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OSCILLATION OF A CLASS OF NEUTRAL DELAY DYNAMIC EQUATIONS OF FOURTH ORDER

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Abstract. In this paper, the oscillation of a class of fourth order nonlinear neutral delay dynamic equations of the form

$$\left(r(t)\left((y(t)+p(t)y(\alpha(t)))^{\Delta^2}\right)\right)^{\Delta^2}+q(t)f(y(\beta(t)))=0$$

is studied on an arbitrary time scale $\mathbb{T}$, under the assumption

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t<\infty, t_0>0,$$

for various ranges of $p(t)$.

1. Introduction

Stefan Hilger [8] has developed the time scales in his Ph.D work. The study of dynamic equations on time scales allows us to avoid proving results twice, once for differential equations and once again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which is a non-empty closed subset of the real numbers $\mathbb{R}$. In this way the results in this paper not only apply to the set of real numbers or set of integers, but also to more general time scales such as $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}=q^{\mathbb{N}_0}=\{t: t=q^k,\ k\in\mathbb{N}_0\}$ with $q>1$ (which has important applications in quantum theory [9]), $\mathbb{T}=\mathbb{N}_0^2=\{t^2: t\in\mathbb{N}_0\}$, $\mathbb{T}=\{\sqrt{n}: n\in\mathbb{N}_0\}$ etc. For basic notations on time scale calculus, we refer the reader to the monographs [3, 4], the survey paper [1], and the references cited therein.

In this work, the author has studied the oscillatory behaviour of solutions of nonlinear delay dynamic equations of the form

$$\left(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2}\right)^{\Delta^2}+q(t)f(y(\beta(t)))=0, \tag{1.1}$$

where $q,\ r\in C_{rd}(\mathbb{T},\mathbb{R}_+)$, $\alpha,\ \beta\in C_{rd}(\mathbb{T},\mathbb{T})$ such that $\alpha(t)\le t,\ \beta(t)\le t$, and $\lim_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\beta(t)$, $f\in C(\mathbb{R},\mathbb{R})$ is a continuous function with the property $uf(u)>0$ for $u\neq 0$, and $p\in C_{rd}(\mathbb{T},\mathbb{R})$, under the assumption

(H_0) $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t<\infty,\ \ t_0>0.$

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If $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$, then (1.1) reduces to

$$(r(t)((y(t)+p(t)y(\alpha(t)))'')''+q(t)f(y(\beta(t)))=0 \tag{1.2}$$

and

$$\Delta^2\left(r(n)(\Delta^2(y(n)+p(n)y(\alpha(n)))\right)+q(n)f(y(\beta(n)))=0 \tag{1.3}$$

respectively.

In the sequel, we assume the following hypotheses on f,α and β:

(H_1) $f(uv)=f(u)f(v),\ for\ u,\ v\in\mathbb{R}\ and\ u,\ v>0,$

(H_2) $f(-u)=-f(u),\ for\ u\in\mathbb{R},$

(H_3) $there\ exist\ \lambda>0,\ such\ that f(u)+f(v)\geq\lambda f(u+v), for\ u,v\in\mathbb{R}\ and\ u,v>0,$

(H_4) $\alpha\ and\ \beta\ are\ bijective\ functions\ satisfying\ the\ properties:$

$\alpha(\beta(t))=\beta(\alpha(t)),\beta^{-1}(\alpha^{-1}(t))=\alpha^{-1}(\beta^{-1}(t)),\beta(\alpha^{-1}(t))=\alpha^{-1}(\beta(t)),$

$\alpha^{-1}(t)\geq t,\beta^{-1}(t)\geq t,\ for\ every\ right-scattered\ point\ t\in[t_0,\ \infty)_{\mathbb{T}},t_0\geq 0.$

Neutral delay differential equations find numerous applications in electric networks. For example, they are frequently used for the study of distributed networks containing lossless transmission lines which arise in high speed computers where the lossless transmission lines are used to interconnect switching circuits (see for e.g.[7]). The problem of obtaining sufficient conditions to ensure the nonlinear neutral delay differential equations are oscillatory has received a great attention. In this work, an attempt is made to investigate the oscillatory character of all solutions of the nonlinear neutral delay dynamic equations of the form (1.1).

In [14, 16], Parhi and Tripathy have considered the equations (1.2) and (1.3) when $\alpha(t)=t-\alpha$ and $\beta(t)=t-\beta$, and established the sufficient conditions for oscillation and asymptotic behaviour of solutions, under the assumptions

(H_5) $\int\limits_0^\infty \frac{t}{r(t)}dt<\infty,$

and its discrete analogue

(H_6) $\sum\limits_{n=0}^\infty \frac{n}{r(n)}<\infty$

respectively. It is interesting to see the unification of continuous and discrete aspects (1.2) and (1.3) through the dynamic equations on time scales in [13]. But, the problem lies there in the works [13], [14] and [16] concerning an *all solution oscillatory*.

The objective of this work is to establish the sufficient conditions for oscillation of all solutions of (1.1) under the assumption (H_0) on an arbitrary time scale $\mathbb{T}$.

In [13], Panigrahi and Reddy have established the conditions for oscillation and asymptotic behaviour of solutions of (1.1) under the assumption

(H_7) $\int\limits_{t_0}^\infty \frac{\sigma(t)}{r(t)}\Delta t<\infty,\ \ t_0>0$

which is comparable with (H_5) and (H_6) when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ respectively. However, the continuity of the problem is slightly deviated when we compare both works [12] and [13]. The fact is that (1.1) is studied in [12] under the condition

(H_8) $\int\limits_{t_0}^\infty \frac{t}{r(t)}\Delta t=\infty,\ \ t_0>0$

with the following result:

Lemma 1.1. [12] *Let* (H_8) *hold. Let* u *be a real valued delta differentiable function on* $[t_0,\infty)_{\mathbb{T}}$ *such that* $L_4u(t) \leq 0$ *for large* t. *If* $u(t) > 0$ *ultimately, then one of cases (a) and (b) holds for large* t, *and if* $u(t) < 0$ *ultimately, then one of cases (b),(c),(d) and (e) holds for large* t, *where*
(a) $L_1u(t) > 0,\ L_2u(t) > 0$ *and* $L_3u(t) > 0$,
(b) $L_1u(t) > 0,\ L_2u(t) < 0$ *and* $L_3u(t) > 0$,
(c) $L_1u(t) < 0,\ L_2u(t) < 0$ *and* $L_3u(t) > 0$,
(d) $L_1u(t) < 0,\ L_2u(t) < 0$ *and* $L_3u(t) < 0$,
(e) $L_1u(t) < 0,\ L_2u(t) > 0$ *and* $L_3u(t) > 0$,
and $L_0u(t) = u(t)$, $L_1u(t) = L_0^{\Delta}u(t)$, $L_2u(t) = r(t)L_1^{\Delta}u(t)$, $L_3u(t) = L_2^{\Delta}u(t)$, $L_4u(t) = L_3^{\Delta}u(t)$.

and in [13] under the condition (H_7). When Lemma 1.1 is with respect to (H_8), the next question is concerning the problem for (H_0), but not for (H_7). Regarding an all solution oscillatory, (1.1) is studied in [19] using (H_8) and in this work we continue the study when (H_0) holds. Based on the proof of Lemma 1.1, we have the following result for our next discussion:

Lemma 1.2. *Let* (H_0) *hold. Let* u *be a real valued delta differentiable function on* $[t_0,\infty)_{\mathbb{T}}$ *such that* $L_4u(t) \leq 0$ *for large* t. *If* $u(t) > 0$ *ultimately, then one of cases (a)-(d) holds for large* t, *and if* $u(t) < 0$ *ultimately, then one of cases (b)-(g) holds for large* t, *where*
(a) $L_1u(t) > 0,\ L_2u(t) > 0$ *and* $L_3u(t) > 0$,
(b) $L_1u(t) > 0,\ L_2u(t) < 0$ *and* $L_3u(t) > 0$,
(c) $L_1u(t) > 0,\ L_2u(t) < 0$ *and* $L_3u(t) < 0$,
(d) $L_1u(t) < 0,\ L_2u(t) > 0$ *and* $L_3u(t) > 0$,
(e) $L_1u(t) < 0,\ L_2u(t) < 0$ *and* $L_3u(t) > 0$,
(g) $L_1u(t) < 0,\ L_2u(t) < 0$ *and* $L_3u(t) < 0$.

Since we are interested in the oscillatory behaviour of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}} = [t_0,\ \infty) \cap \mathbb{T}$. Let $t_{-1} = inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha(t), \beta(t)\}$. By a solution of (1.1) we mean a nontrivial real valued function y on $[T_y,\ \infty)_{\mathbb{T}}$ such that $(y(t) + p(t)y(\alpha(t)) \in C^2_{rd}(\mathbb{T}, \mathbb{R})$, $\left(r(t)(y(t) + p(t)y(\alpha(t)))^{\Delta^2}\right) \in C^2_{rd}(\mathbb{T}, \mathbb{R})$ and satisfies (1.1), for $T_y \geq t_{-1} > t_0 > 0$. In this paper, we do not consider the solutions that eventually vanish identically. A solution y of (1.1) is said to be *oscillatory*, if it is neither eventually positive nor eventually negative and it is *nonoscillatory* otherwise.

We may note that, (1.1) includes a class of differential or difference equations with delay argument of neutral type. In recent years, there has been an increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions of different classes of neutral dynamic equations. We refer the reader to some of the works [2, 5, 6, 10, 11, 17, 18, 20], and the references cited therein.

2. Preliminary Results

In this section we have the following results for our use in the sequel.

Lemma 2.1. *Let (H_0) hold. Let u be a real valued delta differentiable function on $[t_0,\infty)_{\mathbb{T}}$ such that $L_4u(t)\le 0$, for large t. If $u(t)>0$, then there exists a constant $k>0$ such that $u(t)\ge kR_1(t)$, where $R_1(t)=\int_t^\infty R(s)\Delta s=\int_t^\infty\int_s^\infty \frac{\Delta v}{r(v)}\Delta s$.*

Proof. Suppose that $u(t)>0$, for any large t. Consider $Case(d)$ of Lemma 1.3. For $s>\sigma(t)>t>\theta$, $r(s)u^{\Delta^2}(s)>r(t)u^{\Delta^2}(t)$ implies that

$$\int_t^s u^{\Delta^2}(v)\Delta v>r(t)u^{\Delta^2}(t)\int_t^s\frac{\Delta v}{r(v)}$$

and hence

$$-u^{\Delta}(t)\ge u^{\Delta}(s)-u^{\Delta}(t)=\int_t^s u^{\Delta^2}(v)\Delta v>r(t)u^{\Delta^2}(t)\int_t^s\frac{\Delta v}{r(v)}.$$

Consequently,

$$-u^{\Delta}(t)\ge r(t)u^{\Delta^2}(t)\int_t^\infty\frac{\Delta v}{r(v)}.$$

Hence,

$$-\int_\theta^s u^{\Delta}(t)\Delta t\ge\int_\theta^s r(t)u^{\Delta^2}(t)\int_t^\infty\frac{\Delta v}{r(v)}\Delta t,$$

that is,

$$-u(s)+u(\theta)\ge r(\theta)u^{\Delta^2}(\theta)\int_\theta^s\int_t^\infty\frac{\Delta v}{r(v)}\Delta t.$$

Therefore,

$$u(\theta)\ge r(\theta)u^{\Delta^2}(\theta)\int_\theta^\infty R(t)\Delta t=r(\theta)u^{\Delta^2}(\theta)R_1(\theta).$$

We note that (H_0) implies $\int\limits_{t_0}^{\infty}\frac{\Delta t}{r(t)}<\infty$ and hence $R_1(t)\to 0$ as $t\to\infty$. Since $r(t)u^{\Delta^2}(t)$ is nondecreasing, then for a large $t>t_0$ we can find a constant $k_1>0$ such that the last inequality becomes $u(t)\ge k_1R_1(t)$. Using the fact that $R_1(t)$ is nonincreasing and $u(t)$ is nondecreasing in $Cases(a),(b)$ and (c), we can find constants $k_2,k_3>0$ such that for any large $t>t_0$, $u(t)\ge k_2\ge k_3R_1(t)$.

Let $k=\min\{k_1,k_3\}$. Then for all $Cases(a)-(d)$, $u(t)\ge kR_1(t)$ for any large t. This completes the proof of the lemma. □

Lemma 2.2. [15] *Assume that $p(t)>0$, for $t\in[t_0,\infty)_{\mathbb{T}}$. If*

$$\limsup_{t\to\infty}\int_{\tau(t)}^{\sigma(t)}p(s)\Delta s>1,$$

then the inequality

$$x^{\Delta}+p(t)x(\tau(t))\le 0(\ge 0)$$

doesn't admit any eventually positive (negative) solution.

Proof. The proof of the lemma follows from the proof of Theorem 2.4 [15]. Hence the details are omitted.

□

3. Sufficient Conditions for Oscillation

This section deals with the new oscillation criteria for (1.1). Before stating our main results, we assume the following hypotheses for our use in the sequel:

$$A[s,v] = \int_v^s (s-\sigma(t))\frac{(t-v)}{r(t)}\Delta t, s > \sigma(t) > t > v,$$
$$B[v,u] = \int_u^v (\sigma(u)-u)\frac{(t-u)}{r(t)}\Delta t,\ v > \sigma(t) > t > u,$$
$$C[v,u] = \int_u^v (\sigma(t)-u)\frac{(t-u)}{r(t)}\Delta t, v > \sigma(t) > t > u;$$

Theorem 3.1. *Let* $0 \le p(t) \le a < \infty$ *and* $\beta(t) \le \alpha^2(t)$*, for* $t \in [t_0,\ \infty)_{\mathbb{T}}$*. Assume that* $(H_0) - (H_4)$ *hold. If*
(H_9) $Q(t) = \min\{q(t),\ q(\alpha(t))\},\ for\ t \ge t_0,$
(H_{10}) $\frac{f(u)}{u} \ge M_1 > 0,\ for\ u \ne 0,$
(H_{11}) $\limsup_{s\to\infty} \int_{\alpha(s)}^s Q(\theta)f[A(\beta(\theta),\ \beta(s))]\Delta\theta > \frac{1+f(a)}{\lambda M_1},\ \ a > 0,$
(H_{12}) $\limsup_{\theta\to\infty} \int_{\alpha(\theta)}^\theta Q(v)f[C(\beta(v),\ \beta(\theta))]\Delta v > \frac{1+f(a)}{\lambda M_1},\ \ a > 0,$
(H_{13}) $\limsup_{\theta\to\infty} \int_{\beta^2(\theta)}^{\beta(\theta)} Q(v)f[C(\beta(v),\ \beta(\theta))]\Delta v > \frac{(1+f(a))}{\lambda M_1},\ \ a > 0$
and
(H_{14}) $\limsup_{s\to\infty} \int_{\alpha(s)}^s Q(\theta)f[C(\beta(s),\ \beta(\theta))]\Delta\theta > \frac{(1+f(a))}{\lambda M_1}, a > 0$
hold, then (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a non-oscillatory solution of (1.1) on $[t_0,\ \infty)_{\mathbb{T}}$. Without loss of generality, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large such that $y(t) > 0, y(\alpha(t)), y(\beta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Setting $z(t) = y(t) + p(t)y(\alpha(t))$ in (1.1), we get

$$L_4 z(t) = -q(t)f(y(\beta(t))) \le 0. \tag{3.1}$$

Hence, we can find a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $L_i z(t), i = 1, 2, 3$ are eventually of one sign on $[t_2,\ \infty)_{\mathbb{T}}$. In what follows, we consider $Cases(a) - (d)$ of Lemma 1.3.
Case(c) For $v > \sigma(t) > t > u \ge t_2$, it is easy to verify that

$$\begin{aligned} -z(v) &= -z(u) - (v-u)z^{\Delta}(v) + \int_u^v (\sigma(t)-u)z^{\Delta^2}(t)\Delta t \\ &\le \int_u^v (\sigma(t)-u)z^{\Delta^2}(t)\Delta t, \end{aligned} \tag{3.2}$$

and

$$L_2 z(v) - L_2 z(u) = \int_u^v L_3 z(s)\Delta s \le (v-u)L_3 z(u)$$

implies that $L_2z(v) \leq (v-u)L_3z(u)$, that is, $z^{\Delta^2}(v) \leq \frac{(v-u)}{r(v)}L_3z(u)$. Consequently,

$$\begin{aligned} z(v) &\geq \int_u^v (\sigma(t)-u)\frac{(t-u)}{r(t)}(-L_3z(u))\Delta t \\ &= (-L_3z(u))C[v,u],\ for\ v \geq s > \sigma(t) > t > u \geq t_2. \end{aligned}$$

Letting v and u by $\beta(v)$ and $\beta(\theta)$ respectively in the last inequality, we get

$$z(\beta(v)) > (-L_3z(\beta(\theta)))C[\beta(v),\ \beta(\theta)],\ for\ \beta(v) \geq s > \sigma(t) > t > \beta(\theta) \geq t_2. \quad (3.3)$$

Using (1.1), it happens that

$$\begin{aligned} 0 &= L_4z(t) + q(t)f(y(\beta(t))) + f(a)L_4z(\alpha(t)) + f(a)q(\alpha(t))f(y(\beta(\alpha(t)))) \\ &\geq L_4z(t) + f(a)L_4z(\alpha(t)) + Q(t)[f(y(\beta(t))) + f(a)f(y(\alpha(\beta(t))))] \\ &\geq L_4z(t) + f(a)L_4z(\alpha(t)) + \lambda Q(t)f(z(\beta(t))) \end{aligned}$$

due to (H_1), (H_3), (H_4), *and* (H_9), where we have used the fact that $z(t) \leq y(t) + ay(\alpha(t))$. Upon using (3.3) in the last inequality, we obtain

$$\begin{aligned} 0 &\geq L_4z(v) + f(a)L_4z(\alpha(v)) + \lambda Q(v)f(-L_3z(\beta(\theta))C[\beta(v),\ \beta(\theta)]) \\ &\geq L_4z(v) + f(a)L_4z(\alpha(v)) + \lambda Q(v)f(-L_3z(\beta(\theta)))f(C[\beta(v),\ \beta(\theta)]). \end{aligned}$$

Integrating the above inequality from $\beta^2(\theta)$ to $\beta(\theta)$, we obtain

$$\begin{aligned} \lambda\int_{\beta^2(\theta)}^{\beta(\theta)} Q(v)f(-L_3z(\beta(\theta)))f[C(\beta(v),\beta(\theta))]\Delta v &\leq -L_3z(\beta(\theta)) - f(a)L_3z(\alpha(\beta(\theta))) \\ &\leq -(1+f(a))L_3z(\beta(\theta)). \end{aligned}$$

As a result

$$\int_{\beta^2(\theta)}^{\beta(\theta)} Q(v)f[C(\beta(v),\ \beta(\theta))]\Delta v \leq \frac{(1+f(a))}{\lambda}\frac{(-L_3z(\beta(\theta)))}{f(-L_3z(\beta(\theta)))} \leq \frac{(1+f(a))}{\lambda M_1},$$

a contradiction to (H_{13}) due to (H_{10}).

*Case(d)*Let $v > \sigma(t) > t > u \geq t_2$. Then from (3.2) it follows that

$$\begin{aligned} z(u) &= z(v) - (v-u)z^{\Delta}(v) + \int_u^v (\sigma(t)-u)z^{\Delta^2}(t)\Delta t \\ &\geq \int_u^v (\sigma(t)-u)z^{\Delta^2}(t)\Delta t. \end{aligned}$$

Since

$$L_2z(v) - L_2z(u) = \int_u^v L_3z(s)\Delta s \geq (v-u)L_3z(v),$$

then $L_2z(v) \geq (v-u)L_3z(v)$, that is, $z^{\Delta^2}(v) \geq \frac{(v-u)}{r(v)}L_3z(v)$ and

$$\begin{aligned} z(u) &\geq \int_u^v (\sigma(t)-u)\frac{(t-u)}{r(t)}L_3z(t)\Delta t \\ &\geq L_3z(v)\int_u^v (\sigma(t)-u)\frac{(t-u)}{r(t)}\Delta t \\ &= L_3z(v)C[v,u]. \end{aligned}$$

Letting $u = \beta(\theta)$ and $v = \beta(s)$, we get

$$z(\beta(\theta)) \geq L_3 z(\beta(s)) C[\beta(s),\ \beta(\theta)], \tag{3.4}$$

for $\beta(s) > \beta(\theta) \geq t_2$. Using (1.1), it is easy to verify that

$$\begin{aligned} 0 &= L_4 z(t) + q(t) f(y(\beta(t))) + f(a) L_4 z(\alpha(t)) + f(a) q(\alpha(t)) f(y(\beta(\alpha(t)))) \\ &\geq L_4 z(t) + f(a) L_4 z(\alpha(t)) + Q(t)[f(y(\beta(t))) + f(a) f(y(\alpha(\beta(t))))] \\ &\geq L_4 z(t) + f(a) L_4 z(\alpha(t)) + \lambda Q(t) f(z(\beta(t))) \end{aligned}$$

due to (H_1), (H_3), (H_4), *and* (H_5), where we have used the fact that $z(t) \leq y(t) + a y(\alpha(t))$. Using (3.4), the last inequality becomes

$$\begin{aligned} 0 &\geq L_4 z(\theta) + f(a) L_4 z(\alpha(\theta)) + \lambda Q(\theta) f(L_3 z(\beta(s)) C[\beta(s),\ \beta(\theta)]) \\ &\geq L_4 z(\theta) + f(a) L_4 z(\alpha(\theta)) + \lambda Q(\theta) f(L_3 z(\beta(s))) f(C[\beta(s),\ \beta(\theta)]). \end{aligned}$$

Integrating the above inequality from $\alpha(s)$ to s, we obtain

$$\begin{aligned} \lambda \int_{\alpha(s)}^{s} Q(\theta) f(L_3 z(\beta(s))) f[C(\beta(s),\ \beta(\theta))] \Delta\theta &\leq L_3 z(\alpha(s)) + f(a) L_3 z(\alpha(\alpha(s))) \\ &\leq (1 + f(a)) L_3 z(\alpha^2(s)), \end{aligned}$$

where we have used the fact that $\alpha^2(s)) \leq \alpha(s)$. As a result,

$$\lambda f(L_3 z(\beta(s))) \int_{\alpha(s)}^{s} Q(\theta) f[C(\beta(s),\ \beta(\theta))] \Delta\theta \leq (1 + f(a)) L_3 z(\alpha^2(s)),$$

that is,

$$\int_{\alpha(s)}^{s} Q(\theta) f[C(\beta(s),\ \beta(\theta))] \Delta\theta \leq \frac{(1 + f(a)) L_3 z(\alpha^2(s))}{\lambda f(L_3 z(\alpha^2(s)))} \leq \frac{(1 + f(a))}{\lambda M_1},$$

a contradiction to our hypothesis (H_{14}) due to (H_{10}).

$Cases(a)$ and (b) are similar to $Cases(c)$ and (d), and also can be followed from the proof of Theorem 3.1 [19].

If $y(t) < 0$ for sufficiently large t on $[t_0,\ \infty)_{\mathbb{T}}$, then $-y(t)$ is also a solution of (1.1) due to Remark 1.1. Hence the details are omitted. This completes the proof of the theorem. □

Theorem 3.2. *Let* $-1 \leq p(t) \leq 0$ *and* $\beta(t) \leq \alpha^2(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If* $(H_0) - (H_2), (H_4), (H_{10})$ *and*

(H_{15}) $\limsup\limits_{s\to\infty} \int_{\alpha(s)}^{s} q(\theta) f[A(\beta(\theta),\ \beta(s))] \Delta\theta > \frac{1}{M_1}$,

(H_{16}) $\limsup\limits_{\theta\to\infty} \int_{\alpha(\theta)}^{\theta} q(v) f[C(\beta(v),\ \beta(\theta))] \Delta v > \frac{1}{M_1}$,

(H_{17}) $\limsup\limits_{\theta\to\infty} \int_{\beta^2(\theta)}^{\beta(\theta)} q(v) f[C(\beta(v),\ \beta(\theta))] \Delta v > \frac{1}{M_1}$,

(H_{18}) $\limsup\limits_{s\to\infty} \int_{\alpha(s)}^{s} q(\theta) f[C(\beta(s),\ \beta(\theta))] \Delta\theta > \frac{1}{M_1}$,

(H_{19}) $\tau^n(t) = \tau(\tau^{n-1}(t)), \lim\limits_{n\to\infty} \tau^n(t) < \infty$,

(H_{20}) $\limsup\limits_{v\to\infty} \int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u) f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))]) \Delta u > \frac{1}{M_1}$,

(H_{21}) $\limsup_{v\to\infty}\int_v^{\beta^{-1}(\alpha^2(v))} q(\theta)f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta\theta > \frac{1}{M_1}$,

(H_{22}) $\limsup_{\theta\to\infty}\int_{\alpha^{-1}(\beta(\theta))}^{\theta} q(v)f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta v > \frac{1}{M_1}$

hold, then every solution of (1.1) *oscillates.*

Proof. Suppose on the contrary that $y(t)$ is a nonoscillatory solution of (1.1) on $[t_1,\ \infty)_{\mathbb{T}}$. The case $y(t) < 0$ can similarly be dealt with. In what follows, we apply Lemma 1.3, for $t \in [t_2,\ \infty)_{\mathbb{T}}$ with (3.1). Because $z(t)$ is monotonic, then we consider the cases when $z(t) > 0$ and $z(t) < 0$. Suppose there exists a $t_3 \in [t_2,\ \infty)_{\mathbb{T}}$ such that $z(t) > 0$, for $t \geq t_3$. Then $z(t) \leq y(t)$, for $t \in [t_3,\ \infty]_{\mathbb{T}}$ and

$$L_4z(t) + q(t)f(z(\beta(t))) \leq 0. \tag{3.5}$$

Upon applying Lemma 1.3 to (3.5) and then proceeding as in the proof of Theorem 3.1, we get contradictions to $(H_{15}) - -(H_{18})$ due to $\beta(t) \leq \alpha^2(t) \leq \alpha(t)$.

Next, we suppose that $z(t) < 0$, for $t \in [t_3,\ \infty)_{\mathbb{T}}$. Clearly, $z(t) \geq -y(\alpha(t))$, for $t \geq t_3$ implies that there exists a $t_4 \in [t_3,\ \infty)_{\mathbb{T}}$ such that $y(t) \geq -z(\alpha^{-1}(t))$, for $t \in [t_4,\ \infty)_{\mathbb{T}}$ due to (H_4). By Lemma 1.3, any one of $Cases(b) - (g)$ holds on $[t_4,\ \infty)_{\mathbb{T}}$.

In each of $Cases(e)$ and (g), $lim_{t\to\infty}z(t) = -\infty$. However, $z(t) < 0$ for $t \geq t_4$ implies that $y(t) < y(\tau(t))$ and hence

$$y(t) < y(\tau(t)) < y(\tau^2(t)) < .... < y(\tau^n(t)) < ...,$$

that is, $y(t)$ is bounded due to (H_{19}) and so also $z(t)$, a contradiction.

Consider $Case(b)$. Since for $u \geq v > t_4$,

$$L_2z(u) - L_2z(v) = \int_v^u L_3z(s)\Delta s \geq (u-v)L_3z(u),$$

then $-z^{\Delta^2}(v) \geq \frac{(u-v)}{r(v)}L_3z(u)$ which on integration from u to v, we obtain that

$$z^{\Delta}(u) \geq L_3z(u)\int_u^v \frac{(u-t)}{r(t)}\Delta t,$$

that is,

$$\begin{aligned}-z(u) &\geq -z(\sigma(u)) + (\sigma(u)-u)L_3z(u)\int_u^v \frac{(u-t)}{r(t)}\Delta t\\ &\geq L_3z(u)B[v,u] \geq L_3z(v)B[v,u].\end{aligned}$$

Therefore,

$$-z(\alpha^{-1}(\beta(u))) \geq L_3z(\alpha^{-1}(\beta(v)))B[\alpha^{-1}(\beta(v)),\alpha^{-1}(\beta(u))]. \tag{3.6}$$

Since, (1.1) can be viewed as

$$L_4z(u) + q(u)f(-z(\alpha^{-1}(\beta(u)))) \leq 0, \tag{3.7}$$

then using (3.6) and (H_1), (3.7) yields

$$L_4z(u) + q(u)f(L_3z(\alpha^{-1}(\beta(v))))f(B[\alpha^{-1}(\beta(v)),\alpha^{-1}(\beta(u))]) \leq 0.$$

Integrating the last inequality from $\alpha^{-1}(\beta(v))$ to $\alpha^{-1}(v)$, it follows that

$$f(L_3 z(\alpha^{-1}(\beta(v)))) \quad \int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u) f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))])\Delta u \\ \leq L_3 z(\alpha^{-1}(\beta(v)))).$$

Consequently,

$$\int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u) f(B[\alpha^{-1}(\beta(v)), \alpha^{-1}(\beta(u))])\Delta u \leq \frac{1}{M_1}$$

due to (H_{10}), a contradiction to our hypothesis (H_{20}).

Next, we consider $Case(c)$. From (3.2) it follows that

$$\begin{aligned} -z(u) &= -z(v) + (v-u) z^{\Delta}(v) - \int_u^v (\sigma(t) - u) z^{\Delta^2}(t)\Delta t \\ &\geq -\int_u^v (\sigma(t) - u) z^{\Delta^2}(t)\Delta t, \end{aligned}$$

for $v > \sigma(t) > t > u \geq t_4$. Again

$$L_2 z(v) - L_2 z(u) = \int_u^v L_3 z(s)\Delta s \leq (v-u) L_3 z(u)$$

implies that $L_2 z(v) \leq (v-u) L_3 z(u)$ and hence

$$-z(u) \geq -L_3 z(u) \int_u^v (\sigma(t) - u)\frac{(t-u)}{r(t)}\Delta t = -L_3 z(u) C[v, u].$$

We write the above inequality as

$$-z(\alpha^{-1}(\beta(\theta))) \geq -L_3 z(\alpha^{-1}(\beta(\theta))) C[(\alpha^{-1}(\beta(v))), (\alpha^{-1}(\beta(\theta)))], \tag{3.8}$$

for $(\alpha^{-1}\beta(v)) > \sigma(t) > t > (\alpha^{-1}\beta(\theta)) \geq t_4$. Using the relation (3.8) in (3.7) and using (H_1), we obtain

$$L_4 z(\theta) + q(\theta) f(-L_3 z(\alpha^{-1}(\beta(\theta))) f(C[(\alpha^{-1}(\beta(v))), (\alpha^{-1}(\beta(\theta)))]) \leq 0. \tag{3.9}$$

Integrating (3.9) from v to $\beta^{-1}(\alpha^2(v))$, we get

$$\int_v^{\beta^{-1}(\alpha^2(v))} q(\theta) f(-L_3 z(\alpha^{-1}(\beta(\theta))) f(C[(\alpha^{-1}(\beta(v))), (\alpha^{-1}(\beta(\theta)))])\Delta\theta \\ \leq -L_3 z(\beta^{-1}(\alpha^2(v))),$$

that is,

$$f(-L_3 z(\alpha^{-1}(\beta(v))) \int_v^{\beta^{-1}(\alpha^2(v))} q(\theta) f(C[(\alpha^{-1}(\beta(v))), (\alpha^{-1}(\beta(\theta)))])\Delta\theta \\ \leq -L_3 z(\beta^{-1}(\alpha^2(v))).$$

Consequently,

$$f(-L_3 z(\alpha^{-1}(\beta(v))) \int_v^{\beta^{-1}(\alpha^2(v))} q(\theta) f(C[(\alpha^{-1}(\beta(v))), (\alpha^{-1}(\beta(\theta)))])\Delta\theta \\ \leq -L_3 z(\beta^{-1}(\alpha(v))).$$

Since $\beta^{-1}(\alpha(v)) \leq \alpha^{-1}(\beta(v))$ if and only if $\alpha^2(v) \leq v$ due to (H_4), then the last inequality becomes

$$\int_v^{\beta^{-1}(\alpha^2(v))} q(\theta) f(C[(\alpha^{-1}(\beta(v))), (\alpha^{-1}(\beta(\theta)))])\Delta\theta \leq \frac{-L_3 z(\alpha^{-1}(\beta(v)))}{f(-L_3 z(\alpha^{-1}(\beta(v)))} \leq \frac{1}{M_1}$$

which is a contradiction to (H_{21}).

In $Case(d)$, again we use (3.2) and it follows that

$$-z(v) \geq \int_u^v (\sigma(t)-u)z^{\Delta^2}(t)\Delta t,$$

for $v > \sigma(t) > t > u \geq t_4$. Further,

$$L_2 z(v) - L_2 z(u) = \int_u^v L_3 z(s)\Delta s \geq (v-u)L_3 z(v)$$

implies that $L_2 z(v) \geq (v-u)L_3 z(v)$ and hence

$$-z(v) \geq \int_u^v (\sigma(t)-u)\frac{(t-u)}{r(t)} L_3 z(t)\Delta t \geq L_3 z(v)C[v,u].$$

The above inequality can be written as

$$-z(\alpha^{-1}(\beta(v))) \geq L_3 z(\alpha^{-1}(\beta(v)))C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))], \tag{3.10}$$

for $(\alpha^{-1}\beta(v)) > \sigma(t) > t > (\alpha^{-1}\beta(\theta)) \geq t_4$. Using the relation (3.10) in (3.7) and because of (H_1), we obtain

$$L_4 z(v) + q(v)f(L_3 z(\alpha^{-1}(\beta(v)))f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))]) \leq 0. \tag{3.11}$$

Integrating (3.11) from $\alpha^{-1}(\beta(\theta))$ to θ, we get

$$\int_{\alpha^{-1}(\beta(\theta))}^{\theta} q(v)f(L_3 z(\alpha^{-1}(\beta(v)))f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta v \\ \leq L_3 z(\alpha^{-1}(\beta(\theta))),$$

that is,

$$f(L_3 z(\alpha^{-1}(\beta(\theta))) \int_{\alpha^{-1}(\beta(\theta))}^{\theta} q(v)f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta v \\ \leq L_3 z(\alpha^{-1}(\beta(\theta))).$$

Therefore,

$$\int_{\alpha^{-1}(\beta(\theta))}^{\theta} q(v)f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta v \leq \frac{L_3 z(\alpha^{-1}(\beta(\theta)))}{f(L_3 z(\alpha^{-1}(\beta(\theta)))} \leq \frac{1}{M_1}$$

implies a contradiction to (H_{22}). This completes the proof of the theorem. □

Theorem 3.3. *Let $-\infty < -b \leq p(t) \leq -1, b > 0$ and $\beta(t) \leq \alpha^2(t)$, for $t \in [t_0,\infty)_{\mathbb{T}}$. Assume that $(H_0)-(H_2),(H_4)$ and (H_{10}) hold. Furthermore, if*

(H_{23}) $\limsup_{s\to\infty} \int_{\alpha(s)}^{s} q(\theta)f[A(\beta(\theta),\ \beta(s))]\Delta\theta > \frac{1}{f(b^{-1})M_1}$,

(H_{24}) $\limsup_{\theta\to\infty} \int_{\alpha(\theta)}^{\theta} q(v)f[C(\beta(v),\ \beta(\theta))]\Delta v > \frac{1}{f(b^{-1})M_1}$,

(H_{25}) $\limsup_{\theta\to\infty} \int_{\beta^2(\theta)}^{\beta(\theta)} q(v)f[C(\beta(v),\ \beta(\theta))]\Delta v > \frac{1}{f(b^{-1})M_1}$,

(H_{26}) $\limsup_{s\to\infty} \int_{\alpha(s)}^{s} q(\theta)f[C(\beta(s),\ \beta(\theta))]\Delta\theta > \frac{1}{f(b^{-1})M_1}$,

(H_{27}) $\limsup_{v\to\infty} \int_{\alpha^{-1}(\beta(v))}^{\alpha^{-1}(v)} q(u)f(B[\alpha^{-1}(\beta(v)),\alpha^{-1}(\beta(u))])\Delta u > \frac{1}{f(b^{-1})M_1}$,

(H_{28}) $\limsup_{v\to\infty} \int_{v}^{\beta^{-1}(\alpha^2(v))} q(\theta)f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta\theta > \frac{1}{f(b^{-1})M_1}$,

(H_{29}) $\limsup_{\theta\to\infty} \int_{\alpha^{-1}(\beta(\theta))}^{\theta} q(v)f(C[(\alpha^{-1}(\beta(v))),(\alpha^{-1}(\beta(\theta)))])\Delta v > \frac{1}{f(b^{-1})M_1}$,

(H_{30}) $\int_{t_0}^{\infty} q(t)\Delta t = +\infty, \quad t_0 > 0,$
and
(H_{31}) $\limsup_{s\to\infty} \int_{\beta(s)}^{\alpha^{-1}(\beta(s))} q(\theta) f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))])\Delta\theta > \frac{1}{M_1 f(b^{-1})}$
hold, then (1.1) *is oscillatory.*

Proof. The proof of the theorem follows from the proof of Theorem 3.2. We consider $Cases(e)$ and (g) of Lemma 1.3 only when $z(t) < 0$, for $t \in [t_3, \infty)_{\mathbb{T}}$, that is, there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $y(t) \geq (-\frac{1}{b})z(\alpha^{-1}(t))$, for $t \in [t_4, \infty)_{\mathbb{T}}$ due to (H_4) and hence (1.1) reduces to

$$L_4 z(t) + q(t) f(b^{-1}) f(-z(\alpha^{-1}(\beta(t)))) \leq 0. \tag{3.12}$$

In $Case(e)$, $z(t)$ is nonincreasing. So, we can find $t_5 > t_4$ and $L > 0$ such that $z(t) \leq -L$, for $t \geq t_5$. Using (H_1) and therefore, (3.12) yields

$$L_4 z(t) + f(b^{-1}) f(L) q(t) \leq 0, \quad t \geq t_5.$$

Integrating the above inequality from t_5 to ∞, we obtain a contradiction to (H_{30}).

Assume that $Case(g)$ of Lemma 1.3 holds. Proceeding as in $Case(a)$ of Theorem 3.1, we obtain

$$z^{\Delta^2}(u) \leq \frac{(u-v)}{r(u)} L_3 z(v), \tag{3.13}$$

for $u > v > t_4$. For $s > \sigma(t) > t > t_4$, it is easy to verify that

$$z(s) = z(t_4) + (s - t_4) z^{\Delta}(t_2) + \int_{t_4}^{s} (s - \sigma(t)) z^{\Delta^2}(t)\Delta t.$$

Therefore, for $s > v \geq t_4$

$$\begin{aligned} z(s) &\leq \int_{t_4}^{s} (s - \sigma(t)) z^{\Delta^2}(t)\Delta t \\ &\leq \int_{t_4}^{s} (s - \sigma(t)) \frac{(t-v)}{r(t)} L_3 z(v)\Delta t \\ &\leq L_3 z(v) \int_{v}^{s} (s - \sigma(t)) \frac{(t-v)}{r(t)} \Delta t = A[s, v] L_3 z(v) \end{aligned}$$

due to (3.13). Consequently,

$$z(\alpha^{-1}(\beta(\theta))) \leq L_3 z(\alpha^{-1}(\beta(s))) A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))]. \tag{3.14}$$

Using (3.14) in (3.12), it follows that

$$L_4 z(\theta) + q(\theta) f(-\frac{1}{b}) f(L_3 z(\alpha^{-1}(\beta(s)))) f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))]) \leq 0$$

due to (H_1). Integrating the last inequality from $\beta(s)$ to $\alpha^{-1}(\beta(s))$, we obtain that

$$\begin{aligned} f(\tfrac{1}{b}) f(-L_3 z(\alpha^{-1}(\beta(s)))) &\int_{\beta(s)}^{\alpha^{-1}(\beta(s))} q(\theta) f(A[\alpha^{-1}(\beta(\theta)), \alpha^{-1}(\beta(s))])\Delta\theta \\ &\leq -L_3 z(\alpha^{-1}(\beta(s))), \end{aligned}$$

that is,

$$\int_{\beta(s)}^{\alpha^{-1}(\beta(s))} q(\theta)f(A[\alpha^{-1}(\beta(\theta)),\alpha^{-1}(\beta(s))])\Delta\theta \le \frac{-L_3z(\alpha^{-1}(\beta(s)))}{f(\frac{1}{b})f(-L_3z(\alpha^{-1}(\beta(s))))}$$
$$\le \frac{1}{M_1 f(\frac{1}{b})},$$

a contradiction to (H_{31}). This completes the proof of the theorem. □

Theorem 3.4. *Let $0 \le p(t) \le a < \infty$, for $t \in [t_0,\ \infty)_{\mathbb{T}}$. If $(H_0)-(H_4),(H_9)$ and*
(H_{32}) $\int_T^\infty Q(t)f(R_1(\beta(t)))\Delta t = +\infty, T > t_0,$
(H_{33}) $\int_T^\infty \frac{(\sigma(t)-t)^2}{r(\sigma^2(t))}Q(t)f(R_1(\beta(t)))\Delta t = \infty, T > t_0$
hold, then (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 3.1, we consider $Cases(a)-(d)$ of Lemma 1.3. For the said cases,

$$L_4z(t) + f(a)L_4z(\alpha(t)) + \lambda Q(t)f(z(\beta(t))) \le 0 \tag{3.15}$$

holds true, for $t \ge t_3 > t_2$. To (3.15), we apply Lemma 2.1 for $Cases(a),(b)$ and (d) of Lemma 1.3 and therefore,

$$L_4z(t) + f(a)L_4z(\alpha(t)) + \lambda f(k)Q(t)f(R_1(\beta(t))) \le 0 \tag{3.16}$$

due to (H_1). Integrating (3.16) from t_3 to ∞, we get a contradiction to (H_{32}).

For $Case(c)$ of Lemma 1.3, we can write (3.16) as

$$\begin{aligned}
\lambda f(k)Q(t)f(R_1(\beta(t))) &\le -\frac{L_3z(\sigma(t)) - L_3z(t) + f(a)L_3z(\alpha(\sigma(t))) - f(a)L_3z(\alpha(t))}{\sigma(t)-t}\\
&\le -\frac{L_3z(\sigma(t)) + f(a)L_3z(\alpha(\sigma(t)))}{\sigma(t)-t}\\
&\le -\frac{(1+f(a))L_3z(\sigma(t))}{\sigma(t)-t}\\
&= -(1+f(a))\frac{L_2z(\sigma^2(t)) - L_2z(\sigma(t))}{(\sigma(t)-t)^2}\\
&\le -(1+f(a))\frac{L_2z(\sigma^2(t))}{(\sigma(t)-t)^2}\\
&= -(1+f(a))\frac{r(\sigma^2(t))z^{\Delta^2}(\sigma^2(t))}{(\sigma(t)-t)^2},
\end{aligned}$$

for $t \ge t_3 > t_2$. Consequently,

$$\frac{\lambda f(k)}{(1+f(a))}\frac{(\sigma(t)-t)^2}{r(\sigma^2(t))}Q(t)f(R_1(\beta(t))) \le z^{\Delta^2}(\sigma^2(t)). \tag{3.17}$$

Integrating (3.17) from t_3 to ∞, we get a contradiction to (H_{33}). Hence the theorem is proved. □

Theorem 3.5. *Let $-1 \le p(t) \le 0$ and $\beta(\alpha^{-1}(t)) < \sigma^2(t)$, for $t \in [t_0,\infty)_{\mathbb{T}}$. Assume that $(H_0)-(H_2),(H_4),(H_{10})$ and (H_{30}) hold. Furthermore, if*
(H_{34}) $\int_T^\infty q(t)f(R_1(\beta(t)))\Delta t = +\infty, T > t_0,$
(H_{35}) $\int_T^\infty \frac{(\sigma(t)-t)^2}{r(\sigma^2(t))}q(t)f(R_1(\beta(t)))\Delta t = \infty, T > t_0,$

(H_{36}) $\limsup\limits_{t\to\infty} \int_{\beta(\alpha^{-1}(t)}^{\sigma(t)} \frac{q(s)}{r(s)}(\sigma(s)-s)^3\Delta s > \frac{1}{M_1}$
and
(H_{37}) $\liminf\limits_{t\to\infty} \frac{(\sigma(t)-t)^4 q(t)}{r(\sigma^2(t))} > \frac{1}{M_1}$
hold, then every solution of (1.1) *oscillates.*

Proof. On the contrary, we proceed as in Theorem 3.2 to obtain (3.5), for $t \geq t_3$. The rest of this case follows from the proof of Theorem 3.4.

When $z(t) < 0$, for $t \geq t_3$, we consider $Cases(b), (c)$ and (d) of Lemma 1.3 only. Consider $Case(b)$. Using (H_4) in (3.7), it follows that

$$\begin{aligned} q(t)f(-z(\beta(\alpha^{-1}(t)))) &\leq -L_3^{\Delta} z(t) \\ &= \frac{-L_3 z(\sigma(t)) + L_3 z(t)}{\sigma(t)-t} \\ &\leq \frac{L_3 z(t)}{(\sigma(t)-t)} = \frac{L_2^{\Delta} z(t)}{(\sigma(t)-t)} \\ &\leq \frac{-L_2 z(t)}{(\sigma(t)-t)^2}, \end{aligned}$$

for $t \geq t_4 > t_3$. Consequently,

$$\frac{(\sigma(t)-t)^2 q(t)}{r(t)} f(-z(\beta(\alpha^{-1}(t)))) \leq -z^{\Delta^2}(t) \leq \frac{z^{\Delta}(t)}{(\sigma(t)-t)}$$

implies that

$$z^{\Delta}(t) + \frac{(\sigma(t)-t)^3 q(t)}{r(t)} f(z(\beta(\alpha^{-1}(t)))) \geq 0,$$

and because of (H_{10}), the above inequality reduces to

$$z^{\Delta}(t) + M_1 \frac{(\sigma(t)-t)^3 q(t)}{r(t)} z(\beta(\alpha^{-1}(t))) \geq 0 \tag{3.18}$$

which in turn concludes that (3.18) can not have an eventually negative solution (because of Lemma 2.2) due to (H_{36}), a contradiction.

In $Case(c)$, we use the same type of argument as in $Case(b)$ and we obtain the inequality

$$q(t)f(-z(\beta(\alpha^{-1}(t)))) \leq \frac{-r(\sigma^2(t))z(\sigma^2(t))}{(\sigma(t)-t)^4}.$$

Using (H_{10}) to the above inequality, we get

$$z(\sigma^2(t)) - M_1 \frac{(\sigma(t)-t)^4 q(t)}{r(\sigma^2(t))} z(\beta(\alpha^{-1}(t))) \leq 0. \tag{3.19}$$

Applying $\beta(\alpha^{-1}(t)) < \sigma^2(t)$ to (3.19), it follows that

$$\left[1 - M_1 \frac{(\sigma(t)-t)^4 q(t)}{r(\sigma^2(t))}\right] z(\beta(\alpha^{-1}(t))) \leq 0$$

which is a contradiction to (H_{37}). $Case(d)$ follows from $Case(e)$ of Theorem 3.3. Hence, the proof of the theorem is complete. □

Theorem 3.6. *Let* $-\infty \leq -b \leq p(t) \leq -1, b > 0$ *and* $\beta(\alpha^{-1}(t)) < \sigma^2(t)$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *Assume that* $(H_0)-(H_2)$, (H_4), (H_{10}), (H_{30}), (H_{34}) *and* (H_{35}) *hold. Furthermore, if*

(H_{38}) $\limsup_{t\to\infty} \int_{\beta(\alpha^{-1}(t)}^{\sigma(t)} \frac{q(s)}{r(s)}(\sigma(s)-s)^3 \Delta s > \frac{1}{M_1 f(b^{-1})}$

and

(H_{39}) $\liminf_{t\to\infty} \frac{(\sigma(t)-t)^4 q(t)}{r(\sigma^2(t))} > \frac{1}{M_1 f(b^{-1})}$

hold, then every solution of (1.1) *oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 3.5. In case $z(t) < 0$, $Cases(e)$ and (g) of Lemma 1.3 can similarly be dealt with $Case(d)$ of Theorem 3.5. Hence the details are omitted. This completes the proof of the theorem. □

4. Discussion and Examples

Often, it is more challenging to study an all solution oscillatory problem (linear/nonlinear) than a problem (linear/nonlinear) dealing with asymptotic solutions. The later problem may get usual procedure to study than the former one. Even though, (1.1) is highly nonlinear, still all our results are hold true for linear, sublinear and as well as superlinear.

This work deserves a different approach to that of [13] as long as oscillation results are concerned. However, existence of nonoscillation results we take into account. It would be interesting to work out the results of this work for (1.2) and (1.3) respectively. In the following examples, we illustrate our main result:

Example 4.1. Let $\mathbb{T} = \mathbb{Z}$. Consider

$$\triangle^2(ne^n \triangle^2(y(n) + p(n)y(n-1))) + q(n)G(y(n-3)) = 0, \tag{4.1}$$

where $n > 3$, $p(n) = (e^{-2} + e^{-n})$, $q(n) = (e^2-1)^2(e+1)(2e+ne+n)e^n - (e+1)^2(n+1)$, $r(n) = ne^n$ and $G(u) = \frac{4u}{e^2} = \beta u$. Clearly, all conditions of Theorem 3.2 are satisfied. Hence (4.1) is oscillatory. Indeed, $y(n) = (-1)^n$ is one of the oscillatory solutions of (4.1).

Example 4.2. For $\mathbb{T} = \mathbb{R}$, consider

$$\left(t^3\left(y(t) + \left(1 + \frac{1}{t}\right)y(t-\pi)\right)''\right)'' + t^2 y\left(t - 4\pi\right)) = 0, \tag{4.2}$$

for $t > \pi$, where $1 \leq 1 + \frac{1}{t} = p(t) \leq 2$, $r(t) = t^3, \lambda = 1$ and $f(u) = u$. Clearly, all the conditions of Theorem 3.1 are satisfied. Hence (4.2) is oscillatory. In particular, $y(t) = sin\ t$ is such an oscillatory solution of (4.2).

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