-1}}{d\xi_0^{|\boldsymbol{\alpha}|}}(\xi_0^{(m-1)}) = \\
&= \frac{d^{|\boldsymbol{\alpha}|}g_{m-1}}{d\xi_1^{|\boldsymbol{\alpha}|}}(\xi_1^{(m-2)}) - \frac{d^{|\boldsymbol{\alpha}|}g_{m-2}}{d\xi_1^{|\boldsymbol{\alpha}|}}(\xi_1^{(m-2)}) = \sigma_{(m,|\boldsymbol{\alpha}|)}^{(|\boldsymbol{\alpha}|)}, \\
&m = 2, ...
\end{aligned} \tag{4.13}$$

Lemma 4.1. *The matching conditions (4.7) and (4.13) in the area* $\Omega^{(m)}$ *are fulfilled if and only if the matching conditions (4.7) and (4.13) in the area* $\Omega^{(m-1)}$ *are fulfilled.*

Proof. Expressions (4.7) and (4.13) shows that matching conditions on the layer m can be reduced to the matching conditions on the layer $m-1$ and vice a versa. □

This lemma leads directly to the next statement.

Lemma 4.2. *Fulfillment of the conditions (4.7) and (4.13) for* $m = 1$ *is necessary and sufficient for fulfillment of the conditions (4.7) and (4.13) for all* $m = 2, \ldots$.

The matching condition for the $|\boldsymbol{\alpha}|$-order derivative for the functions g_0 and g_1 at the point $\xi_1 = 0$ in explicit form can be presented as following

$$G^{(1)}(0) - \frac{1}{a^{(2)} - a^{(1)}} \left(\frac{d^{|\boldsymbol{\alpha}|}\psi}{d\xi_1^{|\boldsymbol{\alpha}|}}(0) - a^{(1)} \frac{d^{|\boldsymbol{\alpha}|}\varphi}{d\xi_1^{|\boldsymbol{\alpha}|}}(0) \right) = \delta_{|\boldsymbol{\alpha}|}^{(|\boldsymbol{\alpha}|)}. \tag{4.14}$$

Matching condition for the $|\boldsymbol{\alpha}|$-order derivative for the functions f_0 and f_1 at the point $\xi_0 = 0$ in explicit form can be presented as following

$$F^{(1)}(0) - \frac{1}{a^{(2)} - a^{(1)}} \left(-\frac{d^{|\boldsymbol{\alpha}|}\psi}{d\xi_0^{|\boldsymbol{\alpha}|}}(0) + a^{(2)} \frac{d^{|\boldsymbol{\alpha}|}\varphi}{d\xi_0^{|\boldsymbol{\alpha}|}}(0) \right) = \sigma_{|\boldsymbol{\alpha}|}^{(|\boldsymbol{\alpha}|)}. \tag{4.15}$$

Theorem 4.3. *Assuming that* $\mu^{(0)}(x_0) \in C([0, +\infty))$, $\mu^{(l)}(x_0) \in C([-\frac{l}{a^{(1)}}, +\infty))$, $\varphi \in C^{(|\boldsymbol{\alpha}|)}([0, l])$, $\psi \in C^{(|\boldsymbol{\alpha}|-1)}([0, l])$ *then the solution of the problem (2.4),(3.2), (4.5)* $v(\boldsymbol{\xi})$ *exists and is unique, if arbitrary constants are defined according to the rules (4.8), (4.12) and the following statements are equal*

(1) *the solution of the problem (2.4),(3.2), (4.5)* $v(\boldsymbol{\xi}) \in C^{|\boldsymbol{\alpha}|}(\Omega)$*;*
(2) *the matching conditions (4.15),(4.14) are fulfilled when* $\delta_{|\boldsymbol{\alpha}|}^{(|\boldsymbol{\alpha}|)} = \sigma_{|\boldsymbol{\alpha}|}^{(|\boldsymbol{\alpha}|)} = 0$.

Remark 4.4. The solution $v(\boldsymbol{\xi})$ can be transformed to the solution in the terms of variables $\boldsymbol{x}$ by the formula (3.6).

Remark 4.5. If conditions of the theorem 4.3 are fulfilled, but the matching conditions (4.15),(4.14) are not homogeneous, i.e. $\delta_{|\boldsymbol{\alpha}|}^{(|\boldsymbol{\alpha}|)} \neq 0$ or $\sigma_{|\boldsymbol{\alpha}|}^{(|\boldsymbol{\alpha}|)} \neq 0$, then $v(\boldsymbol{\xi}) \in C^{|\boldsymbol{\alpha}|}(\widetilde{\Omega})$, where $\widetilde{\Omega} = \{\boldsymbol{\xi} | \boldsymbol{\xi} \in \Omega \bigwedge \xi_0 \neq \xi_0^{(m)} \bigwedge \xi_1 \neq \xi_1^{(m)}\}$ is an area without characteristics. But in the area Ω the solution $v(\boldsymbol{\xi}) \in C^{|\boldsymbol{\alpha}|-1}$. In this case for numerical solution conjugation conditions on characteristics should be used to obtain correct numerical approximation of the classical solution.

References

1. Korzyuk V.: *Equations of mathematical physics*, Belarus state university, Minsk, 2008.(In Russ.)
2. Korzyuk V.I., Staliarchuk I.I.: Classical solution of the first mixed problem for the Klein-Gordon-Fock eqaution in a half strip *Differential equations* **(50)8** (2014) 1105–1117
3. Korzyuk V.I., Cheb E.S., Karpechina A.A.: Second-order hyperbolic equation for two independent variables, *Proceedings of the National Academy of Sciences of Belarus. Series of Physical-Mathematical Sciences* **1** (2013) 71–80 (In Russ.)

4. Korzyuk V.I., Naumavets S.N.: Classical solution of a mixed problem for a one-dimensional wave equation with high-order derivatives in the boundary conditions, *Reports of the National Academy of Sciences of Belarus* **60(3)** (2016) 11–17. (In Russ.)
5. Korzyuk V., Cheb E., Karpechina A.: Classical solution of the first mixed problem in half-strip for linear second-order hyperbolic equation, *Proceeding of Insitute of Mathematics* **(20)2** (2012) 64–74, Minsk. (In Russ.)
6. Korzyuk V., Cheb E., Karpechina A.: Classical solution of the second mixed problem in half-strip for linear second-order hyperbolic equation, in *Proc. 3rd international scientific conference "Mathematical modeling and differential equations"* (2012) 178–191, Belarus state university, Minsk. (In Russ.)

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