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ON APPLICATION OF MIXED MINKOWSKI VOLUMES IN QUALITATIVE THEORY OF SET DIFFERENTIAL EQUATIONS

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Abstract. For a family of equations with uncertain values of parameters the conditions are given for preservation and decrease of the distance between the mixed Minkowski volumes "filled with" a set of trajectories of the family of equations.

1. Introduction

Among the dynamic properties of a set of trajectories of families of equations the property of stability is of key importance. This is due to the fact that only stable trajectories are observable and therefore they are realizable in real mechanical and other nature systems. For the stability analysis of the families of equations a direct Lyapunov method is developed in terms of auxiliary scalar, vector and matrix-valued functions (see [1–2] and bibliography therein).

Some "similarity" of the properties of Lyapunov functions [3] to the properties of mixed Minkowski volumes (MMV) [4, 5] allows the qualitative analysis of a set of trajectories to be carried out in terms of MMV under certain conditions formulated for the families of equations.

The aim of this paper is a dynamical analysis of a set of trajectories in terms of MMV of one class of equations with uncertain values of parameters.

The paper is arranged as follows. In Section 2 the family of equations under consideration is described and assumptions are presented on the bodies where the sets of trajectories are localized at fixed values of uncertainty parameter.

In Section 3 some properties of non stationary mixed Minkowski volumes are given and their similarity to the properties of Lyapunov functions is discussed.

In Section 4 main results of the paper are set out.

In final Section 5 a discussion of the obtained results is presented and some bibliographic notes are made.

2. A family of uncertain equations

Let $K_c(\mathbb{R}^n)$ be a space of nonempty convex compact subsets in the space $\mathbb{R}^n$. Consider a set of perturbed motion equations of a mechanical or other nature system in the form

$$D_H X = F(t, X, \alpha), \tag{2.1}$$

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$$X(t_0) = X_0 \in K_c(\mathbb{R}^n). \tag{2.2}$$

Here $X \in K_c(\mathbb{R}^n)$ is a states set of system (2.1), $D_H X$ is the derivative of states set, $F \in C(\mathbb{R}_+ \times K_c(\mathbb{R}^n) \times \mathcal{J}, K_c(\mathbb{R}^n))$, $\alpha \in \mathcal{J}$, $\mathcal{J} \subseteq \mathbb{R}^d$ is a compact set of uncertainty parameters.

The mapping $X \in C^1(J, K_c(\mathbb{R}^n))$, where $J = [t_0, t_0 + a]$, $a > 0$, is a solution for the set of equations (2.1) on J, if it satisfies the set of equations (2.1) under the initial conditions (2.2).

The fact that $X(t)$ is a continuously differentiable function on J in the Hukukhara sense [1] implies that

$$X(t) = X_0 + \int_{t_0}^{t} D_H X(s)ds \tag{2.3}$$

and further

$$X(t,\alpha) = X_0 + \int_{t_0}^{t} F(s, X(s), \alpha)ds, \quad t \in J, \tag{2.4}$$

for all $\alpha \in \mathcal{J}$.

We assume on system (2.1) and mappings (2.4) as follows.

A_1. For all $\alpha \in \mathcal{J}$ and $t \in J$ the mapping $F(t, \Theta, \alpha) = \Theta$;

A_2. For the fixed values of $\alpha \in \mathcal{J}$ the mappings $X(t,\alpha)$ are "localized" in the convex "bodies" $P_1(X), \dots, P_n(X)$;

A_3. The bodies $P_i(X) = \Theta$, $i = 1, 2, \dots, n$ iff $X = \Theta$ ($\Theta \in K_c(\mathbb{R}^n)$ is a zero element of the convex set $K_c(\mathbb{R}^n)$).

A_4. The bodies $P_i(X(t,\alpha))$, $i = 1, 2, \dots, n$, are non-vanising and "non-expanding infinity" on finite existence interval of the set of trajectories.

Example 2.1. We consider the set of differential equations (2.1) with $F(t, X, \alpha) = e^{\alpha} X$ and $\mathcal{J} = [0, 1]$. In this case assuming that $\alpha = (0, 1/2, 1) \in [0, 1]$ we will get the bodies $P_1(X) = X(t,\alpha)$ for $\alpha = 0$, $P_2(X) = X(t,\alpha)$ for $\alpha = 1/2$, and $P_3(X) = X(t,\alpha)$ for $\alpha = 1$.

If for the bodies $P_i(X), i = 1, 2, 3$ satisfies the conditions A_2. – A_4. then the set of trajectories of equation (2.1) is localized in the convex bodies $P_i(X(t,\alpha))$, $i = 1, 2, 3$.

It is known (see [1, 2] and bibliography therein) that for the analysis of the set of trajectories of the family of equations (2.1), a generalized Lyapunov function is applied, which possesses the following properties:

P_1. $V(t, X) \in C(\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+)$;

P_2. $V(t, X) = \Theta$ iff $X = \Theta \in K_c(\mathbb{R}^n)$;

P_3. $|V(t, A) - V(t, B)| \le LD(A, B)$, $L > 0$,

for all $A, B \in K_c(\mathbb{R}^n)$, where D is the Hausdorff metric.

An example of the class of functions is $V(t, X) = D(X, \Theta)$ for all $X \in K_c(\mathbb{R}^n)$ with total Dini derivative

$$D^+V(t,X) = \limsup \big\{ \big[D(X + hF(t, X, \alpha)), \Theta) - \\ - D(X, \Theta) \big] h^{-1} : h \to 0^+ \big\}$$

along the solutions of family (2.1).

3. Non stationary mixed Minkowski volumes (cf. [4, 5])

The combination of the fundamental notion of Minkowski addition and the notion of volume yields the notion of mixed volume.

Further we shall consider the MMV for the bodies $P_1, \ldots, P_n$, which are "filled" with the mappings $X \in K_c(\mathbb{R}^n)$ for the fixed values of the uncertainty parameter $\alpha \in \mathcal{J}$.

We designate by $\mathbb{K}^n$ a space of convex bodies $P_i = P_i(X)$ (nonempty compact convex subsets in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$)).

The convex body $P_i(X) \in \mathbb{K}^n$ is defined in the only way by the support function $h_{P_i}\colon S^{n-1} \to \mathbb{R}$, where $h_{P_i}(u) = \max\limits_{x \in P_i} \langle x, u \rangle$, and $\langle \cdot \rangle$ is a standard designation of the scalar product in $\mathbb{R}^n$, and S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Definition 3.1. Let convex bodies $P_1, P_2, \ldots, P_n \in \mathbb{K}^n$ be given for positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. The expression $P = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n$ is called the linear non stationary mixed Minkowski combination and is a convex body with the support function $h_P = \sum\limits_{j=1}^{n} \lambda_j h_{P_j}$.

Let $[n]$ denote a set of positive integers $1, 2, \ldots, n$. The following result holds.

Theorem 3.2. *(cf. [4]) Let $P_1(X), \ldots, P_n(X) \in \mathbb{K}^n$ and $\lambda_1, \ldots, \lambda_n > 0$. Then*

$$Vol(\lambda_1 P_1(X) + \ldots + \lambda_n P_n(X)) = \\ = \sum_{i_1, \ldots, i_n \in [n]} MV(P_{i1}(X), \ldots, P_{in}(X)) \lambda_{i1} \cdots \lambda_{in}, \tag{3.1}$$

where each coefficient $MV(P_{i1}(X), \ldots, P_{in}(X))$ depends only on the bodies $P_{i1}(X)$, $\ldots, P_{in}(X)$.

Definition 3.3. For given $P_1(X), \ldots, P_n(X) \in \mathbb{K}^n$, the coefficient $MV(P_1(X), \ldots, P_n(X))$ is called the non stationary mixed Minkowski volume of convex bodies $P_1(X), \ldots, P_n(X)$ for any values of the uncertainty parameter $\alpha \in \mathcal{J}$.

Further we apply the Hausdorff metric in the space $\mathbb{K}^n$. Let B be a unit sphere in $\mathbb{R}^n$, and $\lambda \geq 0$. The Hausdorff distance between the bodies P_i and P_j is specified by the formula

$$\begin{gathered} \mathcal{D}(P_i(X), P_j(X)) = \inf\{\lambda :\ P_i(X) \subset P_j(X) + \lambda B \quad \text{and} \\ P_j(X) \subset P_i(X) + \lambda B \quad \text{for any} \quad i \neq j,\ i, j, = 1, 2, \ldots, n\}. \end{gathered} \tag{3.2}$$

Together with the metric (3.2) for a pair of bodies in $\mathbb{K}^n$ we shall consider a metric for the body $P(X) = \bigcup\limits_{k=1}^{n} P_k(X)$:

$$\begin{gathered} \mathcal{D}^*(P(X), \Theta) = \inf\{\lambda :\ P(X) \subset \Theta + \lambda B \quad \text{and} \quad \Theta \subset P(X) + \lambda B \\ \text{for any} \quad X \in K_c(\mathbb{R}^n)\}. \end{gathered} \tag{3.3}$$

Note that the Hausdorff distance (3.3) is a metric on $\mathbb{K}^n$ and the pair ($\mathbb{K}^n$, $\mathcal{D}^*(P(X), \Theta)$) is a metric space.

We recall some properties of the non stationary mixed Minkowski volumes:

(1) mixed volume is a non-negative symmetric function on the set $\mathbb{K}^n$, i.e. $MV(P_1(X),\dots,P_n(X))\ge 0$ for all $X\in K_c(\mathbb{R}^n)$;
(2) mixed volumes are monotone with respect to inclusion, i.e. $MV(P_1(X),\dots,P_n(X))\ge MV(P_1^*(X),\dots,P_n^*(X))$ if $P_i(X)\supset P_i^*(X)$ for all $i=1,2,\dots,n$ and for all $X\in K_c(\mathbb{R}^n)$;
(3) $MV(P_1(X),\dots,P_n(X))>0$ iff there exist segments $S_i(X)\subset P_i(X)$, $i=1,2,\dots,n$, whose directions are linearly independent;
(4) mixed volumes are invariant with respect to the permutation of indices, i.e. $MV(P_1(X),\dots,P_n(X))=MV(P_{\sigma(1)}(X),\dots,P_{\sigma(n)}(X))$ for any permutations of σ;
(5) mixed volumes are additive and positive homogeneous, i.e. $MV(\dots,\alpha P_i(X)+\beta P_i^*(X),\dots)=\alpha MV(\dots,P_i(X),\dots)+\beta MV(\dots,P_i^*(X),\dots)$ for any $i=1,2,\dots,n$ and for all $X\in K_c(\mathbb{R}^n)$.

Properties (1) and (3) of MMV imply that when $X\in K_c(\mathbb{R}^n)\cap S_i(X)$ for the MMV $MV(P_1(X),\dots,P_n(X))$ there exists a function $a(r)$ of Hahn class K [6] ($a(0)=0$; $a(r)$ is monotone increasing in r) such that

(A_5) $a(\mathcal{D}(P_i(X),P_j(X)))\le MV(P_1(X),\dots,P_n(X))$ for all $X\in K_c(\mathbb{R}^n)\cap S_i(X)$, $i\ne j$, $i,j=1,2,\dots,n$;
(A_6) besides, note that $MV(P_1(X),\dots,P_n(X))=0$ iff $P_i(X)=0$ for all $X\in K_c(\mathbb{R}^n)$ and all $i=1,2,\dots,n$.

Comparison of the properties of generalized Lyapunov function with the properties of MMV shows that MMV is a positive semi-definite function by virtue of its properties (1), (4) and (5).

This fact allows MMV to be employed as a class of appropriate Lyapunov functions (functionals) in the investigation of sets of trajectories of families of equations.

4. Applications

In addition to the geometric importance of MMV (see [5] and bibliography therein) their application may appear to be useful in the investigation of set of solutions of the family of equations. For example, (see [7]), the numerical construction of solutions to the polynomial systems requires estimating the volume of solution. The theory of MMV allows one to establish boundary for the volume of successive approximations, and therefore, it is of interest for applications to the analysis of set of solutions for either polynomial or other nature systems.

Further we shall designate $\mathcal{D}(P_i(X(t_0)),P_j(X(t_0)))=\mathcal{D}(X(t_0))$ and $\mathcal{D}(P_i(X(t)),P_j(X(t)))=\mathcal{D}(X(t))$ respectively, for all $i\ne j$, $i,j=1,2,\dots,n$.

Definition 4.1. For the solutions of the set of equations (2.1) the distance between the bodies $P_i(X(t))$ does not increase if for given $t_0\in\mathbb{R}_+$ and $\varepsilon>0$ there exists $\delta>0$ such that the condition $\mathcal{D}(X(t_0))<\delta$ implies that $\mathcal{D}(X(t))<\varepsilon$ for all $t\in J$.

Definition 4.2. For the set of solutions of equations (2.1) the distance between the bodies $P_i(X(t))$ decreases if the conditions of Definition 4.1 are satisfies and $\mathcal{D}(X(t))\to 0$ as $t\to\infty$.

Now let us show that the following result takes place.

Theorem 4.3. *Let for the set of equations (2.1) the conditions (A_1)–(A_4) be satisfied and for the generalized bodies $P_1(X)$,..., $P_n(X)$ there exist a MMV $MV(P_1(X), \dots, P_n(X))$ such that the conditions (A_5) and (A_6) are satisfied and, moreover,*

$$MV(P_1(X(t)), \dots, P_n(X(t))) \leq MV(P_1(X_0), \dots, P_n(X_0))$$

along the set of solutions $X(t)$) of the problem (2.1)–(2.2) for all $t \in J$.

Then the Hausdorff distance between the bodies $P_i(X(t))$ and $P_j(X(t))$ does not increase for all $i \neq j \in [1, n]$.

Proof. Let $t_0 \in J$ and $\varepsilon > 0$ be given. Due to the continuity of MMV $MV(P_1(X(t)), \dots, P_n(X(t)))$ and the fact that $MV(P_1(\Theta), \dots, P_n(\Theta)) = 0$ a $\delta = \delta(t_0, \varepsilon)$ is found such that

$$MV(P_1(X_0), \dots, P_n(X_0)) < a(\varepsilon),$$

as soon as $\mathcal{D}(P(X_0)) < \delta$. The condition of Theorem 4.3 implies

$$a(\mathcal{D}(P(t))) \leq MV(P_1(X(t)), \dots, P_n(X(t))) \leq MV(P_1(X_0), \dots, P_n(X_0)) < a(\varepsilon)$$

and further

$$\mathcal{D}(P(t)) \leq a^{-1}\left(MV(P_1(X(t)), \dots, P_n(X(t)))\right) < a^{-1}a(\varepsilon) = \varepsilon$$

for all $t \in J$. This completes the proof of Theorem 4.3. □

Theorem 4.4. *For the Hausdorff distance between the bodies $P_i(X(t))$ and $P_j(X(t))$ vanish for $i \neq j \in [1, n]$, it is sufficient that conditions of Theorem 4.3 be satisfied and, moreover, on the set of solutions $X(t)$ of problem (2.1)–(2.2) the MMV $MV(P_1(X(t)), \dots, P_n(X(t))) \to 0$ as $t \to +\infty$ and for $\mathcal{D}(P(X_0)) < \delta$.*

Proof. Let a MMV exist and possess the properties mentioned in Theorem 4.3. We shall show that the distance between the bodies $P_i(X(t))$ and $P_j(X(t))$ vanishes for $i \neq j \in [1, n]$ as $t \to +\infty$.

In fact, when the conditions of Theorem 4.3 are satisfied, the distance $\mathcal{D}(P(t))$ does not increase. In this case, given $\varepsilon > 0$, one can find a $\delta(t_0, \varepsilon)$ such that for $\mathcal{D}(P(t_0)) < \delta$ one have $\mathcal{D}(P(t)) < \varepsilon$ for all $t \geq t_0$. We shall show that $\mathcal{D}(P(t)) \to 0$ as $t \to +\infty$. Let this be not true. Then, a set of solutions $X(t)$ should be found for the problem (2.1)–(2.2) so that for $\mathcal{D}(P(t_0)) < \delta$ a sequence of moments $t_0 < t_1 < \ldots < t_k$, $t_k \to +\infty$, is found such that $\mathcal{D}(P(t)) > \alpha > 0$. However then $MV(P_1(X(t_k)), \dots, P_n(X(t_k))) > a(\beta) > 0$, $\beta > 0$, which is impossible, since by condition of Theorem 4.4 $MV(P_1(X(t)), \dots, P_n(X(t))) \to 0$ as $t \to \infty$. This completes the proof of Theorem 4.4. □

Example 4.5. (continued) We note that for the bodies $P_1(X), P_2(X), P_3(X)$ we can use the expression (see [8])

$$3!MV(P_1(X), P_2(X), P_3(X)) = \sum_{1 \leq i \leq 3} Vol_3(P_i(X)) -$$

$$- \sum_{1 \leq i_1 < i_2 \leq 3} Vol_3(P_{i_1}(X) + P_{i_2}(X)) + Vol_3(P_1(X) + P_2(X) + P_3(X)).$$

as a function $MV(P_1(X), P_2(X), P_3(X))$.

5. Concluding Remarks

In the classical theory of MMV the convex bodies $P_1, \dots, P_n$ are considered, whose nature is not discussed. Usually it is assumed that these bodies are non-empty compact convex subsets in the space $\mathbb{R}^n$. In this paper we suppose that the convex bodies $P_1(X), \dots, P_n(X)$ are formed by a set of trajectories $X(t) \in K_c(\mathbb{R}^n)$ of the problem (2.1)–(2.2) for fixed values of uncertainty parameters in the set of equations (2.1). This assumption establishes the relationship between the theory of NMV and the qualitative theory of set of trajectories of families of equations.

If the uncertainty parameter in the right-hand part of the family of equations (2.1) is absent, then the convex bodies $P_1(X), \dots, P_n(X)$ are formed by the sets of trajectories $X(t_i) \in K_c(\mathbb{R}^n)$, where $i = 1, 2, \dots, n$. However, in this case the problem of justifying the method of obtaining non-autonomous convex bodies via continuous set of trajectories remains open.

Efficiency of this idea depends on the effective sufficient conditions of decreasing (increasing) of some functional of MMV along the set of trajectories $X(t)$ of problem (2.1)–(2.2).

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