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A TRULY CONFORMABLE CALCULUS ON TIME SCALES

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Abstract. We introduce the definition of conformable derivative on time scales and develop its calculus. Fundamental properties of the conformable derivative and integral on time scales are proved. Linear conformable differential equations with constant coefficients are investigated, as well as hyperbolic and trigonometric functions.

1. Introduction

Local, limit-based, definitions of a so-called conformable derivative on time scales have been recently formulated in [6] by

$$T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}\, t^{1-\alpha}, \quad \alpha \in (0,1], \tag{1.1}$$

and then in [12] by

$$T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t)^\alpha - t^\alpha}, \quad \alpha \in (0,1]. \tag{1.2}$$

Note that if f is Δ-diferentiable at a right-scattered point $t \in \mathbb{T}^\kappa_{[0,\infty)}$ [7], then f is α-differentiable in both cases: for the first definition (1.1) we have

$$T_\alpha(f)(t) = t^{1-\alpha} f^\Delta(t) \tag{1.3}$$

while for the second definition (1.2) one has

$$T_\alpha(f^\Delta)(t) = \frac{\sigma(t) - t}{\sigma^\alpha(t) - t^\alpha} f^\Delta(t), \tag{1.4}$$

where $f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}$. The conformable calculus in the time scale $\mathbb{T} = \mathbb{R}$ is now a well-developed subject: see, e.g., [1, 2, 9] and references therein. For results on arbitrary time scales see [4, 5, 10]. However, the adjective *conformable* may not be appropriate, because $T_0 f \neq f$, that is, letting $\alpha \to 0$ does not result in the identity operator. This is also the case for the recent results of [11]. Moreover, according to (1.3) and (1.4), the variable t must satisfy $t \geq 0$. With this in mind, in this paper we extend the calculus of [3], by considering a truly conformable derivative of order α, $0 \leq \alpha \leq 1$, on an arbitrary time scale $\mathbb{T}$.

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2. Preliminaries

We briefly recall the necessary concepts from the time-scale calculus [7, 8]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The graininess function $\mu : \mathbb{T} \to [0, +\infty[$ is given by

$$\mu(t) = \sigma(t) - t.$$

If $\sigma(t) > t$, then t is said right-scattered, while if $\rho(t) < t$, then t is left-scattered. Moreover, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$, then function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma = f \circ \sigma$.

Definition 2.1. A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $k : [0, 1] \times \mathbb{T} \to [0, \infty)$ is rd-continuous if $k(\alpha, \cdot) : \mathbb{T} \to [0, \infty)$ is rd-continuous for all $\alpha \in [0, 1]$ and $k(\cdot, t) : [0, 1] \to [0, \infty)$ is continuous for all $t \in \mathbb{T}$.

The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by C_{rd}. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$.

The next definition serves as the basis to our notion of conformable differential operator in Section 3.

Definition 2.2 (See [7, 8])**.** Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \epsilon\, |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta derivative of f at t.

3. Main results

We begin by introducing the notion of conformable differential operator of order $\alpha \in [0, 1]$ on an arbitrary time scale $\mathbb{T}$. See Remark 1.5 of [3].

Definition 3.1 (Conformable delta differential operator of order α)**.** Let $\mathbb{T}$ be a time scale and let $\alpha \in [0, 1]$. An operator Δ^α is conformable if and only if Δ^0 is the identity operator and Δ^1 is the standard differential operator on $\mathbb{T}$. Precisely, operator Δ^α is conformable if and only if for a differentiable function f in the sense of Definition 2.2, one has $\Delta^0 f = f$ and $\Delta^1 f = f^\Delta$.

Proposition 3.2 gives an extension of [3] to time scales $\mathbb{T}$: for $\mathbb{T} = \mathbb{R}$, (3.2) subject to (3.1) gives Definition 1.3 of [3].

Proposition 3.2 (A conformable derivative Δ^α on time scales)**.** *Let $\mathbb{T}$ be a time scale, $\alpha \in [0,1]$, and $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to [0,\infty)$ be rd-continuous functions (see Definition 2.1) such that*

$$\begin{gathered} \lim_{\alpha\to 0^+} \kappa_1(\alpha,t) = 1, \quad \lim_{\alpha\to 0^+} \kappa_0(\alpha,t) = 0, \\ \lim_{\alpha\to 1^-} \kappa_1(\alpha,t) = 0, \quad \lim_{\alpha\to 1^-} \kappa_0(\alpha,t) = 1, \\ \kappa_1(\alpha,t) \neq 0, \quad \kappa_0(\alpha,t) \neq 0, \quad \alpha \in (0,1], \end{gathered} \tag{3.1}$$

for all $t \in \mathbb{T}$. Then, the differential operator $\Delta^\alpha f$, defined by

$$\Delta^\alpha f(t) = \kappa_1(\alpha,t) f(t) + \kappa_0(\alpha,t) f^\Delta(t) \tag{3.2}$$

in the class of Δ-differentiable functions f, is conformable in the sense of Definition 3.1.

Proof. The result is a trivial consequence of (3.1)–(3.2): $\Delta^0 f = f$ and $\Delta^1 f = f^\Delta$. □

Remark 3.3. The parameter α has a crucial role. Indeed, α is the order of the operator. Note that given a concrete conformable operator, α is a given fixed constant between zero and one. For example, if $\alpha = 1$, then we get the Hilger derivative [7].

Remark 3.4. Let $\alpha \in (0,1]$, $\kappa_1(\alpha,t) \equiv 0$, and $\kappa_0(\alpha,t) = t^{1-\alpha}$. Then, formally, we recover (1.3) from (3.2). However, such choice of κ_0 and κ_1 is not allowed by (3.1) because (1.3) is not conformable in agreement with Definition 3.1.

Many examples of conformable derivatives on time scales are easily obtained from Proposition 3.2.

Example 3.5. One can take $\kappa_1 \equiv (1-\alpha)\omega^\alpha$ and $\kappa_0 \equiv \alpha\omega^{1-\alpha}$ for $\omega \in (0,\infty)$ or $\kappa_1(\alpha,t) = (1-\alpha)|t|^\alpha$ and $\kappa_0(\alpha,t) = \alpha|t|^{1-\alpha}$ on $\mathbb{T} \setminus \{0\}$ in Proposition 3.2. In this last case,

$$\Delta^\alpha f(t) = (1-\alpha)|t|^\alpha f(t) + \alpha|t|^{1-\alpha} f^\Delta(t).$$

Example 3.6. Similarly to Example 3.5,

$$\Delta^\alpha f(t) = \cos\left(\frac{\alpha\pi}{2}\right)|t|^\alpha f(t) + \sin\left(\frac{\alpha\pi}{2}\right)|t|^{1-\alpha} f^\Delta(t)$$

is a conformable derivative.

Remark 3.7. Let $\alpha, \beta \in [0,1]$. Note that, in general, $\Delta^\alpha\Delta^\beta \neq \Delta^\beta\Delta^\alpha$. Indeed, let functions κ_i, $i = 0, 1$, be Δ_t-differentiable and continuous with respect to α and f be twice Δ-differentiable. We have $\Delta^\alpha f(t) = \kappa_1(\alpha,t) f(t) + \kappa_0(\alpha,t) f^\Delta(t)$ and

$\Delta^\beta f(t) = \kappa_1(\beta,t)f(t) + \kappa_0(\beta,t)f^\Delta(t)$. Therefore,

$$\begin{aligned}
\Delta^\beta\Delta^\alpha f(t) &= \kappa_1(\beta,t)\left(\kappa_1(\alpha,t)f(t)+\kappa_0(\alpha,t)f^\Delta(t)\right)\\
&\quad+\kappa_0(\beta,t)\left(\kappa_1(\alpha,t)f(t)+\kappa_0(\alpha,t)f^\Delta(t)\right)^\Delta\\
&=\kappa_1(\beta,t)\kappa_1(\alpha,t)f(t)+\kappa_1(\beta,t)\kappa_0(\alpha,t)f^\Delta(t)\\
&\quad+\kappa_0(\beta,t)\big[\kappa_1^\Delta(\alpha,t)f^\sigma(t)+\kappa_1(\alpha,t)f^\Delta(t)\\
&\quad+\kappa_0^\Delta(\alpha,t)f^{\Delta^\sigma}(t)+\kappa_0(\alpha,t)f^{\Delta^2}(t)\big]\\
&=\kappa_1(\beta,t)\kappa_1(\alpha,t)f(t)+\kappa_1(\beta,t)\kappa_0(\alpha,t)f^\Delta(t)\\
&\quad+\kappa_0(\beta,t)\kappa_1^\Delta(\alpha,t)f^\sigma(t)+\kappa_0(\beta,t)\kappa_1(\alpha,t)f^\Delta(t)\\
&\quad+\kappa_0(\beta,t)\kappa_0^\Delta(\alpha,t)f^{\Delta^\sigma}(t)+\kappa_0(\beta,t)\kappa_0(\alpha,t)f^{\Delta^2}(t).
\end{aligned}$$

Similar calculations lead us to

$$\begin{aligned}
\Delta^\alpha\Delta^\beta f(t) &= \kappa_1(\alpha,t)\kappa_1(\beta,t)f(t)+\kappa_1(\alpha,t)\kappa_0(\beta,t)f^\Delta(t)\\
&\quad+\kappa_0(\alpha,t)\kappa_1^\Delta(\beta,t)f^\sigma(t)+\kappa_0(\alpha,t)\kappa_1(\beta,t)f^\Delta(t)\\
&\quad+\kappa_0(\alpha,t)\kappa_0^\Delta(\beta,t)f^{\Delta^\sigma}(t)+\kappa_0(\alpha,t)\kappa_0(\beta,t)f^{\Delta^2}(t).
\end{aligned}$$

If $\kappa_0(\alpha,t)\kappa_1^\Delta(\beta,t) \neq \kappa_0(\beta,t)\kappa_1^\Delta(\alpha,t)$ or $\kappa_0(\beta,t)\kappa_0^\Delta(\alpha,t) \neq \kappa_0(\alpha,t)\kappa_0^\Delta(\beta,t)$, then $\Delta^\beta\Delta^\alpha f(t) \neq \Delta^\alpha\Delta^\beta f(t)$.

Example 3.8. Let $\mathbb{T}$ be a time scale, $\kappa_1 \equiv (1-\alpha)\omega^\alpha$ and $\kappa_0 \equiv \alpha\omega^{1-\alpha}$, where $\omega > 0$, $\alpha = \frac{1}{2}$ and $\beta = 1$. If $t \in \mathbb{T}$, $t \geq 0$, and $\sigma(t) \neq t$, then

$$\Delta\Delta^{\frac{1}{2}} f(t) = \frac{1}{2}\left[\frac{\sigma^{\frac{1}{2}}(t)-t^{\frac{1}{2}}}{\mu(t)}\left(f^\sigma(t)+f^{\Delta^\sigma}(t)\right)+t^{\frac{1}{2}}\left(f^\Delta(t)+f^{\Delta^2}(t)\right)\right].$$

On the other hand, we have

$$\Delta^{\frac{1}{2}}\Delta f(t) = \frac{1}{2}t^{\frac{1}{2}}\left(f^\Delta(t)+f^{\Delta^2}(t)\right).$$

In this case, $\Delta\Delta^{\frac{1}{2}} f(t) \neq \Delta^{\frac{1}{2}}\Delta f(t)$.

Definition 3.9 (Conformable exponential function on time scales)**.** Let $\alpha \in (0,1]$, $s,t \in \mathbb{T}$ with $s \leq t$ and let function $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Let $\kappa_0, \kappa_1 : [0,1]\times\mathbb{T} \to [0,\infty)$ be rd-continuous and satisfy (3.1) with $1+\mu(t)\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$. Then, the conformable exponential function on the time scale $\mathbb{T}$ with respect to Δ^α in (3.2) is defined to be

$$E_p(t,s) = e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s), \quad E_0(t,s) = e_{\frac{-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s), \tag{3.3}$$

where $e_{q(t)}(t,s)$ denotes the exponential function on the time scale $\mathbb{T}$ – see definition (2.30) in [7].

Note that if $\mathbb{T} = \mathbb{R}$, then

$$E_p(t,s) = e^{\displaystyle\int_s^t \frac{p(\tau)-\kappa_1(\alpha,\tau)}{\kappa(\alpha,\tau)}d\tau} \quad \text{and} \quad E_0(t,s) = e^{-\displaystyle\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa(\alpha,\tau)}d\tau}.$$

3.1. Fundamental properties of the conformable operators. Using (3.2) and (3.3), we begin by proving several basic but important results.

Theorem 3.10 (Basic properties of conformable derivatives)**.** *Let the conformable differential operator Δ^α be given as in (3.2), where $\alpha \in [0,1]$. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to [0,\infty)$ be rd-continuous and satisfy (3.1) with $1+\mu(t)\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0} \neq 0$ for all $t \in \mathbb{T}^\kappa$. Assume functions f and g are differentiable, as needed. Then,*

(i) $\Delta^\alpha(af+bg) = a\Delta^\alpha(f) + b\Delta^\alpha(g)$ *for all* $a,b \in \mathbb{R}$*;*
(ii) $\Delta^\alpha c = c\kappa_1(\alpha,\cdot)$ *for all constants* $c \in \mathbb{R}$*;*
(iii) $\Delta^\alpha(fg) = f\Delta^\alpha(g) + g^\sigma\Delta^\alpha(f) - fg^\sigma\kappa_1(\alpha,\cdot)$*;*
(iv) $\Delta^\alpha\left(\frac{f}{g}\right) = \frac{g\Delta^\alpha(f)-f\Delta^\alpha(g)}{gg^\sigma} + \frac{f}{g}\kappa_1(\alpha,\cdot)$*;*
(v) $\Delta^\alpha E_p(t,s) = p(t)E_p(t,s)$ *for all* $\alpha \in (0,1]$*;*
(vi) $\Delta^\alpha\left(\int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\right) = f(t)E_0(\sigma(t),t)$ *for all* $\alpha \in (0,1]$*.*

Proof. Relations (i) and (ii) are obvious. From (3.2), it also follows (iii)–(vi):

(iii)

$$\begin{aligned}
\Delta^\alpha(fg) &= \kappa_0(fg^\Delta + f^\Delta g^\sigma) + \kappa_1(fg) \\
&= f\kappa_0 g^\Delta + g^\sigma\kappa_0 f^\Delta + \kappa_1(fg) \\
&= f(\kappa_0 g^\Delta + \kappa_1 g) + g^\sigma(\kappa_0 f^\Delta + \kappa_1 f) - g^\sigma\kappa_1 f \\
&= f\Delta^\alpha g + g^\sigma\Delta^\alpha f - g^\sigma\kappa_1 f;
\end{aligned}$$

(iv)

$$\begin{aligned}
\Delta^\alpha\left(\frac{f}{g}\right) &= \kappa_0\left(\frac{f^\Delta g - fg^\Delta}{gg^\sigma}\right) + \kappa_1\left(\frac{f}{g}\right) \\
&= \frac{\kappa_0(f^\Delta g - fg^\Delta)}{gg^\sigma} + \kappa_1\left(\frac{f}{g}\right) \\
&= \frac{(\kappa_0 f^\Delta + \kappa_1 f)g - \kappa_1 fg - \left(\kappa_0 g^\Delta + \kappa_1 g\right)f + f\kappa_1 g}{gg^\sigma} + \kappa_1\left(\frac{f}{g}\right) \\
&= \frac{g\Delta^\alpha f - f\Delta^\alpha g}{gg^\sigma} + \kappa_1\left(\frac{f}{g}\right);
\end{aligned}$$

(v) $\Delta^\alpha E_{p(t)}(t,s)$

$$\begin{aligned}
&= \kappa_1(\alpha,t)e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) + \kappa_0(\alpha,t)\left(e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s)\right)^\Delta \\
&= \kappa_1(\alpha,t)\left(e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s)\right) + \kappa_0(\alpha,t)\left(\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s)\right) \\
&= p(t)e_{\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}}(t,s) \\
&= p(t)E_p(t,s);
\end{aligned}$$

(vi) we apply Theorem 1.117 of [7]:

$$\begin{aligned}
\Delta^\alpha \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s &= \kappa_1(\alpha,t)\int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s \\
&\quad + \kappa_0(\alpha,t)\left(\frac{f(t)E_0(\sigma(t),t)}{\kappa_0(\alpha,t)} + \int_a^t \frac{f(s)(-\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)})E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\right) \\
&= \kappa_1(\alpha,t)\int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s + f(t)E_0(\sigma(t),t) - \kappa_1(\alpha,t)\int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s \\
&= f(t)E_0(\sigma(t),t).
\end{aligned}$$

The proof is complete. □

Definition 3.11 (Conformable integrals of order α)**.** Let $\alpha \in (0,1]$ and let $\kappa_0, \kappa_1 : [0,1]\times\mathbb{T} \to [0,\infty)$ be rd-continuous and satisfy (3.1) with $1+\mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$ and $t_0 \in \mathbb{T}$. In light of (3.3) and items (v) and (vi) of Theorem 3.10, we define the conformable antiderivative of order α by

$$\int \Delta^\alpha f(t)\Delta^\alpha t = f(t) + cE_0(t,t_0), \quad c \in \mathbb{R}.$$

The conformable α-integral of f over $\mathbb{T}_{[a,t]}$ is defined by

$$\int_a^t f(s)E_0(\sigma(t),s)\Delta^\alpha s := \int_a^t \frac{f(s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s, \tag{3.4}$$

where on the right-hand side we have the standard Δ-integral of time scales [7, 8].

Remark 3.12. It follows from (3.4) that $\Delta^\alpha s = \dfrac{E_0(t,s)}{E_0(\sigma(t),s)\kappa_0(\alpha,s)}\Delta s$.

Theorem 3.13 (Basic properties of the conformable α-integral)**.** *Let the conformable differential operator on time scales Δ^α be given as in (3.2); the integral be given as in (3.4); with $\alpha \in (0,1]$. Let functions κ_0, κ_1 be rd-continuous and satisfy (3.1) with $1+\mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$ and let f and g be Δ-differentiable, as needed. Then,*

(i) *the derivative of the definite integral of f is given by*

$$\Delta^\alpha\left(\int_a^t f(s)E_0(\sigma(t),s)\Delta^\alpha s\right) = f(t)E_0(\sigma(t),t);$$

(ii) *the definite integral of the derivative of f is given by*

$$\int_a^t \Delta^\alpha[f(s)]E_0(t,\sigma(s))\Delta^\alpha s = f(s)E_0(t,s)\mid_{s=a}^{s=t} + \int_a^t f(s)E_0(t,\sigma(s))\kappa_1(\alpha,s)\Delta^\alpha s;$$

(iii) *an integration by parts formula is given by*

$$\begin{aligned}
\int_a^b f(t)\Delta^\alpha[g(t)]E_0(b,\sigma(t))\Delta^\alpha t &= f(t)g(t)E_0(b,\sigma(t))\mid_{t=a}^{t=b} \\
&\quad + \int_a^b \left[(g(t)-g(\sigma(t)))[f(t)]\kappa_1(\alpha,t) - g(\sigma(t))\Delta^\alpha f(t)\right]E_0(b,\sigma(t))\Delta^\alpha t;
\end{aligned}$$

(iv) *a version of the Leibniz rule for differentiation of an integral is given by*

$$\Delta^\alpha\left[\int_a^t f(t,s)E_0(\sigma(t),s)\Delta^\alpha s\right]$$
$$= f(\sigma(t),t)E_0(\sigma(t),t) + \int_a^t \Delta_t^\alpha(f(t,s))E_0(t,s)E_0(\sigma(t),s)\Delta^\alpha s,$$

where the derivative inside the last integral is with respect to t.

Proof. The proof of (i) is clear. The integration by parts formula (ii) follows easily:

$$\begin{aligned}
&\int_a^t \Delta^\alpha\left[f(s)\right]E_0(t,\sigma(s))\Delta^\alpha s\\
&\qquad= \int_a^t \Delta^\alpha\left[f(s)E_0(t,s)\right]\Delta^\alpha s + \int_a^t f(s)E_0(t,\sigma(s))\kappa_1(\alpha,s)\Delta^\alpha s\\
&\qquad= f(s)E_0(t,s)\,|_{s=a}^{s=t} + \int_a^t f(s)E_0(t,\sigma(s))\kappa_1(\alpha,s)\Delta^\alpha s.
\end{aligned}$$

Now we prove (iii):

$$\begin{aligned}
&\int_a^b f(s)\Delta^\alpha(g)(s)E_0(b,s)\Delta^\alpha s\\
&\quad= \int_a^b \Delta^\alpha(fg)(s)E_0(b,\sigma(s))\Delta^\alpha s\\
&\qquad - \int_a^b g^\sigma(s)\left[\Delta^\alpha(f)(s) - f(s)\kappa_1(\alpha,s)\right]E_0(b,\sigma(s))\Delta^\alpha s\\
&\quad= (fg)(s)E_0(b,\sigma(s))|_a^b + \int_a^b (fg)(s)E_0(b,\sigma(s))\kappa_1(\alpha,s)\Delta^\alpha s\\
&\qquad - \int_a^b g^\sigma(s)\left[\Delta^\alpha(f)(s) - f(s)\kappa_1(\alpha,s)\right]E_0(b,\sigma(s))\Delta^\alpha s\\
&\quad= (fg)(s)E_0(b,\sigma(s))|_a^b\\
&\qquad + \int_a^b \left[(fg)(s)\kappa_1(\alpha,s) - g^\sigma(s)(\Delta^\alpha(f)(s) - f(s)\kappa_1(\alpha,s))\right]E_0(b,\sigma(s))\Delta^\alpha s\\
&\quad= (fg)(s)E_0(b,\sigma(s))|_a^b\\
&\qquad + \int_a^b \left[(g(s) - g^\sigma(s))\kappa_1(\alpha,s)f(s) - g^\sigma(s)\Delta^\alpha f(s)\right]E_0(b,\sigma(s))\Delta^\alpha s.
\end{aligned}$$

For (iv), we have:

$$\begin{aligned}
&\Delta^\alpha\left[\int_a^t f(t,s)E_0(\sigma(t),s)\Delta^\alpha s\right]\\
&\quad= \Delta^\alpha \int_a^t f(t,s)E_0(\sigma(t),s)\frac{E_0(t,s)}{E_0(\sigma(t),s)\kappa_0(\alpha,s)}\Delta s\\
&\quad= \Delta^\alpha \int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\\
&\quad= \kappa_0(\alpha,t)\left(\int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\right)^\Delta + \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,t)}\Delta s\\
&\quad= \kappa_0(\alpha,t)\left(\frac{f(\sigma(t),t)E_0(\sigma(t),t)}{\kappa_0(\alpha,t)} + \int_a^t \frac{f^\Delta(t,s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\right)\\
&\qquad+ \kappa_1(\alpha,t)\int_a^t \frac{f(t,s)E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s\\
&\quad= f(\sigma(t),t)E_0(\sigma(t),t) + \int_a^t \frac{\Delta_t^\alpha(f(t,s))E_0(t,s)}{\kappa_0(\alpha,s)}\Delta s,
\end{aligned}$$

where the derivative inside the last integral is with respect to t. □

3.2. Linear 2nd-order conformable differential equations on time scales. Let $\mathbb{T}$ be an arbitrary time scale, $\alpha \in [0,1]$, and let $\kappa_0, \kappa_1 : [0,1]\times\mathbb{T} \to [0,\infty)$ be rd-continuous functions such that (3.1) holds with $1+\mu(t)\frac{\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$. In addition, let Δ^α be as in (3.2), and let $t_0 \in \mathbb{T}$. In this section we are concerned with the following linear second-order conformable dynamic equation on time scales with constant coefficients:

$$\Delta^\alpha\Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = f(t), \quad t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}, \tag{3.5}$$

where we assume $a, b \in \mathbb{R}$, $f \in C_{rd}$. Introduce the operator $L_{2\Delta^\alpha} : C^2_{rd} \to C_{rd}$ by

$$L_{2\Delta^\alpha}(y)(t) = \Delta^\alpha\Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) \tag{3.6}$$

for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$.

Lemma 3.14. *The operator $L_{2\Delta^\alpha}$ defined by* (3.6) *is a linear operator, i.e.,*

$$L_{2\Delta^\alpha}(py_1 + qy_2) = pL_{2\Delta^\alpha}(y_1) + qL_{2\Delta^\alpha}(y_2),$$

where $p, q \in \mathbb{R}$ and $y_1, y_2 \in C^2_{rd}$. If y_1 and y_2 solve the homogeneous equation

$$L_{2\Delta^\alpha} y = 0,$$

then so does $y = py_1 + qy_2$, $p, q \in \mathbb{R}$.

Proof. Using (i) of Theorem 3.10, we find that

$$\begin{aligned}
&L_{2\Delta^\alpha}(py_1+qy_2)(t)\\
&\quad= \Delta^\alpha\Delta^\alpha(py_1(t)+qy_2(t)) + a\Delta^\alpha(py_1(t)+qy_2(t)) + b(py_1(t)+qy_2(t))\\
&\quad= pL_{2\Delta^\alpha}(y_1)(t) + qL_{2\Delta^\alpha}(y_2)(t) = 0
\end{aligned}$$

for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$ and all $p, q \in \mathbb{R}$. □

Definition 3.15. Let $a, b \in \mathbb{R}$ and $f \in C_{rd}$. Equation (3.5) is called regressive if

$$\kappa_0^2 - \mu\kappa_0(a + 2\kappa_1) + \mu^2(b + a\kappa_1 + \kappa_1^2) \neq 0$$

for all $t \in \mathbb{T}^\kappa$.

Theorem 3.16 (Existence and uniqueness of solution)**.** *Let $t_0 \in \mathbb{T}^\kappa$ and functions $\kappa_i(\alpha, t)$, $i = 0, 1$, be Δ_t-differentiable and continuous with respect to α. Assume that the dynamic equation (3.5) is regressive. If $L_{2\Delta^\alpha} y(t) = 0$ admits two solutions y_1 and y_2 with $y_1(t)\Delta^\alpha y_2(t) \neq \Delta^\alpha y_1(t) y_2(t)$ for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$, then the initial value problem*

$$L_{2\Delta^\alpha} y(t) = 0, \quad y(t_0) = y_0, \quad \Delta^\alpha y(t_0) = y_0^\alpha, \tag{3.7}$$

where y_0 and y_0^α are given constants, has a unique solution defined on $\mathbb{T}_{[t_0,\infty)}$.

Proof. If y_1, y_2 are two solutions of $L_{2\Delta^\alpha} y(t) = 0$, then $y(t) = py_1(t) + qy_2(t)$, $p, q \in \mathbb{R}$, is a solution of $L_{2\Delta^\alpha} y(t) = 0$. Therefore, we want to see if we can pick p and q so that $y_0 = y(t_0) = py_1(t_0) + qy_2(t_0)$, $y_0^\alpha = p\Delta^\alpha y_1(t_0) + q\Delta^\alpha y_2(t_0)$. Let

$$\mathbf{M} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ \Delta^\alpha y_1(t_0) & \Delta^\alpha y_2(t_0) \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} y_0 \\ y_0^\alpha \end{pmatrix}.$$

System $M \times X = B$ has a unique solution because we are assuming matrix M to be invertible. □

Definition 3.17. For two Δ^α-differentiable functions on $\mathbb{T}_{[t_0,\infty)}$ y_1 and y_2, we define the Wronskian $W = W(y_1, y_2)$ by

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ \Delta^\alpha y_1(t) & \Delta^\alpha y_2(t) \end{pmatrix}, \quad t \in \mathbb{T}_{[t_0,\infty)}.$$

We say that two solutions y_1 and y_2 of $L_{2\Delta^\alpha} y = 0$ form a fundamental set of solutions for $L_{2\Delta^\alpha} y = 0$ provided $W(y_1, y_2)(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$.

Theorem 3.18. *If the pair of functions y_1 and y_2 form a fundamental system of solutions for $L_{2\Delta^\alpha} y = 0$, $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$, then*

$$y(t) = py_1(t) + qy_2(t), \quad p, q \in \mathbb{R}, \tag{3.8}$$

is a general solution of $L_{2\Delta^\alpha} y = 0$, $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$. In particular, the solution of the initial value problem (3.7) is given by

$$y(t) = \frac{\Delta^\alpha y_2(t_0) y_0 - y_2(t_0) y_0^\alpha}{W(y_1, y_2)(t_0)} y_1(t) + \frac{y_1(t_0) y_0^\alpha - \Delta^\alpha y_1(t_0) y_0}{W(y_1, y_2)(t_0)} y_2(t).$$

Remark 3.19. By general solution we mean that every function of form (3.8) is a solution and every solution is of this form.

Proof. The proof is similar to the one of Theorem 3.7 of [7]. □

3.3. Hyperbolic and trigonometric functions. Now we consider the linear second-order homogeneous dynamic conformable equation with constant coefficients

$$\Delta^\alpha \Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = 0, \quad a, b \in \mathbb{R}, \quad t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}. \tag{3.9}$$

We assume (3.9) to be regressive, i.e., $\kappa_0 - \mu(a + 2\kappa_1) + \mu^2(b + a\kappa_1 + \kappa_1^2) \neq 0$, $t \in \mathbb{T}^\kappa$. Let $\lambda \in \mathbb{C}$ be such that $1 + \mu(t)\frac{\lambda - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$, $t \in \mathbb{T}^\kappa$, and $y(t) = E_\lambda(t, t_0)$, $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$, be a solution of (3.9). If $y(t) = E_\lambda(t, t_0)$, then

$$\Delta^\alpha \Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = (\lambda^2 + a\lambda + b)E_\lambda(t, t_0)$$

and, because $E_\lambda(t, t_0) \neq 0$, $y(t) = E_\lambda(t, t_0)$ is a solution of (3.9) if and only if λ satisfies the characteristic equation of (3.9):

$$\lambda^2 + a\lambda + b = 0. \tag{3.10}$$

The solutions λ_1 and λ_2 of (3.10) are given by

$$\lambda_1 = \frac{-a - \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a + \sqrt{a^2 - 4b}}{2} \tag{3.11}$$

and, since (3.9) is regressive, $1 + \mu(t)\frac{\lambda_1 - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ and $1 + \mu(t)\frac{\lambda_2 - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ for all $t \in \mathbb{T}^\kappa$.

Theorem 3.20. *Suppose equation* (3.9) *is regressive and* $a^2 - 4b \neq 0$. *Then,* $E_{\lambda_1}(\cdot, t_0)$ *and* $E_{\lambda_2}(\cdot, t_0)$ *form a fundamental system of* (3.9), *where* $t_0 \in \mathbb{T}$ *and* λ_1 *and* λ_2 *are given as in* (3.11), *and the solution of the initial value problem*

$$\Delta^\alpha \Delta^\alpha y(t) + a\Delta^\alpha y(t) + by(t) = 0, \quad y(t_0) = y_0, \quad \Delta^\alpha y_{t_0} = y_0^\alpha, \tag{3.12}$$

is given by

$$y_0(t) = \frac{E_{\lambda_1}(t, t_0) + E_{\lambda_2}(t, t_0)}{2} + \frac{a y_0 + 2y_0^\alpha}{\sqrt{a^2 - 4b}} \frac{E_{\lambda_2}(t, t_0) - E_{\lambda_1}(t, t_0)}{2}, \quad t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}.$$

Proof. Since λ_1 and λ_2, given by (3.11), are solutions of the characteristic equation (3.10), we know that both $E_{\lambda_1}(\cdot, t_0)$ and $E_{\lambda_2}(\cdot, t_0)$ are solutions of (3.9). Moreover,

$$\begin{aligned} W\left(E_{\lambda_1}(t, t_0), E_{\lambda_2}(t, t_0)\right) &= \det \begin{pmatrix} E_{\lambda_1}(t, t_0) & E_{\lambda_2}(t, t_0) \\ \lambda_1 E_{\lambda_1}(t, t_0) & \lambda_2 E_{\lambda_2}(t, t_0) \end{pmatrix} \\ &= \lambda_2 E_{\lambda_1}(t, t_0) E_{\lambda_2}(t, t_0) - \lambda_1 E_{\lambda_1}(t, t_0) E_{\lambda_2}(t, t_0) \\ &= (\lambda_2 - \lambda_1) E_{\lambda_1}(t, t_0) E_{\lambda_2}(t, t_0) \\ &= \sqrt{a^2 - 4b} E_{\lambda_1}(t, t_0) E_{\lambda_2}(t, t_0), \end{aligned}$$

so that $W(E_{\lambda_1}(t, t_0) E_{\lambda_2}(t, t_0)) \neq 0$ for all $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$, unless $a^2 - 4b = 0$. Having obtained a fundamental system $y_1 = E_{\lambda_1}(\cdot, t_0)$ and $y_2 = E_{\lambda_2}(\cdot, t_0)$ of (3.9), now we obtain a solution of (3.12), namely $y(t) = c_1 y_1(t) + c_2 y_2(t)$. For that we solve the linear system of equations

$$\begin{cases} y_0 = c_1 y_1(t_0) + c_2 y_2(t_0) \\ \Delta^\alpha y(t_0) = \lambda_1 c_1 y_1(t_0) + \lambda_2 c_2 y_2(t_0) \end{cases}$$

in the unknowns c_1 and c_2, obtaining $c_1 = \frac{y_0}{2} - \frac{a y_0 + 2y_0^\alpha}{2\sqrt{a^2 - 4b}}$ and $c_2 = \frac{y_0}{2} + \frac{a y_0 + 2y_0^\alpha}{2\sqrt{a^2 - 4b}}$. $\square$

Hyperbolic functions are associated with the case $a = 0$ and $b < 0$.

Definition 3.21 (Hyperbolic functions)**.** Let $\mathbb{T}$ be a time scale and $t_0 \in \mathbb{T}$. If $p \in C_{rd}$ and $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-p^2 + \kappa_1^2) \neq 0$ for all $t \in \mathbb{T}^\kappa$, then we define the hyperbolic functions $\cosh_{p\Delta^\alpha}(\cdot, t_0)$ and $\sinh_{p\Delta^\alpha}(\cdot, t_0)$ on $\mathbb{T}_{[t_0,\infty)}$ by

$$\cosh_{p\Delta^\alpha}(\cdot, t_0) = \frac{E_p(\cdot, t_0) + E_{-p}(\cdot, t_0)}{2} \quad \text{and} \quad \sinh_{p\Delta^\alpha}(\cdot, t_0) = \frac{E_p(\cdot, t_0) - E_{-p}(\cdot, t_0)}{2}.$$

Remark 3.22. The condition $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-p^2 + \kappa_1^2) \neq 0$ of Definition 3.21 is equivalent to $1 + \mu(t)\frac{p(t)-\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$ and $1 - \mu(t)\frac{p(t)+\kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$.

Lemma 3.23. *Let $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-p^2 + \kappa_1^2) \neq 0$ for all $t \in \mathbb{T}^\kappa$. Then,*

$$\begin{aligned} \Delta^\alpha \cosh_{p\Delta^\alpha}(\cdot, t_0) &= p \sinh_{p\Delta^\alpha}(\cdot, t_0), \\ \Delta^\alpha \sinh_{p\Delta^\alpha}(\cdot, t_0) &= p \cosh_{p\Delta^\alpha}(\cdot, t_0), \\ \cosh^2_{p\Delta^\alpha}(\cdot, t_0) - \sinh^2_{p\Delta^\alpha}(\cdot, t_0) &= E_p(\cdot, t_0)E_{-p}(\cdot, t_0), \end{aligned}$$

for all $t \in \mathbb{T}_{[t_0,\infty)}$.

Proof. The first two formulas are trivially verified. The last relation follows from

$$\begin{aligned} &(\cosh^2_{p\Delta^\alpha} - \sinh^2_{p\Delta^\alpha})(\cdot, t_0) \\ &\quad = \left(\frac{E_p(\cdot, t_0) + E_{-p}(\cdot, t_0)}{2}\right)^2 - \left(\frac{E_p(\cdot, t_0) - E_{-p}(\cdot, t_0)}{2}\right)^2 \\ &\quad = \frac{E_p^2(\cdot, t_0) + 2E_p(\cdot, t_0)E_{-p}(\cdot, t_0) + E_{-p}^2(\cdot, t_0)}{4} \\ &\qquad - \frac{E_p^2(\cdot, t_0) - 2E_p(\cdot, t_0)E_{-p}(\cdot, t_0) + E_{-p}^2(\cdot, t_0)}{4} \\ &\quad = E_p(\cdot, t_0)E_{-p}(\cdot, t_0) \end{aligned}$$

for all $t \in \mathbb{T}_{[t_0,\infty)}$. □

Theorem 3.24. *If $\gamma \in \mathbb{R}\backslash\{0\}$, $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-\gamma^2 + \kappa_1^2) \neq 0$, and $t_0 \in \mathbb{T}^\kappa$, then*

$$y(t) = c_1 \cosh_{\gamma\Delta^\alpha}(t, t_0) + c_2 \sinh_{\gamma\Delta^\alpha}(t, t_0)$$

is a general solution of

$$\Delta^\alpha\Delta^\alpha y - \gamma^2 y = 0 \tag{3.13}$$

on $t \in \mathbb{T}^{\kappa^2}_{[t_0,\infty)}$, where c_1 and c_2 are arbitrary constants.

Proof. One can easily prove that $\cosh_{\gamma\Delta^\alpha}(\cdot, t_0)$ and $\sinh_{\gamma\Delta^\alpha}(\cdot, t_0)$ are solutions of (3.13). We prove that they form a fundamental set of solutions for (3.13):

$$\begin{aligned} W\left(\cosh_{\gamma\Delta^\alpha}(t, t_0), \sinh_{\gamma\Delta^\alpha}(t, t_0)\right) &= \det \begin{pmatrix} \cosh_{\gamma\Delta^\alpha}(t, t_0) & \sinh_{\gamma\Delta^\alpha}(t, t_0)) \\ \gamma\sinh_{\gamma\Delta^\alpha}(t, t_0)) & \gamma\cosh_{\gamma\Delta^\alpha}(t, t_0) \end{pmatrix} \\ &= \gamma\cosh^2_{\gamma\Delta^\alpha}(t, t_0) - \gamma\sinh^2_{\gamma\Delta^\alpha}(t, t_0) \\ &= \gamma\left(\cosh^2_{\gamma\Delta^\alpha}(t, t_0) - \sinh^2_{\gamma\Delta^\alpha}(t, t_0)\right) \\ &= \gamma E_\gamma(t, t_0)E_{-\gamma}(t, t_0) \end{aligned}$$

is different from zero for all $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$, unless $\gamma = 0$. □

Example 3.25. Let $\mathbb{T}$ be a time scale, $t_0 \in \mathbb{T}^\kappa$. If $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(-\gamma^2 + \kappa_1^2) \neq 0$, with $\gamma \in \mathbb{R}\backslash\{0\}$, then the solution of the initial value problem

$$\Delta^\alpha \Delta^\alpha y(t) - \gamma^2 y(t) = 0, \quad y(t_0) = y_0, \quad \Delta^\alpha y(t_0) = y_0^\alpha,$$

is given by

$$y(t) = y_0 \cosh_{\gamma\Delta^\alpha}(t, t_0) + \frac{y_0^\alpha}{\gamma} \sinh_{\gamma\Delta^\alpha}(t, t_0)$$

for all $t \in \mathbb{T}_{[t_0,\infty)}^{\kappa^2}$.

Definition 3.26 (Trigonometric functions)**.** Let $\mathbb{T}$ be a time scale, $p \in C_{rd}$, $t_0 \in \mathbb{T}$, and $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(p^2 + \kappa_1^2) \neq 0$ for all $t \in \mathbb{T}^\kappa$. Then we define the trigonometric functions $\cos_{p\Delta^\alpha}(\cdot, t_0)$ and $\sin_{p\Delta^\alpha}(\cdot, t_0)$ by

$$\cos_{p\Delta^\alpha}(\cdot, t_0) = \frac{E_{ip}(\cdot, t_0) + E_{-ip}(\cdot, t_0)}{2} \quad \text{and} \quad \sin_{p\Delta^\alpha}(\cdot, t_0) = \frac{E_{ip}(\cdot, t_0) - E_{-ip}(\cdot, t_0)}{2i}.$$

Remark 3.27. The condition $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(p^2 + \kappa_1^2) \neq 0$ of Definition 3.26 is equivalent to $1 + \mu(t)\left(\frac{ip(t) - \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)}\right) \neq 0$ and $1 - \mu(t)\frac{ip(t) + \kappa_1(\alpha,t)}{\kappa_0(\alpha,t)} \neq 0$.

Lemma 3.28. *Let $p \in C_{rd}$. If $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(p^2 + \kappa_1^2) \neq 0$ for all $t \in \mathbb{T}^\kappa$, then*

$$\begin{gathered}
\Delta^\alpha \cos_{p\Delta^\alpha}(\cdot, t_0) = -p \sin_{p\Delta^\alpha}(\cdot, t_0), \\
\Delta^\alpha \sin_{p\Delta^\alpha}(\cdot, t_0) = p \cos_{p\Delta^\alpha}(\cdot, t_0), \\
\cos^2_{p\Delta^\alpha}(\cdot, t_0) + \sin^2_{p\Delta^\alpha}(\cdot, t_0) = E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0).
\end{gathered}$$

Remark 3.29. If $\alpha = 1$, then $E_{ip}(\cdot, t_0) = e_{ip}(\cdot, t_0) = \cos_p(\cdot, t_0) + i \sin_p(\cdot, t_0)$.

Proof. Similarly to Lemma 3.23, the first two formulas are easily verified. We have

$$\begin{aligned}
&\cos^2_{p\Delta^\alpha}(\cdot, t_0) + \sin^2_{p\Delta^\alpha}(\cdot, t_0) \\
&\quad = \left(\frac{E_{ip}(\cdot, t_0) + E_{-ip}(\cdot, t_0)}{2}\right)^2 + \left(\frac{E_{ip}(\cdot, t_0) - E_{-ip}(\cdot, t_0)}{2i}\right)^2 \\
&\quad = \frac{E_{ip}^2(\cdot, t_0) + 2E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0) + E_{-ip}^2(\cdot, t_0)}{4} \\
&\qquad\quad - \frac{E_{ip}^2(\cdot, t_0) - 2E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0) + E_{-ip}^2(\cdot, t_0)}{4} \\
&\quad = E_{ip}(\cdot, t_0)E_{-ip}(\cdot, t_0)
\end{aligned}$$

and the last relation also holds. ∎

Example 3.30. Let $\mathbb{T} = \mathbb{R}$, $\gamma \in \mathbb{R}$, and $t_0 \in \mathbb{T}$. Then, the conformable trigonometric functions cosine and sine are given by

$$\begin{aligned}
\cos_{\gamma\Delta^\alpha}(t,t_0) &= \frac{E_{i\gamma}(t,t_0) + E_{-i\gamma}(t,t_0)}{2} \\
&= \frac{e^{\int_{t_0}^{t} \frac{i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} + e^{\int_{t_0}^{t} \frac{-i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds}}{2} \\
&= \frac{e^{i\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds} e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} + e^{-i\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds} e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds}}{2} \\
&= \frac{e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} \left(2\cos(\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds)\right)}{2} \\
&= e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} \cos\left(\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds\right)
\end{aligned}$$

and

$$\begin{aligned}
\sin_{\gamma\Delta^\alpha}(t,t_0) &= \frac{E_{i\gamma}(t,t_0) - E_{-i\gamma}(t,t_0)}{2i} \\
&= \frac{e^{\int_{t_0}^{t} \frac{i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} - e^{\int_{t_0}^{t} \frac{-i\gamma - \kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds}}{2i} \\
&= \frac{e^{i\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds} e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} - e^{-i\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds} e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds}}{2i} \\
&= e^{-\int_{t_0}^{t} \frac{\kappa_1(s,t_0)}{\kappa_0(s,t_0)} ds} \sin\left(\int_{t_0}^{t} \frac{\gamma}{\kappa_0(s,t_0)} ds\right).
\end{aligned}$$

Theorem 3.31. *Let* $\mathbb{T}$ *be a time scale and* $t_0 \in \mathbb{T}^\kappa$. *If* $\kappa_0^2 - 2\mu\kappa_0\kappa_1 + \mu^2(\gamma^2 + \kappa_1^2) \neq 0$, $\gamma \in \mathbb{R}\backslash\{0\}$, *then* $y(t) = c_1 \cos_{\gamma\Delta^\alpha}(t,t_0) + c_2 \sin_{\gamma\Delta^\alpha}(t,t_0)$ *is a general solution of*

$$\Delta^\alpha \Delta^\alpha y + \gamma^2 y = 0, \quad t \in \mathbb{T}^{\kappa^2}. \tag{3.14}$$

Proof. One can easily show that $\cos_{\gamma\Delta^\alpha}(\cdot,t_0)$ and $\sin_{\gamma\Delta^\alpha}(\cdot,t_0)$ are solutions of (3.14). We prove that they form a fundamental set of solutions for (3.14): for $\gamma \neq 0$,

$$\begin{aligned}
W\left(\cos_{\gamma\Delta^\alpha}(t,t_0), \sin_{\gamma\Delta^\alpha}(t,t_0)\right) &= \det \begin{pmatrix} \cos_{\gamma\Delta^\alpha}(t,t_0) & \sin_{\gamma\Delta^\alpha}(t,t_0)) \\ -\gamma \sin_{\gamma\Delta^\alpha}(t,t_0)) & \gamma \cos_{\gamma\Delta^\alpha}(t,t_0) \end{pmatrix} \\
= \gamma\left(\cos^2_{\gamma\Delta^\alpha}(t,t_0) + \sin^2_{\gamma\Delta^\alpha}(t,t_0)\right)^2 &= \gamma E_{ip}(t,t_0) E_{-ip}(t,t_0) \neq 0
\end{aligned}$$

for all $t \in \mathbb{T}^\kappa_{[t_0,\infty)}$. We conclude that $y(t) = c_1 \cos_{\gamma\Delta^\alpha}(t,t_0) + c_2 \sin_{\gamma\Delta^\alpha}(t,t_0)$, $t \in \mathbb{T}_{[t_0,\infty)}$, is a general solution of (3.14). □

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