Global and Stochastic Analysis
Vol. 5, No. 1, June (2018), 31-38
Special Issue: *In honor of Professor Ioannis P. Stavroulakis*

MUK PUBLICATIONS
Open Access Publisher

NONTRIVIAL SOLUTIONS OF SYSTEMS OF NONLOCAL CAPUTO FRACTIONAL BVPS

GENNARO INFANTE AND SAMIRA RIHANI

Abstract. In this short note, we illustrate how recent results by Infante and Pietramala, valid for systems of Hammerstein integral equations, can be used to discuss the existence, non-existence and multiplicity of nontrivial solutions for systems of Caputo fractional differential equations subject to nonlocal boundary conditions.

1. Introduction

Nieto and Pimentel [23], motivated by earlier work of Infante and Webb [12] on nonlocal problems for ODEs, studied the existence of at least one positive solution of the fractional differential equation

$$^{C}D^{\alpha}u(t) + f(t,u(t)) = 0,\ t \in (0,1), \tag{1.1}$$

subject to the nonlocal boundary conditions (BCs)

$$u'(0) = 0,\ \beta\, {}^{C}D^{\alpha-1}u(1) + u(\eta) = 0, \tag{1.2}$$

where $1 < \alpha \leq 2$, ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α, $\beta > 0$, $0 \leq \eta \leq 1$ and f is continuous. The study of Nieto and Pimentel was continued by Cabada and Infante [4] who, by means of fixed point index theory, studied the existence of *multiple* positive solutions of (1.1) under some BCs that involve Riemann-Stieltjes integrals and cover the ones in (1.2) as a special case.

Note that the BCs that occur in (1.2) are of *nonlocal* type. The study of nonlocal BCs goes back, as far as we know, to Picone [27] and has been widely developed during the years. We refer the reader to the reviews by Whyburn [33], Conti [6], Ma [22], Ntouyas [26] and Štikonas [31] and the papers by Karakostas and Tsamatos [15, 16] and by Webb and Infante [32]. We mention that Costabile and Napoli [7] also contributed to the study of nonlocal problems in the context of ODEs under a numerical point of view.

The problem of existence of solutions for *systems* of Caputo fractional differential equations under a variety of BCs has been investigated by a number of authors; for example local BCs have been investigated by Khan and ur Rehman [17] and Lan and Lin [21] and nonlocal BCs have been studied by Ahmad and Nieto [1], ur Rehman and co-authors [29] and Zhao and Gong [35].

Date: submitted November 20, 2016; accepted April 18, 2017; communicated by Sandra Pinelas
2010 *Mathematics Subject Classification.* Primary 34A08, secondary 34B10, 45G15, 47H30.
Key words and phrases. Nontrivial solution, nonlocal boundary condition, fractional differential equation, fixed point index, cone.

In this paper we discuss the existence, non-existence and multiplicity of *non-trivial* solutions for the system of fractional differential equations

$$\begin{aligned} {}^C D^{\alpha_1} u(t) + f_1(t, u(t), v(t)) &= 0,\ t \in (0,1), \\ {}^C D^{\alpha_2} v(t) + f_2(t, u(t), v(t)) &= 0,\ t \in (0,1), \end{aligned} \tag{1.3}$$

subject the BCs

$$\begin{aligned} u'(0) = 0,\ \beta_1 {}^C D^{\alpha_1 - 1} u(1) + u(\eta_1) = 0, \\ v'(0) = 0,\ \beta_2 {}^C D^{\alpha_2 - 1} v(1) + v(\eta_2) = 0, \end{aligned} \tag{1.4}$$

where $1 < \alpha_1, \alpha_2 < 2$, $\beta > 0$, $0 \le \eta_1, \eta_2 \le 1$ and f_1, f_2 are continuous. In order to do this, we associate to the system (1.3)-(1.4) a system of Hammerstein integral equations of the type

$$\begin{aligned} u(t) &= \int_0^1 k_1(t,s) f_1(s, u(s), v(s))\, ds, \\ v(t) &= \int_0^1 k_2(t,s) f_2(s, u(s), v(s))\, ds, \end{aligned} \tag{1.5}$$

and we make use of recent results of Infante and Pietramala [10], that rely on the fixed point index theory. We work in a suitable cone in a product space of continuous functions, where functions are allowed to *change sign*.

Our results complement those in [23], providing multiplicity and nonexistence results in the context of nontrivial solutions.

2. Existence and nonexistence results

We begin by recalling some results from [10] and we make the following assumptions on the terms that occur in the system (1.5).

- For every $i = 1, 2$, $f_i : [0,1] \times (-\infty, \infty) \times (-\infty, \infty) \to [0, \infty)$ is continuous.
- For every $i = 1, 2$, $k_i : [0,1] \times [0,1] \to (-\infty, \infty)$ is continuous.
- For every $i = 1, 2$, there exist a subinterval $[a_i, b_i] \subseteq [0,1]$, a function $\Phi_i \in L^\infty[0,1]$, and a constant $c_i \in (0,1]$, such that

$$\begin{aligned} |k_i(t,s)| \le \Phi_i(s) &\text{ for } t \in [0,1] \text{ and a. e. } s \in [0,1], \\ k_i(t,s) \ge c_i \Phi_i(s) &\text{ for } t \in [a_i, b_i] \text{ and a. e. } s \in [0,1]. \end{aligned}$$

- For every $i = 1, 2$, we have $\int_{a_i}^{b_i} \Phi_i(s)\, ds > 0$.

We work in the space $C[0,1] \times C[0,1]$ endowed with the norm

$$\|(u,v)\| := \max\{\|u\|_\infty, \|v\|_\infty\},$$

where $\|w\|_\infty := \max\{|w(t)|, t \in [0,1]\}$, define the sets

$$\tilde{K}_i := \{w \in C[0,1] : \min_{t \in [a_i, b_i]} w(t) \ge c_i \|w\|_\infty\},$$

and consider the cone K in $C[0,1] \times C[0,1]$ defined by

$$K := \{(u,v) \in \tilde{K}_1 \times \tilde{K}_2\}.$$

By a *nontrivial* solution of the system (1.5) we mean a solution $(u,v) \in K$ of (1.5) such that $\|(u,v)\| \neq 0$. Note that the functions in $\tilde{K}_i$ are positive on the subinterval $[a_i, b_i]$ but are allowed to change sign in $[0,1]$. This type of cone has been introduced by Infante and Webb in [11] and is similar to a cone of *non-negative* functions firstly used by Krasnosel'skiĭ, see e.g. [18], and D. Guo, see e.g. [13].

Note that, under the assumptions above (see Lemma 2.1 of [10]), the integral operator

$$T(u,v)(t) := \begin{pmatrix} \int_0^1 k_1(t,s) f_1(s,u(s),v(s))\, ds \\ \int_0^1 k_2(t,s) f_2(s,u(s),v(s))\, ds \end{pmatrix} \tag{2.1}$$

leaves the cone K invariant and is compact.

We use the following open bounded sets (relative to K):

$$K_{\rho_1,\rho_2} = \{(u,v) \in K : \|u\|_\infty < \rho_1 \text{ and } \|v\|_\infty < \rho_2\},$$

and

$$V_{\rho_1,\rho_2} = \{(u,v) \in K : \min_{t\in[a_1,b_1]} u(t) < \rho_1 \text{ and } \min_{t\in[a_2,b_2]} v(t) < \rho_2\}.$$

Note that $K_{\rho_1,\rho_2} \subset V_{\rho_1,\rho_2} \subset K_{\rho_1/c_1,\rho_2/c_2}$; this is a key property used in order to prove the multiplicity results. When $\rho_1 = \rho_2 = \rho$ we write K_ρ and V_ρ. The set V_ρ (in the context of systems) was introduced by Infante and Pietramala in [14] and is equal to the set called $\Omega^{\rho/c}$ in [9]. $\Omega^{\rho/c}$ is an extension to the case of systems of a set given by Lan [20].

The next Lemma summarizes some sufficient conditions from [10] regarding the index computations.

Lemma 2.1. *The following hold.*

(1) *Assume that*

$(\mathrm{I}^1_{\rho_1,\rho_2})$ *there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1,2$*

$$f_i^{\rho_1,\rho_2} < m_i \tag{2.2}$$

where

$$f_i^{\rho_1,\rho_2} = \sup\Big\{\frac{f_i(t,u,v)}{\rho_i} :\ (t,u,v) \in [0,1]\times[-\rho_1,\rho_1]\times[-\rho_2,\rho_2]\Big\}$$

and

$$\frac{1}{m_i} = \sup_{t\in[0,1]} \int_0^1 |k_i(t,s)|\, ds.$$

Then the fixed point index of T relative to K_ρ, $i_K(T, K_{\rho_1,\rho_2})$ is equal to 1.

(2) *Assume that*

$(\mathrm{I}^0_{\rho_1,\rho_2})$ *there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1,2$*

$$f_{i,(\rho_1,\rho_2)} > M_i, \tag{2.3}$$

where

$$f_{1,(\rho_1,\rho_2)} = \inf\Big\{\frac{f_1(t,u,v)}{\rho_1}:\ (t,u,v)\in[a_1,b_1]\times[\rho_1,\rho_1/c_1]\times[-\rho_2/c_2,\rho_2/c_2]\Big\},$$

$$f_{2,(\rho_1,\rho_2)} = \inf\Big\{\frac{f_2(t,u,v)}{\rho_2}:\ (t,u,v)\in[a_2,b_2]\times[-\rho_1/c_1,\rho_1/c_1]\times[\rho_2,\rho_2/c_2]\Big\},$$

$$\frac{1}{M_i} = \inf_{t\in[a_i,b_i]}\int_{a_i}^{b_i} k_i(t,s)\,ds.$$

Then $i_K(T,V_{\rho_1,\rho_2})=0$.

(3) *Assume that*

$(\mathrm{I}^0_{\rho_1,\rho_2})^\star$ *there exist* $\rho_1,\rho_2>0$ *such that for some* $i\in\{1,2\}$ *we have*

$$f^*_{i,(\rho_1,\rho_2)} > M_i, \tag{2.4}$$

where

$$f^*_{1,(\rho_1,\rho_2)} = \inf\Big\{\frac{f_1(t,u,v)}{\rho_1}:\ (t,u,v)\in[a_1,b_1]\times[0,\rho_1/c_1]\times[-\rho_2/c_2,\rho_2/c_2]\Big\}.$$

$$f^*_{2,(\rho_1,\rho_2)} = \inf\Big\{\frac{f_2(t,u,v)}{\rho_2}:\ (t,u,v)\in[a_2,b_2]\times[-\rho_1/c_1,\rho_1/c_1]\times[0,\rho_2/c_2]\Big\}.$$

Then $i_K(T,V_{\rho_1,\rho_2})=0$.

Remark 2.2. In the previous Lemma the condition $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$ allows the nonlinearities to have a different growth, at the cost of having to deal with a larger domain. Nonlinearities with different growths were considered, with different approaches, in [5, 24, 25, 34].

Using the Lemma 2.1 it is possible to prove, by means of fixed point index, a result regarding the existence of at least one, two or three nontrivial solutions. Note that it is also possible to state a results for four or more nontrivial solutions, in the line of the paper [19].

Theorem 2.3. [10] *The system* (1.5) *has at least one nontrivial solution in* K *if one of the following conditions holds.*

- (S_1) *For* $i=1,2$ *there exist* $\rho_i,r_i\in(0,\infty)$ *with* $\rho_i/c_i<r_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ [*or* $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$], $(\mathrm{I}^1_{r_1,r_2})$ *hold.*
- (S_2) *For* $i=1,2$ *there exist* $\rho_i,r_i\in(0,\infty)$ *with* $\rho_i<r_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ *hold.*

The system (1.5) *has at least two nontrivial solutions in* K *if one of the following conditions holds.*

- (S_3) *For* $i=1,2$ *there exist* $\rho_i,r_i,s_i\in(0,\infty)$ *with* $\rho_i/c_i<r_i<s_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$, [*or* $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$], $(\mathrm{I}^1_{r_1,r_2})$ *and* $(\mathrm{I}^0_{s_1,s_2})$ *hold.*
- (S_4) *For* $i=1,2$ *there exist* $\rho_i,r_i,s_i\in(0,\infty)$ *with* $\rho_i<r_i$ *and* $r_i/c_i<s_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ *and* $(\mathrm{I}^1_{s_1,s_2})$ *hold.*

The system (1.5) *has at least three nontrivial solutions in* K *if one of the following conditions holds.*

(S_5) *For* $i = 1,2$ *there exist* $\rho_i, r_i, s_i, \sigma_i \in (0,\infty)$ *with* $\rho_i/c_i < r_i < s_i$ *and* $s_i/c_i < \sigma_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ *[or* $(\mathrm{I}^0_{\rho_1,\rho_2})^\star$*],* $(\mathrm{I}^1_{r_1,r_2})$, $(\mathrm{I}^0_{s_1,s_2})$ *and* $(\mathrm{I}^1_{\sigma_1,\sigma_2})$ *hold.*

(S_6) *For* $i = 1,2$ *there exist* $\rho_i, r_i, s_i, \sigma_i \in (0,\infty)$ *with* $\rho_i < r_i$ *and* $r_i/c_i < s_i < \sigma_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$, $(\mathrm{I}^1_{s_1,s_2})$ *and* $(\mathrm{I}^0_{\sigma_1,\sigma_2})$ *hold.*

The following result provides some sufficient conditions for the non-existence of nonzero solutions.

Theorem 2.4. [10] *Assume that one of the following conditions holds:*

(1) *For* $i = 1,2$,

$$f_i(t,u_1,u_2) < m_i|u_i| \text{ for every } t \in [0,1] \text{ and } u_i \neq 0. \tag{2.5}$$

(2) *For* $i = 1,2$,

$$f_i(t,u_1,u_2) > M_i u_i \text{ for every } t \in [a_i,b_i] \text{ and } u_i > 0. \tag{2.6}$$

(3) *There exists* $i = 1,2$ *such that* (2.5) *is verified for* f_i *and for* $j \neq i$ *condition* (2.6) *is verified for* f_j*.*

Then there is no non-trivial solution of the system (1.5) *in* K*.*

We now turn our attention back to the fractional system (1.3)-(1.4) and illustrate the applicability of the above results. Firstly we recall the definition of the Caputo derivative, for its properties we refer to the books [3, 8, 28, 30].

Definition 2.5. For a function $y : [0,+\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $2 > \alpha > 1$ is given by

$${}^C D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds, \quad n = [\alpha]+1,$$

where Γ denotes the Gamma function, that is

$$\Gamma(s) = \int_0^{+\infty} x^{s-1} e^{-x} dx,$$

and $[\alpha]$ denotes the integer part of a number α.

Observe that, by direct calculation (see Lemma 2.4 of [23]), the (mild) solution of the linear equation

$${}^C D^\alpha w(t) + y(t) = 0,$$

under the BCs

$$w'(0) = 0,\ \beta\, {}^C D^{\alpha-1} w(1) + w(\eta) = 0,$$

can be written in the form

$$w(t) = \beta \int_0^1 y(s) ds + \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds.$$

The solution of the system (1.3)-(1.4) is understood in the mild sense, that is given by the solution of the associated integral system

$$\begin{aligned} u(t) &= \int_0^1 k_1(t,s) f_1(s,u(s),v(s))\,ds, \\ v(t) &= \int_0^1 k_2(t,s) f_2(s,u(s),v(s))\,ds, \end{aligned} \tag{2.7}$$

where

$$k_i(t,s) = \beta_i + \frac{1}{\Gamma(\alpha_i)} \begin{cases} (\eta_i - s)^{\alpha_i - 1}, s \le \eta_i, \\ 0, \qquad\qquad s > \eta_i, \end{cases} - \frac{1}{\Gamma(\alpha_i)} \begin{cases} (t-s)^{\alpha_i-1}, s \le t, \\ 0, \qquad\qquad s > t; \end{cases}$$

we refer to the work of Lan and Lin [21] for comments on regularity issues for equations involving Caputo derivatives.

Here we focus on the case

$$\beta_i \Gamma(\alpha_i) < (1-\eta_i)^{\alpha_i - 1},$$

and seek solutions of the integral equations that are positive on a subinterval of $[0,1]$ and are allowed to change sign elsewhere. In this case the interval $[a_i,b_i]$ can be chosen equal to $[0,b_i]$, where b_i is such that $\eta_i \le b_i < 1$ and $\beta_i\Gamma(\alpha_i) > (b_i - \eta_i)^{\alpha_i - 1}$.

Upper and lower bounds for $k_i(t,s)$ were given in [23], as follows:

$$\Phi_i(s) = \begin{cases} \frac{(1-\eta_i)^{\alpha_i-1}}{\Gamma(\alpha_i)} - \beta_i, & s > \eta_i, \\ \beta_i + \frac{(\eta_i - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & s \le \eta_i, \end{cases} \tag{2.8}$$

$$c_i = \min\Big\{ \frac{\beta_i\Gamma(\alpha_i) - (b_i-\eta_i)^{\alpha_i-1}}{(1-\eta_i)^{\alpha_i-1} - \beta_i\Gamma(\alpha_i)}, \frac{\beta_i\Gamma(\alpha_i) - (b_i-\eta_i)^{\alpha_i-1}}{\beta_i\Gamma(\alpha_i) + \eta_i^{\alpha_i-1}} \Big\}.$$

Thus we work in the cone

$$K := \{(u,v) \in \tilde{K_1} \times \tilde{K_2}\},$$

where

$$\tilde{K_i} := \{w \in C[0,1] : \min_{t\in[0,b_i]} w(t) \ge c_i \|w\|_\infty\}.$$

Theorems 2.3 and 2.4 can be applied to the system (1.3)-(1.4), provided that the nonlinearities have a suitable growth. We stress that the constants that occur in our theory can either be computed or estimated. In the next example we show precisely this situation. The numbers are rounded to the third decimal place, unless exact.

Example 2.6. Consider the data

$$\alpha_1 = 3/2,\ \beta_1 = 1/5,\ \eta_1 = 3/4,\ \alpha_2 = 5/4,\ \beta_2 = 2/5,\ \eta_2 = 2/3.$$

In this case, one may use the intevals

$$[a_1,b_1] = [0,31/40],\ [a_2,b_2] = [0,41/60],$$

and this choice leads to

$$c_1 = 0.018,\ c_2 = 0.002.$$

We make use of quantities

$$\frac{1}{\hat{M}_i} := \int_{a_i}^{b_i} c_i \Phi_i(s)\,ds, \;\; \frac{1}{\hat{m}_i} := \int_0^1 \Phi_i(s)\,ds$$

and note that

$$M_i \le \hat{M}_i \text{ and } m_i \ge \hat{m}_i.$$

With the data above we obtain

$$\hat{M}_1 = 84.192, \;\; \hat{m}_1 = 1.370, \;\; \hat{M}_2 = 482.545 \text{ and } \hat{m}_2 = 1.058.$$

Acknowledgments. G. Infante was partially supported by G.N.A.M.P.A. - INdAM (Italy). This paper has been presented at the conference New Trends in Numerical Analysis: Theory, Methods, Algorithms and Applications (NETNA2015), dedicated to Professor Francesco A. Costabile on the occasion of his 70th birthday. The authors would like to thank Professor Sandra Pinelas for the opportunity to include this manuscript in the special issue in honour of Professor Emeritus Ioannis P. Stavroulakis.

References

1. Ahmad, B. and Nieto, J. J.: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.* **58** (2009), 1838–1843.
2. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.* **18** (1976) 620–709.
3. Anastassiou, G. A.: *Fractional differentiation inequalities*, Springer, Dordrecht, 2009.
4. Cabada, A. and Infante, G.: Positive solutions of a nonlocal Caputo fractional BVP, *Dynam. Systems Appl.* **23** (2014) 715–722.
5. Cheng, X. and Zhang, Z.: Existence of positive solutions to systems of nonlinear integral or differential equations, *Topol. Methods Nonlinear Anal.* **34** (2009) 267–277.
6. Conti, R.: Recent trends in the theory of boundary value problems for ordinary differential equations, *Boll. Un. Mat. Ital.* **22** (1967) 135–178.
7. Costabile, F. and Napoli, A.: A method for high-order multipoint boundary value problems with Birkhoff-type conditions, *Int. J. Comput. Math.* **92** (2015), 192–200.
8. Diethelm, K.: *The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type*, Lecture Notes in Mathematics, **2004**, Springer-Verlag, Berlin, 2010.
9. Franco, D., Infante, G. and O'Regan, D.: Nontrivial solutions in abstract cones for Hammerstein integral systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **14** (2007), 837–850.
10. Infante, G. and Pietramala, P.: Nonzero radial solutions for a class of elliptic systems with nonlocal BCs on annular domains, *NoDEA Nonlinear Differential Equations Appl.* **22** (2015) 979–1003.
11. Infante, G. and Webb, J. R. L.: Three point boundary value problems with solutions that change sign, *J. Integral Equations Appl.* **15** (2003) 37–57.
12. Infante, G. and Webb, J. R. L.: Loss of positivity in a nonlinear scalar heat equation, *NoDEA Nonlinear Differential Equations Appl.* **13** (2006) 249–261.
13. Guo, D. and Lakshmikantham, V.: *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.
14. Infante, G. and Pietramala, P.: Eigenvalues and non-negative solutions of a system with nonlocal BCs, *Nonlinear Stud.* **16** (2009) 187–196.
15. Karakostas, G. L. and Tsamatos, P. Ch.: Existence of multiple positive solutions for a nonlocal boundary value problem, *Topol. Methods Nonlinear Anal.*, **19** (2002), 109–121.

16. Karakostas, G. L. and Tsamatos, P. Ch.: Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electron. J. Differential Equations* **2002** No. 30 17 pp.
17. Khan, R. A. and ur Rehman, M.: Existence of multiple positive solutions for a general system of fractional differential equations, *Comm. Appl. Nonlinear Anal.* **18** (2011) 25–35.
18. Krasnosel'skiĭ, M. A. and Zabreĭko, P. P.: *Geometrical methods of nonlinear analysis*, Springer-Verlag, Berlin, 1984.
19. Lan, K. Q.: Multiple positive solutions of Hammerstein integral equations with singularities, *Diff. Eqns and Dynam. Syst.* **8** (2000) 175–195.
20. Lan, K. Q.: Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc.* **63** (2001) 690–704.
21. Lan, K. Q. and Lin, W.: Positive solutions of systems of Caputo fractional differential equations, *Commun. Appl. Anal.* **17** (2013) 61–86.
22. Ma,R.: A survey on nonlocal boundary value problems, *Appl.Math.E-Notes* **7** (2007)257–279.
23. Nieto, J. and Pimentel, J.: Positive solutions of a fractional thermostat model, *Bound. Value Probl.* (2013), 2013:5, 11.
24. Precup, R.: Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications, *Mathematical models in engineering, biology and medicine*, 284–293, AIP Conf. Proc., 1124, Amer. Inst. Phys., Melville, NY, 2009.
25. Precup, R.: Existence, localization and multiplicity results for positive radial solutions of semilinear elliptic systems, *J. Math. Anal. Appl.* **352** (2009) 48–56.
26. Ntouyas, S. K.: Nonlocal initial and boundary value problems: a survey, in: *Handbook of differential equations: ordinary differential equations. Vol. II*, 461–557, Elsevier B. V., Amsterdam, 2005.
27. Picone, M.: Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **10** (1908), 1–95.
28. Podlubny, I.: *Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.
29. Ur Rehman, M., Khan, R. A. and Eloe, P.: Positive solutions of nonlocal boundary value problem for higher order fractional differential system, *Dynam. Systems Appl.* **20** (2011) 169–182.
30. Samko, S. G., Kilbas, A. A. and Marichev, O. I.: *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993.
31. Štikonas, A.: A survey on stationary problems, Green's functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control* **19** (2014) 301–334.
32. Webb, J. R. L. and Infante, G.: Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.* **74** (2006) 673–693.
33. Whyburn, W. M.: Differential equations with general boundary conditions, *Bull. Amer. Math. Soc.* **48** (1942) 692–704.
34. Yang, Z.: Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlinear Anal.* **62** (2005) 1251–1265.
35. Zhao, K. and Gong, P.: Positive solutions of Riemann-Stieltjes integral boundary problems for the nonlinear coupling system involving fractional-order differential, *Adv. Difference Equ.* (2014) 2014:254 18 pp.

Gennaro Infante, Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy. *E-mail address*: gennaro.infante@unical.it

Samira Rihani, Faculty of Mathematics, USTHB, El Alia Bab Ezzouar Algiers 16111, Algeria. *E-mail address*: maths_samdz@yahoo.fr