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Heat Kernel Estimates on a Connected Sum of Two Copies of $\mathbb{R}^n$ Along a Surface of Revolution

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Contents

Abstract

We prove sharp two sided heat kernel estimates on a connected sum of two copies of $\mathbb{R}^n$ along a surface of revolution taking into account a bottleneck effect. In the proof, estimates of the hitting probability of a non-compact set play a crucial role. For the heat kernel upper bound, we use isoperimetric inequalities on connected sums. For the heat kernel lower bound, we use a lower bound of the Dirichlet heat kernel in the exterior of a non-compact set.

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1 Introduction

Let M be a connected, geodesically complete non-compact Riemannian manifold and Δ be the (positive definite) Laplace operator associated with its Riemannian metric. The heat kernel $p(t,x,y)$ is defined as the minimal positive fundamental solution of the heat equation

$$(\partial_t + \Delta)u(t,x) = 0 \tag{1.1}$$

on $(0,\infty)\times M$. From the probabilistic point of view, $p(t,x,y)$ can be regarded as the transition density of the Brownian motion $(\{X_t\},\{\mathbb{P}_x\})$ on M.

It is interesting to study the relationship between the long time behavior of $p(t,x,y)$ and global geometric properties of M. Many authors have considered this problem for various classes of manifolds – see [8], [12], [23] and literatures therein.

Let $d(x,y)$ be the geodesic distance on M and $V(x,r)$ be the Riemannian volume of the geodesic ball $B(x,r)$ of radius r centered at x. One of the most interesting heat kernel estimates is the following *Li-Yau type* estimate

$$p(t,x,y) \asymp \frac{C}{V(x,\sqrt{t})}\exp\left(-b\frac{d(x,y)^2}{t}\right) \quad \forall x,y\in M,\ t>0, \tag{1.2}$$

where $C,b>0$ are constants and the sign $\asymp$ means that both $\leqslant$ and $\geqslant$ are satisfied but possibly with different values of the constants C,b. For example, (1.2) is obviously true for $\mathbb{R}^n$, since $V(x,\sqrt{r}) = \mathrm{const}\, t^{n/2}$. Moreover, the same estimate is true for the heat kernel of uniformly elliptic operators in divergence form in $\mathbb{R}^n$ ([1]). Li and Yau [20] proved the estimate (1.2) for the manifolds with non-negative Ricci curvature.

A complete characterization of manifolds admitting (1.2) is known and follows from the works of Fabes, Stroock [10], Grigor'yan [13], Saloff-Coste [22], [23] (see Theorem 7.1 in Appendix).

On the other hand, there are many interesting manifolds where the heat kernel does not satisfy (1.2). Let M_1 and M_2 be two Riemannian manifolds of the same dimension, and let A_1, A_2 be closed subsets of M_1, M_2 respectively. Let A_1, A_2 have non-empty interiors and smooth boundaries. Let J be a manifold with boundary so that ∂J is isometric to the disjoint union $\partial A_1 \amalg \partial A_2$.

Then we define the connected sum $M = M_1 \#_J M_2$ as the disjoint union of $M_1 \setminus A_1, M_2 \setminus A_2$ and J with identification of $\partial A_1 \amalg \partial A_2$ and ∂J. The Riemannian metric of M on $M_i \setminus A_i$ is defined to be the metric of M_i, and the Riemannian metric on J is chosen so that the metric on M is smooth.

We are interested in heat kernel bounds on the connected sum $M = M_1 \#_J M_2$ assuming that the heat kernels on M_1 and M_2 satisfy the Li-Yau estimate (1.2). If A_1, A_2, J are compact then the value of the heat kernel $p(t,x,y)$ for x and y at the different ends of M may be significantly smaller than predicted by (1.2), which is due to a bottleneck effect. The first author and Saloff-Coste proved in [16] that the heat kernel $p(t,x,y)$ on $\mathbb{R}^n \#_J \mathbb{R}^n$ with compact A_1, A_2, J and $n \geqslant 3$ satisfies the following estimate:

$$p(t,x,y) \asymp Ct^{-n/2} \left(\frac{1}{d(x,J)^{n-2}} + \frac{1}{d(y,J)^{n-2}} \right) \exp\left(-b\frac{d(x,y)^2}{t} \right) \tag{1.3}$$

assuming that x, y belong to different copies of $\mathbb{R}^n$ and $d(x,J), d(y,J), t$ are large enough. The terms $d(x,J)^{2-n}$ and $d(y,J)^{2-n}$ arise from the bottleneck effect and give a quantitative meaning to the latter.

Now assume that A_1, A_2 are non-compact subsets of $\mathbb{R}^n$. In this case the heat kernel behavior can be depend on the structure of the joint J. The full extent of this dependence is not yet clear. To avoid complications arising from the structure of J, let us assume that $A_1 = A_2$ and that J is defined by embedding of the two copies of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as on Figure 1.

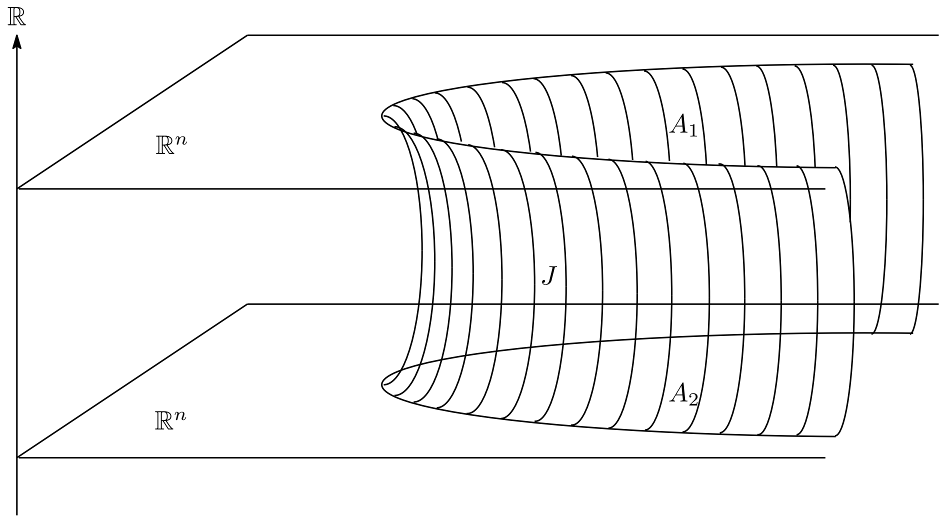

Figure 1: Definition of J.

It follows from the results of [19] and [6], that if, for some $\varepsilon > 0$ and for any Euclidean ball $B(x,r)$ with $r \geqslant 1$ and $x \in A_i$,

$$\mu\left(B\left(x,r\right) \cap A_i\right) \asymp cr^{n-2+\varepsilon}, \tag{1.4}$$

then M satisfies the Poincaré inequality (7.1) and consequently the Li-Yau estimate (1.2) (cf. Theorem 7.1 in Appendix). In this case there is no bottleneck effect, due to the fact that A_1, A_2 are fat enough.

The purpose of this paper is to obtain two-sided estimates of the heat kernel on $M = \mathbb{R}^n \#_J \mathbb{R}^n$ where J connects two non-compact domains of revolution A_1, A_2 that however are small enough so that the Poincaré inequality fails and the heat kernel bounds become non-trivial.

Fix two integers m and n so that $0 \leqslant m \leqslant n-1$. For every $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define functions $r\left(x\right)$ and $h\left(x\right)$ by

$$r(x) = \sqrt{\sum_{1 \leqslant i \leqslant m} x_i^2 + 1}, \quad h(x) = \sqrt{\sum_{m+1 \leqslant i \leqslant n} x_i^2}. \tag{1.5}$$

For any $\alpha \geqslant 0$, define a domain of revolution $A\left(m,\alpha\right)$ in $\mathbb{R}^n$ by

$$A(m,\alpha) = \{x \in \mathbb{R}^n \mid h(x) \leqslant r(x)^\alpha\} \tag{1.6}$$

(see Figure 2). If $m = 0$ then $r\left(x\right) \equiv 1$ so that $A\left(m,\alpha\right)$ does not depend on

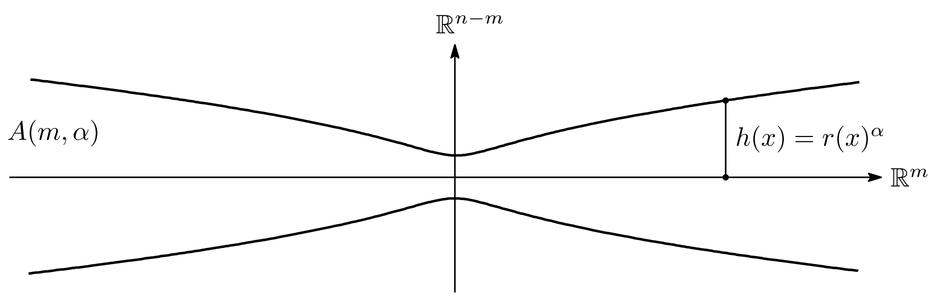

Figure 2: $A(m,\alpha)$

α. In this case we always take $\alpha = 0$.

Now consider two copies of $\mathbb{R}^n$: $M_1 = M_2 = \mathbb{R}^n$ and denote by A_1, A_2 two copies of the set $A\left(m,\alpha\right)$ on M_1 and M_2, respectively. Define a connected sum

$$M^n_{m,\alpha} = M_1 \#_J M_2 = \mathbb{R}^n \#_J \mathbb{R}^n.$$

The joint J can be taken again as on Figure 1 (see Sections 3 and 6 for rigorous definition of J).

If either $m = n-1$ or $\alpha \geqslant 1$, then the condition (1.4) is satisfied and, hence, $M^n_{m,\alpha}$ admits Li-Yau bounds (1.2). In this paper we treat the case

$$0 \leqslant m \leqslant n-3, \quad 0 \leqslant \alpha < 1, \tag{1.7}$$

while postponing the remaining critical case $m = n-2$ to another opportunity.

To state our main result, let us introduce the following notation. As above set $A = A(m,\alpha)$ and, for any $L \geqslant 0$, define the set

$$E^L = \{x \in \mathbb{R}^n | \, d(x,A) \geqslant Lr(x)^\alpha\}.$$

Denote by E^L_k a copy of E^L on M_k, $k = 1,2$ so that E^L_1 and E^L_2 can be regarded as disjoint subsets of $M^n_{m,\alpha}$. We use the same notation $r(x)$, $h(x)$ for $x \in M_k \setminus A_k$ if there is no confusion.

Our main result is as follows.

Theorem 1.1. *Let m,n,α be as in* (1.7) *(if $m = 0$ then set $\alpha = 0$). Then there exist constants $L, T \geqslant 1$ such that the heat kernel on $M = M^n_{m,\alpha}$ satisfies the following estimates:*

(i) For all $x, y \in M \setminus E^L_2$ and $t > T\left(d(x,E^L_1) + d(y,E^L_1)\right)^2$,

$$p(t,x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-b\frac{d(x,y)^2}{t}\right). \tag{1.8}$$

(ii) For all $x \in E^L_1$, $y \in E^L_2$ and $t > T\left(d(x,J) + d(y,J)\right)^{2\alpha}$,

$$\begin{aligned} p(t,x,y) \;\; \asymp \;\; & \frac{C}{t^{n/2}}\Bigg\{\left(\frac{r(x)^\alpha}{d(x,J)}\right)^{n-m-2} + \frac{1}{d(x,J)^{(1-\alpha)(n-m-2)}} \\ & + \left(\frac{r(y)^\alpha}{d(y,J)}\right)^{n-m-2} + \frac{1}{d(y,J)^{(1-\alpha)(n-m-2)}}\Bigg\} e^{-b\frac{d(x,y)^2}{t}} \end{aligned} \tag{1.9}$$

In particular, if $d(x,J) \geqslant L'r(x)$, $d(y,J) \geqslant L'r(y)$ for some $L' > 1$, then

$$p(t,x,y) \asymp \frac{C}{t^{n/2}}\left(\frac{1}{d(x,J)^{(1-\alpha)(n-m-2)}} + \frac{1}{d(y,J)^{(1-\alpha)(n-m-2)}}\right) e^{-b\frac{d(x,y)^2}{t}}. \tag{1.10}$$

It is easy to see that the cases (i), (ii) cover all possible locations of the points x, y on $M^n_{m,\alpha}$, up to switching the indices $1,2$.

Remark 1.2. As we see from the last statement of the above theorem, the bottleneck effect, given by the estimate (1.10), manifests itself in the situation when x and y are far enough from the joint J. If x and y are close enough to the boundaries of E^L_1 and E^L_2, respectively, then $d(x,J) \asymp r(x)^\alpha$, $d(y,J) \asymp r(y)^\alpha$, and (1.9) amounts to the Li-Yau estimate (1.8). The estimate (1.9) can be regarded as an interpolation between the Li-Yau estimate (1.8) and the estimate (1.10) (see Figure 3).

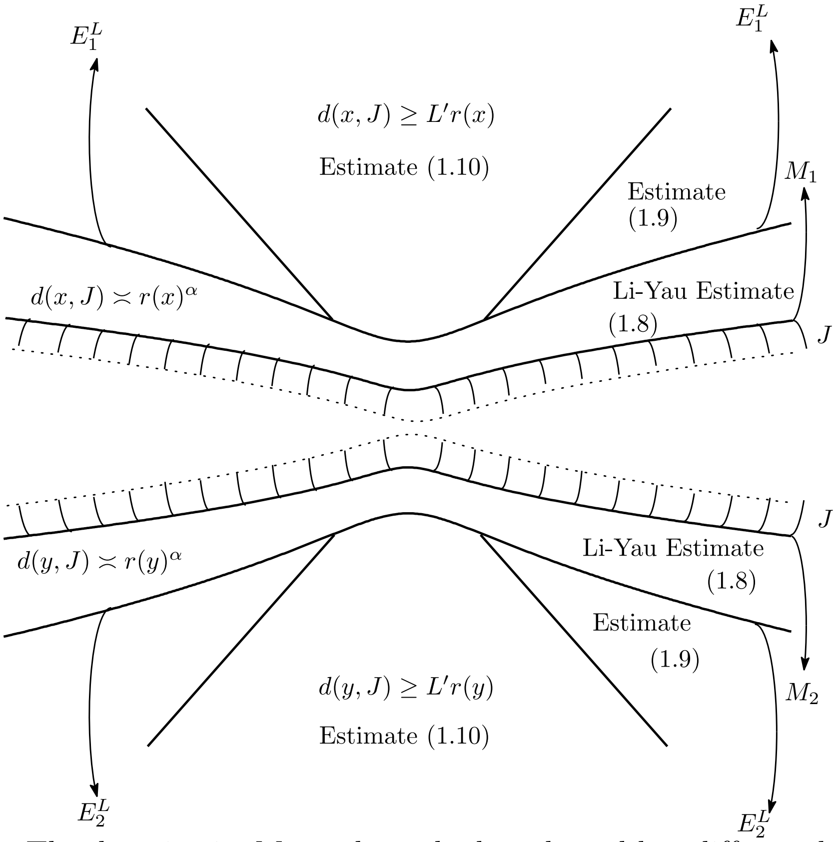

Figure 3: The domains in $M^n_{m,\alpha}$ where the heat kernel has different behavior.

Remark 1.3. In the case $\alpha = 0$, the above estimates follow from already known results. If $m = 0$ then $A(0,0) = B(1)$, that is, $M^n_{0,0}$ is the connected sum along the surface of the unit ball. Then the above estimate follows from the estimate (1.3). In the case $m \geqslant 1$, we have $A(m,0) = B(\mathbb{R}^m, 1)$, that is, $M^n_{m,0}$ is the connected sum along the 1-neighborhood of a m-dimensional subspace. Since

$$M^n_{m,0} = M^{n-m}_{0,0} \times \mathbb{R}^m,$$

the above heat kernel bound follows from the heat kernel estimate on $M^{n-m}_{0,0}$ and a simple formula for the heat kernel on Riemannian products (see [12, Section 9.2.1]).

Remark 1.4. Our theorem can be applied to connected sums of two copies of $\mathbb{R}^n$ along $A_k(m,\alpha) \cap Q$, $k = 1,2$, where Q is a union of some quadrants of $\mathbb{R}^m$ (together with smoothing deformation). For example, we can obtain a sharp heat kernel bound on the connected sum of two copies of $\mathbb{R}^4$ along a paraboloid of revolution:

$$\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \,|\, x_2^2 + x_3^2 + x_4^2 \leqslant x_1\}.$$

NOTATION. Throughout this article, the letters c, C, b, B... denote positive constants whose values may be different at different instances. When the value of a constant is significant, it will be clearly stated.

2 Hitting probability of a non-compact set

Let M be a geodesically complete non-parabolic Riemannian manifold. For any closed set $A \subset M$ define the first hitting time τ_A of A by

$$\tau_A = \inf\{t > 0 : X_t \in A\}.$$

The main purpose of this section is to estimate the probability $\mathbb{P}_x(\tau_A < t)$ of hitting A before time t assuming that the process X_t starts at a point x.

2.1 General estimates

For a precompact set $F \subset M$ and an open set U containing $\overline{F}$, the capacity of the capacitor (F, U) is defined by

$$\operatorname{cap}(F, U) = \inf_{\substack{\varphi \in Lip_0(U) \\ \varphi|_F = 1}} \int_U |\nabla\varphi|^2 d\mu \tag{2.1}$$

(cf. [15]). In the case $U = M$, we use the abbreviation $\operatorname{cap}(F, M) \equiv \operatorname{cap}(F)$. Grigor'yan and Saloff-Coste proved the following estimate of $\mathbb{P}_x(\tau_A < t)$ in [15, Theorems 3.5, 3.7].

Theorem 2.1. *Let A be a compact subset of M and U be an open set containing A. Then, for all $x \in M \setminus U$, $t > 0$, the following estimate holds:*

$$\begin{aligned} \operatorname{cap}(A) \int_0^t \inf_{z \in \partial A} p(s, x, z) ds &\leqslant \mathbb{P}_x(\tau_A < t) \\ &\leqslant 2\operatorname{cap}(A, U) \int_0^t \sup_{z \in U \setminus A} p_{M \setminus A}(s, x, z) ds. \end{aligned} \tag{2.2}$$

In this section, we obtain estimates $\mathbb{P}_x(\tau_A < t)$ for non-compact A. The following elementary lemma will be useful for estimating of certain integrals.

Lemma 2.2. *Let f be a positive function on $(0, \infty)$ satisfying*

$$\frac{f(D)}{f(d)} \geqslant \kappa \left(\frac{D}{d}\right)^\beta \quad \forall D \geqslant d > 0 \tag{2.3}$$

with some $\kappa > 0$, $\beta > 2$. Then there exists $c > 0$ such that, for all $d, t > 0$,

$$\int_0^t \frac{1}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \geqslant c\frac{d^2}{f(d)} \exp\left(-\frac{2d^2}{t}\right). \tag{2.4}$$

In addition, if f satisfies

$$\frac{f(D)}{f(d)} \leqslant \kappa' \left(\frac{D}{d}\right)^{\beta'} \qquad \forall D \geqslant d > 0 \tag{2.5}$$

with some $\kappa' > 0$, $\beta' \geqslant \beta > 2$, then there exists $C > 0$ such that, for all $d, t > 0$,

$$\int_0^t \frac{1}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \leqslant C \frac{d^2}{f(d)} \exp\left(-\frac{d^2}{2t}\right). \tag{2.6}$$

Proof. Let us first prove (2.4). If $t \leqslant d^2$, then by (2.3)

$$\begin{aligned}\int_0^t \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \geqslant& \kappa \int_{t/2}^t \left(\frac{d^2}{s}\right)^{\beta/2} \exp\left(-\frac{d^2}{s}\right) ds \\ \geqslant& \kappa \exp\left(-\frac{2d^2}{t}\right) \int_{t/2}^t \left(\frac{d^2}{s}\right)^{\beta/2} ds.\end{aligned}$$

Using again that $t \leqslant d^2$, we obtain

$$\int_{t/2}^t \left(\frac{d^2}{s}\right)^{\beta/2} ds = \frac{2^{\frac{\beta}{2}} - 1}{\frac{\beta}{2} - 1} \frac{d^\beta}{t^{\beta/2-1}} \geqslant cd^2,$$

where we have also used that $\beta > 2$. Then (2.4) follows.

In the case $t > d^2$ we have

$$\begin{aligned}\int_0^t \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \geqslant& \kappa \int_{d^2/2}^{d^2} \left(\frac{d^2}{s}\right)^{\beta/2} \exp\left(-\frac{d^2}{s}\right) ds \\ \geqslant& \kappa e^{-2} \int_{d^2/2}^{d^2} ds = cd^2,\end{aligned}$$

which proves (2.4).

To prove (2.6), we consider the same two cases again. If $t \leqslant d^2$ then by (2.5)

$$\int_0^t \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \leqslant \kappa' \exp\left(-\frac{d^2}{2t}\right) \int_0^{d^2} \left(\frac{d^2}{s}\right)^{\beta'/2} \exp\left(-\frac{d^2}{2s}\right) ds.$$

By changing the variable $s = \frac{d^2}{u}$, we obtain

$$\int_0^{d^2} \left(\frac{d^2}{s}\right)^{\beta'/2} \exp\left(-\frac{d^2}{2s}\right) ds \leqslant d^2 \int_1^\infty u^{\frac{\beta'}{2}-2} \exp(-u) du = Cd^2,$$

whence (2.6) follows.

In the case $t > d^2$ we have

$$\exp\left(-\frac{1}{2}\right) \leqslant \exp\left(-\frac{d^2}{2t}\right),$$

so that it suffices to show that

$$\int_0^t \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \leqslant Cd^2.$$

We split the integral as follows:

$$\int_0^{d^2} \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds + \int_{d^2}^t \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds.$$

The first term has the desired bound by the previous argument for $t \leqslant d^2$. By (2.5) the second term can be estimated as follows:

$$\int_{d^2}^t \frac{f(d)}{f(\sqrt{s})} \exp\left(-\frac{d^2}{s}\right) ds \leqslant \kappa^{-1} \exp\left(-\frac{d^2}{t}\right) \int_{d^2}^t \left(\frac{d^2}{s}\right)^{\beta/2} ds.$$

Using $\beta > 2$, we obtain

$$\int_{d^2}^t \left(\frac{d^2}{s}\right)^{\beta/2} ds \leqslant \int_{d^2}^\infty \left(\frac{d^2}{s}\right)^{\beta/2} ds = \frac{d^2}{\frac{\beta}{2} - 1},$$

which together the previous lines finishes the proof. $\square$

Lemma 2.3. *Let us fix a closed set $A \subset M$ and two families $\{F_i\}_{i\in I}$ and $\{U_i\}_{i\in I}$ of subsets of M such that F_i are compact, U_i are open, $F_i \subset U_i$ and*

$$A \subset \bigcup_{i\in I} F_i.$$

Let x be a point in $M \setminus \bigcup_{i\in I} U_i$. Then, for all $t > 0$, the following estimate holds

$$\mathbb{P}_x(\tau_A < t) \leqslant 2 \sum_{i\in I} \operatorname{cap}(F_i, U_i) \int_0^t \sup_{z\in U_i\setminus F_i} p(s, x, z) ds. \tag{2.7}$$

Moreover, if the heat kernel $p(t, x, y)$ of M satisfies the Gaussian upper estimate

$$p(t, x, y) \leqslant \frac{C}{V(x, \sqrt{t})} \exp\left(-b\frac{d(x, y)^2}{t}\right) \quad x, y \in M,\ t > 0 \tag{2.8}$$

and the volume function $V(x, R)$ of M satisfies the conditions

$$\kappa \left(\frac{D}{d}\right)^\beta \leqslant \frac{V(x, D)}{V(x, d)} \leqslant k' \left(\frac{D}{d}\right)^{\beta'}, \quad \forall D \geqslant d > 0, \tag{2.9}$$

with some constants $\kappa, \kappa' > 0$ and $\beta' \geqslant \beta > 2$, then

$$\mathbb{P}_x(\tau_A < t) \leqslant C_1 \sum_{i \in I} \operatorname{cap}(F_i, U_i) \frac{u_i(x)^2}{V(x, u_i(x))} \exp\left(-b_1 \frac{u_i(x)^2}{t}\right), \tag{2.10}$$

where $u_i(x) = d(x, U_i)$ and the constants C_1, b_1 depend only on the constants $\kappa, \kappa', \beta', \beta$ and on the constants C, b from (2.8).

Proof. Set $A_i = F_i \cap A$. Since the sample paths of Brownian motion X_t are continuous, the first hitting point X_{τ_A} belongs to A and, hence, to one of the sets A_i. It is obvious that

$$\tau_A < t \text{ and } X_{\tau_A} \in A_i \Rightarrow \tau_{A_i} < t,$$

which implies

$$\begin{aligned}
\mathbb{P}_x(\tau_A < t) &\leqslant \sum_{i \in I} \mathbb{P}_x(\tau_A < t \text{ and } X_{\tau_A} \in A_i) \\
&\leqslant \sum_{i \in I} \mathbb{P}_x(\tau_{A_i} < t) \\
&\leqslant \sum_{i \in I} \mathbb{P}_x(\tau_{F_i} < t),
\end{aligned}$$

where we have also used that $A_i \subset F_i$. Estimating $\mathbb{P}_x(\tau_{F_i} < t)$ by (2.2), we obtain (2.7).

Under the additional conditions (2.8) and (2.9), we have

$$\sup_{z \in U_i \setminus F_i} p(s, x, z) ds \leqslant \frac{C}{V(x, \sqrt{s})} \exp\left(-b \frac{u_i(x)^2}{s}\right)$$

and, by Lemma 2.2,

$$\int_0^t \frac{C}{V(x, \sqrt{s})} \exp\left(-b \frac{u_i(x)^2}{s}\right) ds \leqslant C' \frac{u_i(x)^2}{V(x, u_i(x))} \exp\left(-b' \frac{u_i(x)^2}{t}\right).$$

Substituting into (2.7), we obtain (2.10). □

By using the monotonicity of the hitting probability, (2.2) and Lemma 2.2, we obtain the following lower estimate of the hitting probability:

Lemma 2.4. *Let A be a closed subset of M, K be a compact subset of A, and x be a point in $M \setminus A$. Then for all $t > 0$,*

$$\mathbb{P}_x(\tau_A < t) \geqslant \operatorname{cap}(K) \int_0^t \inf_{z \in \partial K} p(s, x, z) ds. \tag{2.11}$$

Moreover, suppose that M admits the Li-Yau bound (1.2) *and* (2.9). *Set*

$$D = \sup_{z\in\partial K} d(x,z).$$

Then the following estimate is true for all $t>0$:

$$\mathbb{P}_x(\tau_A < t) \geqslant c_1 \mathrm{cap}(K) \frac{D^2}{V(x,D)} \exp\left(-B_1 \frac{D^2}{t}\right), \tag{2.12}$$

where the constants $c_1, B_1 > 0$ depend only on κ, β and on the constants in (1.2).

2.2 Hitting probability of the set $A(m,\alpha)$ in $\mathbb{R}^n$

First we prove some capacity estimates in $\mathbb{R}^n$. Let $B_d(\ell) \subset \mathbb{R}^d$ be the d-dimensional ball of radius ℓ centered at the origin. Fix some integers $0 \leqslant m < n$ and real $0 < h < R$, and set

$$\begin{aligned}
\mathcal{D}_0 &= B_m(R) \times B_{n-m}(h)\\
\mathcal{D}_0' &= B_m(2R) \times B_{n-m}(2h),\\
\mathcal{D}_1 &= (B_m(2R) \setminus B_m(R)) \times B_{n-m}(h)\\
\mathcal{D}_1' &= \left(B_m(4R) \setminus \overline{B_m(R/2)}\right) \times B_{n-m}(2h).
\end{aligned}$$

Note that $\mathcal{D}_0 \subset \mathcal{D}_0'$ and $\mathcal{D}_1 \subset \mathcal{D}_1'$. In the case $m=0$, the balls $B_0(R)$ are identical to $\{0\}$ and the annuli $B_0(2R) \setminus B_0(R)$ are empty, so that

$$\mathcal{D}_0 = B_n(h), \mathcal{D}_0' = B_n(2h), \mathcal{D}_1 = \mathcal{D}_1' = \emptyset.$$

Denote by cap_d the capacity in $\mathbb{R}^d$ and by μ_d the Lebesgue measure in $\mathbb{R}^d$.

Lemma 2.5. *If $0 \leqslant m \leqslant n-3$, then the following estimates hold*

$$c_1 R^m h^{n-m-2} \leqslant \mathrm{cap}_n(\mathcal{D}_0) \leqslant \mathrm{cap}_n(\mathcal{D}_0, \mathcal{D}_0') \leqslant C_1 R^m h^{n-m-2}. \tag{2.13}$$

If in addition $m \geqslant 1$ then also

$$c_1 R^m h^{n-m-2} \leqslant \mathrm{cap}_n(\mathcal{D}_1) \leqslant \mathrm{cap}_n(\mathcal{D}_1, \mathcal{D}_1') \leqslant C_1 R^m h^{n-m-2}. \tag{2.14}$$

The constants $c_1, C_1 > 0$ depend on n, m only.

Proof. It is known that, for any $d \geqslant 3$,

$$\mathrm{cap}_d(B_d(r)) = a_d r^{d-2}, \tag{2.15}$$

$$\mathrm{cap}_d(B_d(r), B_d(R)) = a_d \left(\frac{1}{r^{d-2}} - \frac{1}{R^{d-2}}\right)^{-1}, \tag{2.16}$$

where $a_d > 0$ (see [14, Example 4.2]). Then the estimate (2.13) in the case $m = 0$ follows from (2.15)-(2.16) with $d = n$.

In the case $m \geqslant 1$ we use the following estimates of the capacity of product sets:

$$\begin{aligned}\mu_m(F)\mathrm{cap}_{n-m}(G) \leqslant \mathrm{cap}(F \times G) &\leqslant \mathrm{cap}(F \times G, F' \times G') \\ &\leqslant \mathrm{cap}_m(F, F')\mu_{n-m}(G') + \mu_m(F')\mathrm{cap}_{n-m}(G, G')\end{aligned} \quad (2.17)$$

where (F, F') is a capacitor $\mathbb{R}^m$, (G, G') is a capacitor in $\mathbb{R}^{n-m}$, and $F \subset F'$, $G \subset G'$. Combining these estimates with (2.15)-(2.16), we obtain (2.13).

It follows from (2.17) and (2.15) that

$$\mathrm{cap}_n(\mathcal{D}_1) \geqslant \mu_m\left(B_m(2R) \setminus B_m(R)\right)\mathrm{cap}_{n-m}(B_{n-m}(h)) \geqslant c_1 R^m h^{n-m-2},$$

which proves the lower estimate in (2.13). For the upper bound, we have

$$\begin{aligned}\mathrm{cap}(\mathcal{D}_1, \mathcal{D}_1') \;\leqslant\; & \left[\mathrm{cap}_m\left(B_m(R/2), B_m(R)\right) + \mathrm{cap}_m\left(B_m(2R), B_m(4R)\right)\right] \\ & \times \mu_{n-m}\left(B_{n-m}(2h)\right) \\ & + \mu_m\left(B_m(4R) \setminus B_m(R/2)\right)\mathrm{cap}_{n-m}\left(B_{n-m}(h), B_{n-m}(2h)\right) \\ \leqslant\; & C_1 R^m h^{n-m-2},\end{aligned}$$

which finishes the proof. $\square$

Our next goal is to estimate the hitting probability of the set $A = A(m, \alpha)$ in $\mathbb{R}^n$, which was defined by (1.6).

Theorem 2.6. *Let $1 \leqslant m \leqslant n-3$, $0 \leqslant \alpha < 1$ or $m = 0$, $\alpha = 0$. There exists $L \geqslant 1$ such that for all $x \in \mathbb{R}^n$ with $d := d(x, A) > Lr(x)^\alpha$ and for all $t > 0$*

$$\mathbb{P}_x(\tau_A < t) \asymp C\left(\left(\frac{r(x)^\alpha}{d}\right)^{n-m-2} + \frac{1}{d^{(1-\alpha)(n-m-2)}}\right)\exp\left(-b\frac{d^2}{t}\right). \quad (2.18)$$

In particular, there is a constant $L' > 1$ such that if $d(x, A) \geqslant L'r(x)$ then

$$\mathbb{P}_x(\tau_A < t) \asymp \frac{C}{d^{(1-\alpha)(n-m-2)}}\exp\left(-b\frac{d^2}{t}\right). \quad (2.19)$$

Proof. Let us fix $x \in \mathbb{R}^n$ for the entire proof and denote by x' the orthogonal projection of x onto $\mathbb{R}^m \subset \mathbb{R}^n$. First we prove the upper bound in (2.18). Set

$$R_i = \begin{cases} 0 & i = 0 \\ 2^i d & i \in \mathbb{N}, \end{cases}$$

and define a sequence of compact sets $\{F_i\}_{i=0}^\infty$ by

$$F_i = \left\{z \in A \,:\, R_i \leqslant |z' - x'| \leqslant R_{i+1}\right\}, \quad (2.20)$$

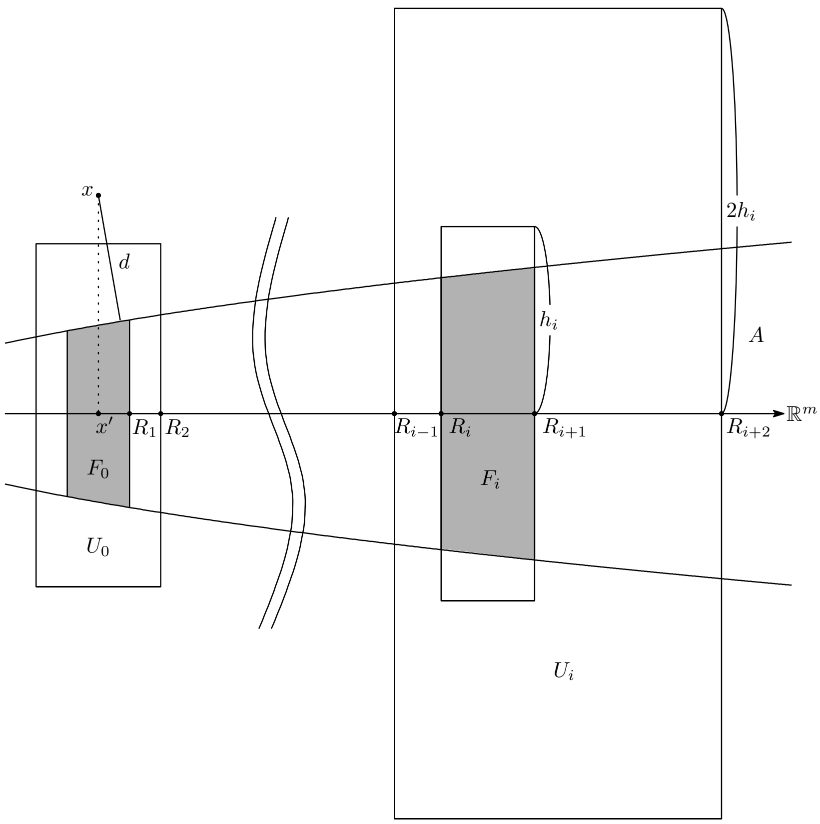

Figure 4: Sequences F_i and U_i

where $|\cdot|$ is the Euclidean norm (see Figure 4).

Set

$$h_i := \left(r(x) + 2^{i+1}d\right)^\alpha = (r(x) + R_{i+1})^\alpha$$

and observe that

$$F_i \subset x' + \left(\overline{B_m(R_{i+1})} \setminus B_m(R_i)\right) \times \overline{B_{n-m}(h_i)}.$$

Consider also the sets

$$U_i = x' + \left(B_m(2R_{i+1}) \setminus \overline{B_m(R_i/2)}\right) \times B_{n-m}(2h_i). \tag{2.21}$$

Taking $L \geqslant (8\cdot 2^\alpha)^{\frac{1}{1-\alpha}}$, we obtain that, for all $x \in M \setminus A$ satisfying $d \geqslant Lr(x)^\alpha$,

$$2h_0 \leqslant 2r(x)^\alpha + 2^{1+\alpha}d^\alpha \leqslant \frac{d}{2},$$

whence we have

$$d(x, U_0) = |x - x'| - 2h_0 \geqslant d - 2h_0 \geqslant \frac{d}{2} = \frac{R_1}{4}.$$

Furthermore, we have

$$d(x, U_i) \geqslant d(x', U_i) = \frac{R_{i+1}}{4} \quad \text{for all } i \geqslant 1. \tag{2.22}$$

Applying the estimate (2.10) of Lemma 2.3 and the estimates of capacity of Lemma 2.5, we obtain

$$\begin{aligned}
\mathbb{P}_x(\tau_A < t) \leqslant& C' \sum_{i=0}^{\infty} R_{i+1}^m h_i^{n-m-2} \frac{1}{R_{i+1}^{n-2}} \exp\left(-b\frac{R_{i+1}^2}{t}\right) \\
\leqslant& C' \sum_{i=0}^{\infty} \left(\frac{h_i}{R_{i+1}}\right)^{n-m-2} \exp\left(-b\frac{d^2}{t}\right) \\
\leqslant& C'' \left(\sum_{i=0}^{\infty} \left(\frac{r(x)^\alpha}{R_{i+1}}\right)^{n-m-2} + \sum_{i=0}^{\infty} \frac{1}{R_{i+1}^{(1-\alpha)(n-m-2)}}\right) \exp\left(-b\frac{d^2}{t}\right) \\
\leqslant& C \left(\left(\frac{r(x)^\alpha}{d}\right)^{n-m-2} + \frac{1}{d^{(1-\alpha)(n-m-2)}}\right) \exp\left(-b\frac{d^2}{t}\right),
\end{aligned} \tag{2.23}$$

where in the last line we have summed up a geometric series.

Next we prove the lower bound in (2.18). For any $z \geqslant 0$, define a point $x_z \in \mathbb{R}^n$ by

$$x_z = \begin{cases} \left(1 + \frac{z}{|x'|}\right) x', \text{ if } x' \neq 0, \\ (z, 0, ..., 0), \text{ if } x' = 0. \end{cases} \tag{2.24}$$

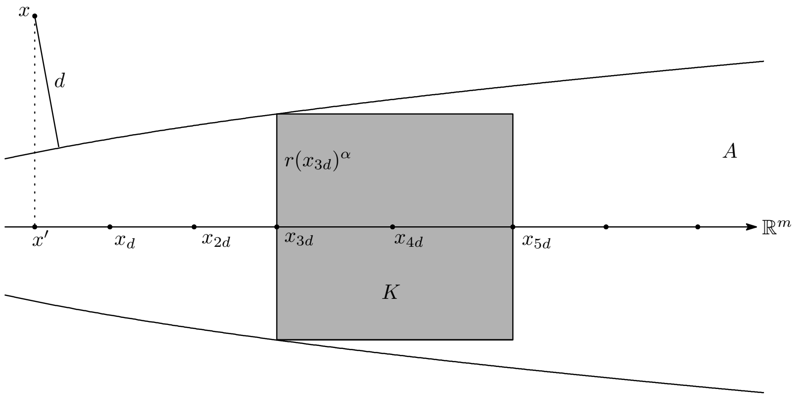

Figure 5: Compact subset K of A

Define a compact set $K \subset \mathbb{R}^n$ by

$$K = x_{4d} + \overline{B_m(d)} \times \overline{B_{n-m}\left(r\left(x_{3d}\right)^\alpha\right)} \tag{2.25}$$

and observe that $K \subset A$ (see Figure 5).

By Lemma 2.4, $\mathbb{P}_x\left(\tau_A < t\right)$ can be estimated via $\operatorname{cap}(K)$. By the estimate (2.13) of Lemma 2.5, we have

$$\operatorname{cap}(K) \geqslant c d^m r(x_{3d})^{\alpha(n-m-2)}.$$

Observe that $|x_z| = |x'| + z$ and

$$r\left(x_z\right) = \sqrt{1+\left|x_z\right|^2} = \sqrt{1+\left(\left|x'\right|+z\right)^2} > \frac{\sqrt{1+\left|x'\right|^2}+z}{2} = \frac{r\left(x\right)+z}{2},$$

in particular, we have

$$r\left(x_{3d}\right) \geqslant \frac{r\left(x\right)+3d}{2}.$$

Taking $L \geqslant 1$ large enough, we obtain that, for all $x \in M \setminus A$ satisfying $d \geqslant L r(x)^\alpha$,

$$D := \sup_{z \in \partial K} d(x,z) \leqslant 7d + r(x_{3d})^\alpha \leqslant Cd.$$

Hence, applying the estimate (2.12) of Lemma 2.4, we obtain

$$\begin{aligned}
\mathbb{P}_x\left(\tau_A < t\right) &\geqslant c' d^m r(x_{3d})^{\alpha(n-m-2)} \frac{1}{D^{n-2}} e^{-B'D^2/t} \\
&\geqslant \frac{c''}{d^{n-m-2}} \left(\frac{r\left(x\right)+3d}{2}\right)^{\alpha(n-m-2)} e^{-Bd^2/t} \\
&\geqslant c\left(\left(\frac{r(x)^\alpha}{d}\right)^{n-m-2} + \frac{1}{d^{(1-\alpha)(n-m-2)}}\right) e^{-Bd^2/t}. \square
\end{aligned}$$

3 Isoperimetric inequality on connected sums

Let N be a n-dimensional Riemannian manifold, possibly with boundary ∂N. We say that N satisfies the isoperimetric inequality if there exists a constant $c > 0$ such that

$$\mu_{n-1}(\partial\Omega) \geqslant c\mu_n(\Omega)^{\frac{n-1}{n}} \tag{3.1}$$

for all compact sets $\Omega \subset N$ whose topological boundary $\partial\Omega$ is a C^1-smooth hypersurface in N (see Figure 6).

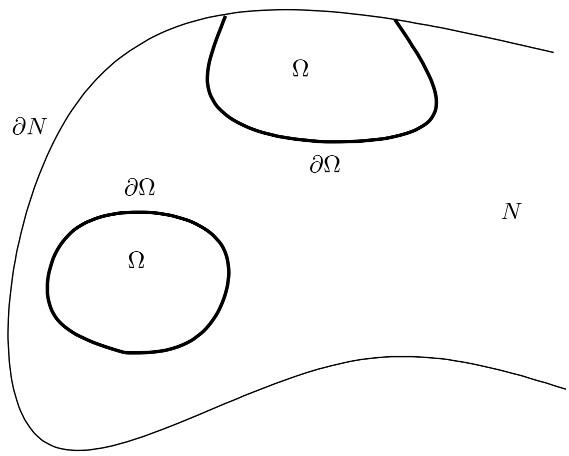

Figure 6: Boundary of N (thin line) and boundary of Ω (thick line).

Here μ_n is the Riemannian measure μ on N and μ_{n-1} is the $(n-1)$-dimensional induced Riemannian measure on $(n-1)$-dimensional hypersurfaces in N (see, for example, [3]).

It should be noted that, if N is complete, i.e. $\partial N = \emptyset$, then the isoperimetric inequality on N implies the global Gaussian upper bound for the heat kernel of N :

$$p(t,x,y) \leqslant ct^{-n/2}e^{-bd(x,y)^2/t} \tag{3.2}$$

(cf. [23]). From the point of a bottleneck effect arising from the connected sum, this estimate is too rough. Nevertheless, this estimate plays a crucial role in the proof of sharper upper estimate of the heat kernel in Lemma 4.1 (cf. [16, Section 4] and [2]).

Let us first prove the following lemma.

Lemma 3.1. *Let M be a Riemannian manifold without boundary and N_1, N_2 be two closed subsets of M that have C^1-smooth boundaries. Assume that $N_1 \cup N_2 = M$ and that both N_1, N_2 considered as manifolds with boundaries, satisfy the isoperimetric inequality (3.1). Then M also satisfies (3.1).*

Proof. For any compact subset Ω of M with C^1-boundary, set

$$\Omega_1 = \Omega \cap N_1, \quad \Omega_2 = \Omega \cap N_2.$$

Clearly, Ω_i is a closed subset of the manifold N_i, and Ω_i has in N_i a C^1-boundary $\partial\Omega_i = \partial\Omega \cap N_i$. Without loss of generality, we assume that

$$\mu_n(\Omega_1) \geqslant \mu_n(\Omega_2).$$

By the isoperimetric inequality (3.1) on N_1, we have

$$\begin{aligned}\mu_{n-1}(\partial\Omega) \geqslant& \mu_{n-1}(\partial\Omega_1)\\ \geqslant& c\mu_n(\Omega_1)^{\frac{n-1}{n}}\\ \geqslant& c'\left(\mu_n(\Omega_1) + \mu_n(\Omega_2)\right)^{\frac{n-1}{n}}\\ =& c'\mu_n(\Omega)^{\frac{n-1}{n}},\end{aligned}$$

which was to be proved. □

Fix parameters m, n, α such that $0 \leqslant m \leqslant n-1$, $0 < \alpha \leqslant 1$ or $m = 0$, $\alpha = 0$ and consider $A = A(m,\alpha) \subset \mathbb{R}^n$ defined in (1.6). In this section, we consider also both the cases of $m = n-1$ and $\alpha = 1$. Let $M_1 = M_2 = \mathbb{R}^n$ and denote by A_1, A_2 the copies of A on M_1, M_2, respectively. Here we define the joint J to be isomorphic to $\partial A(m,\alpha) \times [0,1]$ and its Riemannian metric satisfies that $M^n_{m,\alpha} \setminus E^0_2 = M_1 \setminus A_1 \amalg J$ and $M^n_{m,\alpha} \setminus E^0_1 = M_1 \setminus A_2 \amalg J$ are quasi-isometric to $\mathbb{R}^n \setminus A'$, where

$$A' = \left\{ x \in \mathbb{R}^n \mid h(x) \leqslant \frac{1}{2} r(x)^\alpha \right\}.$$

See Section 6 for additional condition for the joint J.

Then we prove the following:

Theorem 3.2. *The connected sum $M^n_{m,\alpha}$ satisfies the isoperimetric inequality (3.1).*

Proof. We note that the isoperimetric inequality (3.1) is invariant under quasi-isometry up to constant. Since $M_1 \setminus A_1 \amalg J$, $M_2 \setminus A_2 \amalg J$ are quasi-isometric to $\mathbb{R}^n \setminus A'$, by the previous lemma, it suffices to show that (3.1) on $\mathbb{R}^n \setminus A'$.

For $\kappa = (k_1, k_2, \ldots, k_n) \in \{-1, 1\}^n$, let

$$Q_\kappa = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid k_i x_i \geqslant 0\}$$

be a quadrant of $\mathbb{R}^n$. Set

$$\mathbf{q} = (\underbrace{0, 0, \ldots, 0}_{m}, \underbrace{k_{m+1}, k_{m+2} \ldots, k_n}_{n-m}) \in \mathbb{R}^n$$

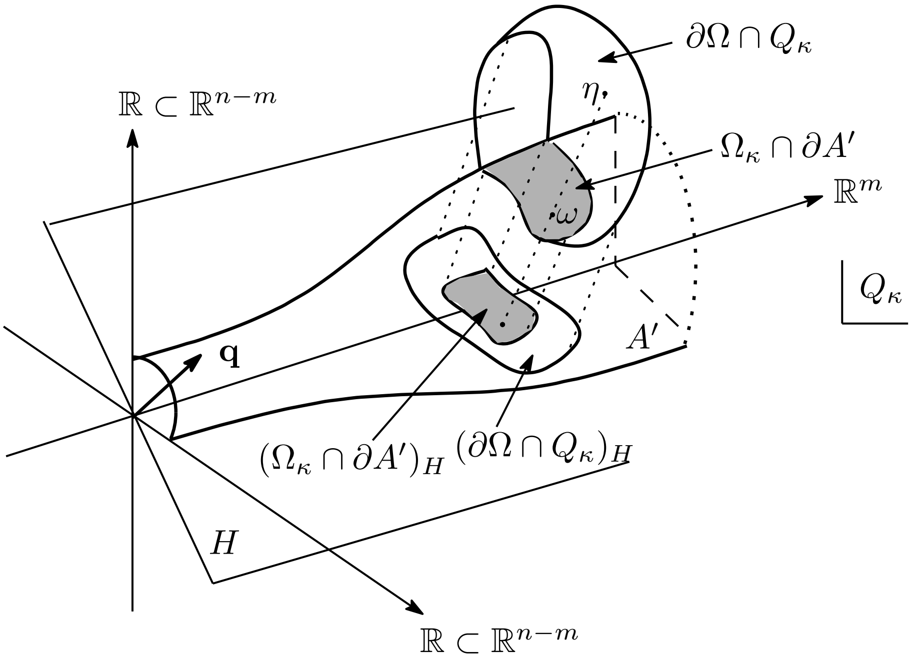

Figure 7: $Q_\kappa \backslash A'$

and $H = \mathbf{q}^\perp$, that is, the orthogonal complement of $\mathbf{q}$ (see Figure 7). For $\omega \in Q_\kappa$, we denote by ω_H the orthogonal projection of ω onto H.

Let Ω be a compact subset of $\mathbb{R}^n \setminus A'$ and set $\Omega_\kappa = \Omega \cap Q_\kappa$. Since

$$\omega \in \Omega_\kappa \cap \partial A' \Rightarrow \exists \eta \in \partial\Omega \cap Q_\kappa,\ \eta_H = \omega_H$$

(see Figure 7), we have

$$\mu_{n-1}(\partial\Omega \cap Q_\kappa) \geqslant \mu_{n-1}\left((\partial\Omega \cap Q_\kappa)_H\right) \geqslant \mu_{n-1}\left((\Omega_\kappa \cap \partial A')_H\right).$$

Here $0 \leqslant \alpha \leqslant 1$ implies that the Jacobian of the map $*_H : Q_\kappa \cap \partial A' \to H$ is uniformly non-degenerate. Therefore there exists $\epsilon > 0$ such that, for every compact set $U \subset Q_\kappa \cap \partial A'$,

$$\mu_{n-1}(U_H) \geqslant \epsilon \mu_{n-1}(U).$$

Then we obtain

$$\mu_{n-1}(\partial\Omega \cap Q_\kappa) \geqslant \epsilon \mu_{n-1}\left(\Omega_\kappa \cap \partial A'\right). \tag{3.3}$$

Summing up (3.3) for $\kappa \in \{-1,1\}^n$, we obtain

$$\begin{aligned}\mu_{n-1}(\partial\Omega) =& \sum_{\kappa\in\{-1,1\}^n} \mu_{n-1}(\partial\Omega \cap Q_\kappa)\\ \geqslant& \epsilon \sum_{\kappa\in\{-1,1\}} \mu_{n-1}(\Omega_\kappa \cap \partial A)\\ =& \epsilon\mu_{n-1}(\Omega\cap\partial A').\end{aligned}$$

By the isoperimetric inequality (3.1) on $\mathbb{R}^n$, we have

$$\begin{aligned}\mu_{n-1}(\partial\Omega) \geqslant& \frac{1}{2}\mu_{n-1}(\partial\Omega) + \frac{\epsilon}{2}\mu_{n-1}(\Omega\cap\partial A')\\ \geqslant& c\mu_{n-1}\left(\partial\Omega\cup(\Omega\cap\partial A')\right)\\ \geqslant& c'\mu_n(\Omega)^{\frac{n-1}{n}},\end{aligned}$$

which concludes the isoperimetric inequality on $\mathbb{R}^n\setminus A'$. □

We remark that the above theorem implies that the connected sum $M^n_{m,\alpha}$ admits the global Gaussian heat kernel upper bound (3.2).

4 Heat kernel upper bound

4.1 General estimates

Let M_1, M_2 be geodesically complete non-parabolic Riemannian manifolds. We denote by $d_k(x,y)$, $V_k(x,r)$ and $p_k(t,x,y)$ the geodesic distance, the Riemannian volume of the ball and the heat kernel on M_k, respectively. For closed sets $A_1\subset M_1$ and $A_2\subset M_2$ of non-empty interior, let us consider the connected sum $M = M_1\#_J M_2$ along ∂A_1 and ∂A_2 by a joint J. Then the heat kernel $p(t,x,y)$ on M satisfies the following upper estimate:

Lemma 4.1. *Let us fix two families $\{F_i\}_{i\in I_1}$ and $\{U_i\}_{i\in I_1}$ of subsets of M_1 such that F_i are compact, U_i are open, $F_i\subset U_i$ and*

$$A_1\subset\bigcup_{i\in I_1}F_i.$$

We set also two families $\{K_j\}_{j\in I_2}$, $\{V_j\}_{j\in I_2}$ of subsets of M_2 by the same manner. Let x be a point in $M_1\setminus\bigcup_{i\in I_1}U_i$ and y be a point in $M_2\setminus\bigcup_{j\in I_2}V_j$. Then, for all $t>0$, the following estimate holds

$$\begin{aligned}p(t,x,y)\leqslant& \sum_{i\in I_1}\mathrm{cap}(F_i,U_i)\int_0^{t/2}\sup_{z\in U_i\setminus F_i}p_1(s,x,z)ds\sup_{\substack{0\leqslant s\leqslant t/2\\ z\in\partial A_1\cap F_i}}p(s,z,y)\\ &+\sum_{j\in I_2}\mathrm{cap}(K_j,V_j)\int_0^{t/2}\sup_{z\in V_j\setminus K_j}p_2(s,y,z)ds\sup_{\substack{0\leqslant s\leqslant t/2\\ z\in\partial A_2\cap K_j}}p(s,z,x). \quad (4.1)\end{aligned}$$

Moreover, suppose that the heat kernels $p_1(t,x_1,x_2)$ of M_1 and $p_2(t,y_1,y_2)$ of M_2 satisfy the Gaussian upper estimate (2.8) and the volume functions $V_1(x,R)$ of M_1, $V_2(y,R)$ of M_2 satisfy the volume doubling property (2.9). For $i \in I_1$, let $u_i(x) = d_1(x,U_i)$ and

$$f_i(y) = d(y, \partial A_1 \cap F_i).$$

For $j \in I_2$, we set $v_j(y)$, $k_j(x)$ by the same manner. Then

$$p(t,x,y) \leqslant C_1 t^{-n/2} \left(\sum_{i \in I_1} \frac{u_i(x)^2}{V_1(x,u_i(x))} \exp\left(-b_1 \frac{u_i(x)^2 + f_i(y)^2}{t} \right) \right. \\ \left. + \sum_{j \in I_2} \frac{v_j(y)^2}{V_2(y,v_j(y))} \exp\left(-b_1 \frac{v_j(y)^2 + k_j(x)^2}{t} \right) \right), \tag{4.2}$$

where the constants C_1, b_1 depend only on the constants $\kappa, \kappa', \beta, \beta'$ in (2.9) and C, b in (2.8).

Proof. From the argument in the proof of [16, Lemma 3.3], we have

$$p(t,x,y) \leqslant \mathbb{E}_x \left(1_{\{\tau_A < t/2\}} p(t-\tau_A, X_{\tau_A}, y) \right) + \mathbb{E}_y \left(1_{\{\tau_A < t/2\}} p(t-\tau_A, X_{\tau_A}, x) \right). \tag{4.3}$$

Since $A_1 \subset \bigcup_{i \in I_1} F_i$, by using the same argument as in the proof of Lemma 2.3, the first expectation in (4.3) can be estimated by

$$\mathbb{E}_x(1_{\{\tau_A < t/2\}} p(t-\tau_A, X_{\tau_A}, y)) \leqslant \\ \sum_{i \in I_1} \mathbb{E}_x \left(1_{\{\tau_A < t/2\} \cap \{X_{\tau_A} \in \partial A_1 \cap F_i\}} p(t-\tau_A, X_{\tau_A}, y) \right).$$

Then the strong Markov property yields

$$\mathbb{E}_x(1_{\{\tau_A < t/2\} \cap \{X_{\tau_A} \in \partial A_1 \cap F_i\}} p(t-\tau_A, X_{\tau_A}, y)) \\ \leqslant \mathbb{P}_x(\{\tau_A < t/2\} \cap \{X_{\tau_A} \in \partial A_1 \cap F_i\}) \sup_{\substack{0 \leqslant s \leqslant t/2 \\ z \in \partial A_1 \cap F_i}} p(s,z,y).$$

By the same argument as in the proof of Lemma 2.3, we have

$$\mathbb{P}_x \left(\{\tau_A < t/2\} \cap \{X_{\tau_A} \in \partial A_1 \cap F_i\} \right) \leqslant \mathbb{P}_x^{M_1} \left(\tau_{F_i} < t/2 \right).$$

Applying the estimate of the hitting probability (2.2), we obtain

$$\mathbb{E}_x \left(1_{\{\tau_A < t/2\}} p(t-\tau_A, X_{\tau_A}, y) \right) \leqslant \\ \sum_{i \in I_1} \operatorname{cap}(F_i, U_i) \int_0^{t/2} \sup_{z \in U_i \setminus F_i} p_1(s,x,z) ds \sup_{\substack{0 \leqslant s \leqslant t/2 \\ z \in \partial A_1 \cap F_i}} p(s,z,y).$$

Similarly we have

$$\mathbb{E}_y \left(1_{\{\tau_A < t/2\}} p(t-\tau_A, X_{\tau_A}, x)\right) \leqslant$$
$$\sum_{j \in I_2} \operatorname{cap}(K_j, V_j) \int_0^{t/2} \sup_{z \in V_j \setminus K_j} p_2(s,y,z) ds \sup_{\substack{0 \leqslant s \leqslant t/2 \\ z \in \partial A_2 \cap K_j}} p(s,z,x),$$

whence we obtain (4.1).

Under the additional conditions (2.8) and (2.9), we have

$$\sup_{\substack{0 \leqslant s \leqslant t/2 \\ z \in \partial A_1 \cap F_i}} p(s,z,y) \leqslant \frac{C}{V(y,\sqrt{t})} \exp\left(-b\frac{f_i(y)^2}{t}\right)$$

and, by Lemma 2.3,

$$\int_0^{t/2} \sup_{z \in U_i \setminus F_i} p_1(s,x,z) ds \leqslant \frac{u_i(x)^2}{V_1(x, u_i(x))} \exp\left(-b\frac{u_i(x)^2}{t}\right).$$

Substituting them into (4.1), we obtain (4.2). □

4.2 Heat kernel upper bound on $M^n_{m,\alpha}$

Fix parameters $1 \leqslant m \leqslant n-3$, $0 \leqslant \alpha < 1$, or $m = 0$, $\alpha = 0$, $n \geqslant 3$ and recall that the subset $A = A(m,\alpha)$ of $\mathbb{R}^n$ given by

$$A(m,\alpha) = \{x \in \mathbb{R}^n \mid h(x) \leqslant r(x)^\alpha\}.$$

We consider $M_1 = M_2 = \mathbb{R}^n$ and denote by A_1 and A_2 the two copies of $A(m,\alpha)$ on M_1 and M_2, respectively. Then $M^n_{m,\alpha}$ denotes $M_1 \#_J M_2$, the connected sum of $M_1 \setminus A_1$ and $M_2 \setminus A_2$ by a joint J. Our goal of this section is to prove the following upper estimate of the heat kernel $p(t,x,y)$ on $M^n_{m,\alpha}$:

Theorem 4.2. *There exists $L \geqslant 1$ such that for all $x \in M_1 \setminus A_1$, $y \in M_2 \setminus A_2$ satisfying $d(x,A) > Lr(x)^\alpha$, $d(y,A) > Lr(x)^\alpha$ and $t > 0$,*

$$p(t,x,y) \leqslant Ct^{-n/2}\Bigg(\left(\frac{r(x)^\alpha}{d(x,A)}\right)^{n-3} + \frac{1}{d(x,A)^{(1-\alpha)(n-3)}} + \left(\frac{r(y)^\alpha}{d(y,A)}\right)^{n-3} + \frac{1}{d(y,A)^{(1-\alpha)(n-3)}}\Bigg) e^{-bd(x,y)^2/t}.$$

Proof. Given a point $x \in M_1$, define the sequence of the couples $F_i \subset U_i$ by (2.20), (2.21). Such sequence can be defined in the same way for any point $y \in M_2$; in this case we denote the couples by $K_j \subset V_j$.

Since by Theorem 3.2 $M^n_{m,\alpha}$ satisfies the isoperimetric inequality (3.1), Lemma 4.1 implies that

$$\begin{aligned} p(t,x,y) \leqslant & Ct^{-n/2} \left(\sum_{i\in\{0\}\cup\mathbb{N}} \frac{\operatorname{cap}(F_i,U_i)}{u_i(x)^{n-2}} \exp\left(-b_1 \frac{u_i(x)^2+f_i(y)^2}{t}\right) \right. \\ & \left. + \sum_{j\in\{0\}\cup\mathbb{N}} \frac{\operatorname{cap}(K_j,V_j)}{v_j(y)^{n-2}} \exp\left(-b_1 \frac{v_j(y)^2+k_j(x)^2}{t}\right) \right). \end{aligned}$$

From the estimates in (2.22), by taking L large enough, for $x \in M_1$ with $d(x,A) > Lr(x)^\alpha$,

$$\operatorname{diam} U_i \leqslant C_1 2^i d(x,A) \leqslant C_2 d(x,U_i) = C_2 u_i(x)$$

for some $C_1, C_2 > 0$. Then we have

$$\begin{aligned} d(x,y) \leqslant & u_i(x) + \operatorname{diam} U_i + f_i(y) \\ \leqslant & (1+C_2)u_i(x) + f_i(y), \end{aligned}$$

which implies that

$$\exp\left(-b_1 \frac{u_i(x)^2+f_i(y)^2}{t}\right) \leqslant \exp\left(-b\frac{d(x,y)^2}{t}\right)$$

for some $b > 0$.

By the same argument, for $y \in M_2$ with $d(y,A) \geqslant Lr(y)^\alpha$,

$$\exp\left(-b_1 \frac{v_j(y)^2+k_j(x)^2}{t}\right) \leqslant \exp\left(-b\frac{d(x,y)^2}{t}\right).$$

Using the estimate (2.23) from the proof Theorem 2.6, we conclude that

$$\begin{aligned} p(t,x,y) \leqslant Ct^{-n/2} & \left(\left(\frac{r(x)^\alpha}{d(x,A)}\right)^{n-3} + \frac{1}{d(x,A)^{(1-\alpha)(n-3)}} \right. \\ & \left. + \left(\frac{r(y)^\alpha}{d(x,A)}\right)^{n-3} + \frac{1}{d(y,A)^{(1-\alpha)(n-3)}} \right) \exp\left(-b\frac{d(x,y)^2}{t}\right). \end{aligned}$$

□

5 Dirichlet heat kernel in the exterior of a non-compact set

In this section, we study the Gaussian lower bound of the Dirichlet heat kernel in the exterior of a non-compact set. This is a generalization of such an estimate on the exterior of a compact set proved in [15]. The Gaussian lower bound of the Dirichlet heat kernel plays a crucial role for the lower estimates of the heat kernel on connected sums (Lemma 6.1).

Let M be a geodesically complete non-parabolic Riemannian manifold that admits Li-Yau estimate (1.2), that is,

$$\frac{c_0}{V(x,\sqrt{t})}\exp\left(-B_0\frac{d(x,y)^2}{t}\right)\leqslant p(t,x,y)\leqslant\frac{C_0}{V(x,\sqrt{t})}\exp\left(-b_0\frac{d(x,y)^2}{t}\right),\tag{5.1}$$

for some constants $b_0, B_0, c_0, C_0 > 0$. By Theorem 7.1, (5.1) implies the volume doubling property

$$V(x,2r)\leqslant D_0V(x,r),\tag{5.2}$$

for all $x\in M$, $r>0$ and with some constant $D_0>1$. For any closed set $A\subset M$, define the hitting probability of A by

$$\Psi_A(x)=\mathbb{P}_x(\tau_A<\infty)$$

(cf. Section 2).

We say that a set $\Omega\subset M$ is a *good domain* with respect to $A\subset M$ if

$$\Psi_A(x)<\frac{c_0}{4D_0^2C_0}e^{-B_0}\quad\text{for all } x\in\Omega\tag{5.3}$$

and

$$V_\Omega(x,r):=\mu(B(x,r)\cap\Omega)>cV(x,r)\quad\forall x\in\Omega,\forall r>0\tag{5.4}$$

for some $c>0$. Clearly, (5.4) implies the volume doubling property for $V_\Omega(x,r)$ on Ω. Denote by d_Ω the geodesic distance on Ω.

In the next theorem, we obtain the Gaussian lower estimate of the Dirichlet heat kernel $p_{M\setminus A}(t,x,y)$.

Theorem 5.1. *Let Ω be a good domain with respect to $A\subset M$. Then for all $x,y\in\Omega$ and $t>0$,*

$$p_{M\setminus A}(t,x,y)\geqslant\frac{c_1}{V_\Omega(x,\sqrt{t})}\exp\left(-B_1\frac{d_\Omega(x,y)^2}{t}\right),\tag{5.5}$$

where the constants c_1, B_1 depend only on the constants b_0, B_0, c_0, C_0 from (5.1) and on the constant D_0 from (5.2).

Proof. First we assume that $x,y\in\Omega$ satisfies

$$d(x,y)\leqslant d_\Omega(x,y)\leqslant\sqrt{t}.\tag{5.6}$$

[16, Lemma 3.3] implies that

$$p_{M\setminus A}(t,x,y)\geqslant p(t,x,y)-\sup_{\substack{t/2\leqslant s\leqslant t\\ v\in\partial A}}p(s,v,y)\Psi_A(x)$$

$$- \sup_{\substack{t/2 \leqslant s \leqslant t \\ \omega \in \partial A}} p(s, \omega, x) \Psi_A(y).$$

From the assumption of the Gaussian lower bound (1.2) for $p(t,x,y)$ and (5.6),

$$p_{M\setminus A}(t,x,y) \geqslant \frac{c_0}{V(x,\sqrt{t})} e^{-B_0} - \sup_{t/2 \leqslant s \leqslant t} \frac{C_0}{V(y,\sqrt{t})} \Psi_A(x) - \sup_{t/2 \leqslant s \leqslant t} \frac{C_0}{V(x,\sqrt{t})} \Psi_A(y).$$

Since

$$B(x,r) \subset B(y, r + d(x,y)) \subset B(y, r+\sqrt{t}),$$

by the volume doubling property on M,

$$\frac{1}{V\left(y, \sqrt{\frac{t}{2}}\right)} \leqslant \frac{V(x,\sqrt{t})}{V\left(y, \frac{\sqrt{t}}{2}\right)} \frac{1}{V(x,\sqrt{t})} \leqslant \frac{V(y, 2\sqrt{t})}{V\left(y, \frac{\sqrt{t}}{2}\right)} \frac{1}{V(x,\sqrt{t})} \leqslant \frac{D_0^2}{V(x,\sqrt{t})}$$

and

$$\frac{1}{V\left(x, \sqrt{\frac{t}{2}}\right)} \leqslant \frac{V(x,\sqrt{t})}{V\left(x, \frac{\sqrt{t}}{2}\right)} \frac{1}{V(x,\sqrt{t})} \leqslant \frac{D_0}{V(x,\sqrt{t})}.$$

Then we get

$$p_{M\setminus A}(t,x,y) \geqslant \frac{1}{V(x,\sqrt{t})} \left(c_0 e^{-B_0} - D_0^2 C_0 (\Psi_A(x) + \Psi_A(y)) \right).$$

From the assumption (5.3) of Ω, we have

$$D_0^2 C_0 (\Psi_A(x) + \Psi_A(y)) < \frac{c_0}{2} e^{-B_0}$$

and then

$$p_{M\setminus A}(t,x,y) \geqslant \frac{c_0}{2V(x,\sqrt{t})} e^{-B_0} \geqslant \frac{c c_0}{2V_{\Omega(A)}(x,\sqrt{t})} e^{-B_0}$$

for $x, y \in \Omega$ with $d_\Omega(x,y) \leqslant \sqrt{t}$.

Since $V_\Omega(x,r)$ satisfies the volume doubling condition (5.4), applying the usual chaining argument (cf. [18]) to arbitrary $x, y \in \Omega$, we conclude the theorem. □

Remark 5.2. Sharp Dirichlet heat kernel estimate on inner uniform domains is obtained in [17].

6 Heat kernel lower bound

Let M_1 and M_2 be geodesically complete non-parabolic Riemannian manifolds. In this section we consider a lower bound of the heat kernel $p(t,x,y)$ on a connected sum $M = M_1 \#_J M_2$ of M_1 and M_2 along the boundary of $A_1 \subset M_1$ and $A_2 \subset M_2$ by J.

Lemma 6.1. *Let U be an open subset of M_1 so that $U \cap A_1 \neq \emptyset$ and let F be a compact subset of $A_1 \cap U$. We set W, $K \subset M_2$ by the same manner. Let*

$$M_1' = (M_1 \setminus A_1) \cup U, \quad M_2' = (M_2 \setminus A_2) \cup W,$$

and denote by $p_{M_i'}(t,x,y)$ the Dirichlet heat kernel on M_i'. Then for all $x \in M_1'$, $y \in M_2'$ and $t > 0$, we have

$$\begin{aligned} p(t,x,y) \;&\geqslant\; \frac{1}{2}\mathrm{cap}\,(F, M_1') \int_0^{t/2} \inf_{z \in \partial F} p_{M_1'}(s,x,z)ds \inf_{\substack{t/2 \leqslant s \leqslant t \\ z \in \partial A_1 \cap U}} p(s,z,y) \\ &\quad + \frac{1}{2}\mathrm{cap}\,(K, M_2') \int_0^{t/2} \inf_{z \in \partial K} p_{M_2'}(s,y,z)ds \inf_{\substack{t/2 \leqslant s \leqslant t \\ z \in \partial A_2 \cap W}} p(s,z,x). \quad (6.1) \end{aligned}$$

Moreover, assume that the heat kernels $p_1(t,x_1,x_2)$ on M_1 and $p_2(t,y_1,y_2)$ on M_2 satisfy the Li-Yau bound (1.2) and the volume functions $V_1(x,R)$ of M_1 and $V_2(y,R)$ of M_2 satisfy the volume doubling property (2.9). Let Ω_1, Ω_2 be good domains in M_1, M_2 with respect to $A_1 \setminus U$, $A_2 \setminus W$, respectively. We suppose that $F \subset \Omega_1$ and $K \subset \Omega_2$. For $x \in \Omega_1$, $y \in \Omega_2$, we set

$$\begin{aligned} F(x) &= \sup\{d_{\Omega_1}(x,z) \mid z \in F\}, \\ K(y) &= \sup\{d_{\Omega_2}(y,z) \mid z \in K\}. \end{aligned}$$

Then, for all $x \in \Omega_1$, $y \in \Omega_2$ and $t > 0$,

$$\begin{aligned} p(t,x,y) \;&\geqslant\; c_1 \mathrm{cap}\,(F, M_1') \frac{F(x)^2}{V_1(x, F(x))} e^{-B_1 \frac{F(x)^2}{t}} \inf_{\substack{t/2 \leqslant s \leqslant t \\ z \in \partial A_1 \cap U}} p(s,z,y) \\ &\quad + c_1 \mathrm{cap}(K, M_2') \frac{K(y)^2}{V_2(y, K(y))} e^{-B_1 \frac{K(y)^2}{t}} \inf_{\substack{t/2 \leqslant s \leqslant t \\ z \in \partial A_2 \cap W}} p(s,z,x). \quad (6.2) \end{aligned}$$

where the constants $B_1, c_1 > 0$ depend only on constants κ, β from (2.9), and on constants C, b from (1.2).

Proof. By using [16, Lemma 3.1], the strong Markov property yields

$$\begin{aligned} p(t,x,y) &\geqslant \mathbb{E}_x(1_{\{\tau_A < t/2\}} p(t - \tau_A, X_{\tau_A}, y)) \\ &\geqslant \mathbb{E}_x(1_{\{\tau_A < t/2\} \cap \{X_{\tau_A} \in \partial A_1 \cap U\}} p(t - \tau_A, X_{\tau_A}, y)) \\ &\geqslant \mathbb{P}_x\left(\{\tau_{A_1} < t/2\} \cap \{X_{\tau_{A_1}} \in A_1 \cap U\}\right) \inf_{\substack{t/2 \leqslant s \leqslant t \\ z \in \partial A_1 \cap U}} p(s,z,y). \end{aligned}$$

Since $F \subset A_1 \cap U$, we note that

$$\begin{aligned}\tau_{A_1\setminus U} \geqslant \frac{t}{2} \text{ and } \tau_F < \frac{t}{2} \Rightarrow & \tau_{A_1} < \frac{t}{2} \text{ and } X_{\tau_{A_1}} \in A_1 \cap U \\ \Leftrightarrow & \tau_{A_1\setminus U} \geqslant \frac{t}{2} \text{ and } \tau_{A_1\cap U} < \frac{t}{2}.\end{aligned}$$

Then the hitting probability

$$\mathbb{P}_x\left(\{\tau_{A_1} < t/2\} \cap \{X_{\tau_{A_1}} \in A_1 \cap U\}\right)$$

can be estimated from below by the hitting probability to F by time $t/2$ in $(A_1 \setminus U)^c = M_1'$ with Dirichlet boundary condition. By using the estimate of the hitting probability (2.2) on M_1', the following lower estimate holds:

$$\mathbb{P}_x\left(\{\tau_{A_1} < t/2\} \cap \{X_{\tau_{A_1}} \in \partial A_1 \cap U\}\right) \geqslant \operatorname{cap}(F, M_1') \int_0^{t/2} \inf_{z\in\partial F} p_{M_1'}(s,x,z)ds.$$

Hence we obtain

$$p(t,x,y) \geqslant \operatorname{cap}(F, M_1') \int_0^{t/2} \inf_{z\in\partial F} p_{M_1'}(s,x,z)ds \inf_{\substack{t/2\leqslant s\leqslant t \\ z\in\partial A_1\cap U}} p(s,z,y). \tag{6.3}$$

By the symmetry of $p(t,x,y)$ with respect to x, y, the estimate (6.1) follows.

Under the additional assumptions (1.2), (2.9) of M and $F \subset \Omega_1$, we have

$$\inf_{z\in\partial F} p_{M_1'}(s,x,z)ds \geqslant \frac{c}{V_1(x,\sqrt{s})} \exp\left(-B\frac{F(x)^2}{s}\right)$$

by Theorem 5.1, and

$$\int_0^t \frac{c}{V_1(x,\sqrt{s})} \exp\left(-B\frac{F(x)^2}{s}\right) \geqslant \frac{c' F(x)^2}{V_1(x,F(x))} \exp\left(-B_1\frac{F(x)^2}{t}\right)$$

by Lemmas 2.2, 2.4. Substituting into (6.3), we obtain

$$\begin{aligned}&p(t,x,y) \geqslant \\ &c' \operatorname{cap}(F, M_1') \frac{F(x)^2}{V_1(x,F(x))} \exp\left(-B_1\frac{F(x)^2}{t}\right) \inf_{\substack{t/2\leqslant s\leqslant t \\ z\in\partial A_1\cap U}} p(s,z,y).\end{aligned}$$

By the symmetry of $p(t,x,y)$ with respect to x, y, we conclude the lemma. $\square$

Next we need the lower bound for

$$\inf_{\substack{t/2\leqslant s\leqslant t \\ z\in\partial A_1\cap U}} p(s,z,y).$$

Suppose that the heat kernel $p_2(t, y_1, y_2)$ on M_2 satisfies the Li-Yau bound (1.2). Let Ω_2 be a good domain in M_2 with respect to J and let us assume the parabolic Harnack inequality (7.3) for all balls in $M \setminus \Omega_2$ which do not intersect the boundary.

For $z \in \partial A_1 \cap U$, set $w = w(z)$ in Ω_2 and fix a continuous curve γ_z between z and w of length ℓ_z. For $r > 0$, let γ_1 be a connected component of

$$\gamma_z \setminus B(\Omega_2, 2r)$$

containing z, and γ_2 be a connected component of

$$\gamma_z \setminus B(J, 2r)$$

containing w. We set

$$\Gamma_1(r) = B(\gamma_1, r),\ \Gamma_2(r) = B(\gamma_2, r).$$

We denote by ρ_z the supremum of $r > 0$ so that

$$\Gamma_1(r) \cap \Gamma_2(r) \neq \emptyset$$

(see Figure 8).

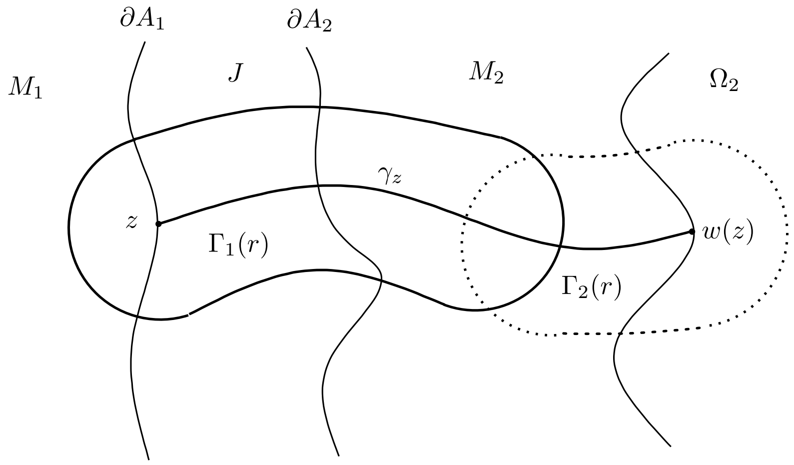

Figure 8: $\Gamma_1(r)$ and $\Gamma_2(r)$

Set

$$\mathcal{W} = \bigcup_{z \in \partial A_1 \cap U} w(z),$$
$$\ell = \sup_{z \in \partial A_1 \cap U} \ell_z,$$
$$\rho = \frac{1}{2} \inf_{z \in \partial A_1 \cap U} \rho_z.$$

For $y \in \Omega_2$, set also

$$\mathcal{W}(y) = \sup_{w \in \mathcal{W}} d_{\Omega_2}(y, w).$$

Then we obtain the following:

Lemma 6.2. *For all $y \in \Omega_2$ and $t > 2\ell^2$,*

$$\inf_{z \in \partial A_1 \cap U} p(t, z, y) \geqslant \exp\left(-H'\left(1 + \frac{\ell^2}{\rho^2}\right)\right) \frac{c}{V(y, \sqrt{t})} \exp\left(-B\frac{\mathcal{W}(y)^2}{t}\right),$$

where the constant H' depends only on the constants H from (7.3) on $M \setminus \Omega_2$ and on M_2, and the constants c, B depend only on the constants C, b from (1.2).

Proof. It should be noted that the Harnack inequality (7.3) holds on M_2 from the assumption of the Li-Yau estimate (1.2). Conjunction with the assumption of the Harnack inequality on $M \backslash \Omega_2$ for all balls which do not intersect the boundary, for any $z \in \partial A_1 \cap U$, we can apply [23, Corollary 5.4.4] on $\Gamma_1(\rho)$ and $\Gamma_2(\rho)$. Hence there exists $H' > 0$ such that for all $t > 2\ell^2$,

$$\begin{aligned} p(t, z, y) &\geqslant \exp\left(-H'\left(1 + \frac{\ell_z^2}{\rho_z^2}\right)\right) p(t - \ell_z^2, w, y) \\ &\geqslant \exp\left(-H'\left(1 + \frac{\ell^2}{\rho^2}\right)\right) p_{M_2 \backslash A_2}(t - \ell_z^2, w, y). \end{aligned}$$

Since $w, y \in \Omega_2$, Theorem 5.1 implies that

$$p_{M_2 \backslash A_2}(t - \ell_z^2, w, y) \geqslant \frac{c}{V(y, \sqrt{t - \ell_z^2})} \exp\left(-B\frac{d_{\Omega_2}(w, y)^2}{t - \ell_a^2}\right).$$

By the volume doubling property (7.2), for $t > 2\ell^2 \geqslant 2\ell_z^2$, we obtain

$$\frac{c}{V(y, \sqrt{t - \ell_z^2})} \exp\left(-B\frac{d_{\Omega_2}(w, y)^2}{t - \ell_z^2}\right) \geqslant \frac{c'}{V(y, \sqrt{t})} \exp\left(-B'\frac{\mathcal{W}(y)^2}{t}\right)$$

which concludes the lemma. □

Let n, m, α be as in (1.7) and let $A = A(m, \alpha)$ where the latter is defined by (1.6). Consider two copies of $\mathbb{R}^n$: $M_1 = M_2 = \mathbb{R}^n$ and denote by A_1, A_2 the copies of the set A on M_1 and M_2, respectively. Consider the connected sum

$$M^n_{m,\alpha} = M_1 \#_J M_2 = \mathbb{R}^n \#_J \mathbb{R}^n$$

between $M_1 \setminus A_1$ and $M_2 \setminus A_2$ by J. Here the joint J is defined so that for all $L \geqslant 0$, there exists a quasi-isometry

$$f_k^L : \mathbb{R}^n \setminus A' \to M^n_{m,\alpha} \setminus E_k^L,$$

where

$$A' = \left\{ x \in \mathbb{R}^n \mid h(x) \leqslant \frac{1}{2} r(x)^\alpha \right\}$$

and

$$E_k^L = \{ x \in M_k \mid d(x, A) \geqslant L r(x)^\alpha \} .$$

Note that, by Theorem 2.6, there exists $L_0 > 0$ such that $E_1^L \subset M_1$, $E_2^L \subset M_2$ are good domains with respect to $A_1 = A_2 = A(m, \alpha)$ for all $L \geqslant L_0$, respectively. Then we obtain the following lower bound of the heat kernel $p(t, x, y)$ on $M_{m,\alpha}^n$ assuming that x and y belong to different copies of $\mathbb{R}^n$ and $d(x, J)$, $d(y, J)$, t are large enough:

Theorem 6.3. *There exist $L \geqslant L_0$, $T > 1$ such that, for all $x \in E_1^L$, $y \in E_2^L$ and $t > T \left(d(x, J) + d(y, J) \right)^{2\alpha}$,*

$$\begin{aligned} p(t, x, y) \geqslant c t^{-n/2} \Bigg\{ & \left(\frac{r(x)^\alpha}{d(x, J)} \right)^{n-m-2} + \frac{1}{d(x, J)^{(1-\alpha)(n-m-2)}} \\ & + \left(\frac{r(y)^\alpha}{d(y, J)} \right)^{n-m-2} + \frac{1}{d(y, J)^{(1-\alpha)(n-m-2)}} \Bigg\} e^{-B \frac{d(x,y)^2}{t}} . \end{aligned}$$

Proof. As we have taken in (2.24), recall

$$x_z = \begin{cases} \left(1 + \frac{z}{|x'|} \right) x', \text{ if } x' \neq 0, \\ (z, 0, ..., 0), \ \text{ if } x' = 0. \end{cases}$$

For $d = d(x, J)$, we define

$$U = x_{4d} + B_m(3d) \times B_{n-m} \left((r(x) + 7d)^\alpha \right) \subset M_1$$

and a compact set

$$F = x_{4d} + \overline{B_m(d)} \times \overline{B_{n-m} \left(r(x_{3d})^\alpha \right)},$$

which has been taken in (2.25). By the same argument as in Theorem 2.6, the hitting probability

$$\mathbb{P}_z(\tau_{A_1 \setminus U} < \infty)$$

has the same upper bound with respect to the distance $d(z, A_1 \setminus U)$. Since

$$d(F, A_1 \backslash U) = 2d,$$

by taking $L > 1$ large enough, for all $x \in M_1 \setminus A_1$ with $d \geqslant L r(x)^\alpha$,

$$\Omega = E_1^L \cup \{ z \in M_1 \, : \, |z' - x_{4d}| \leqslant 2d \}$$

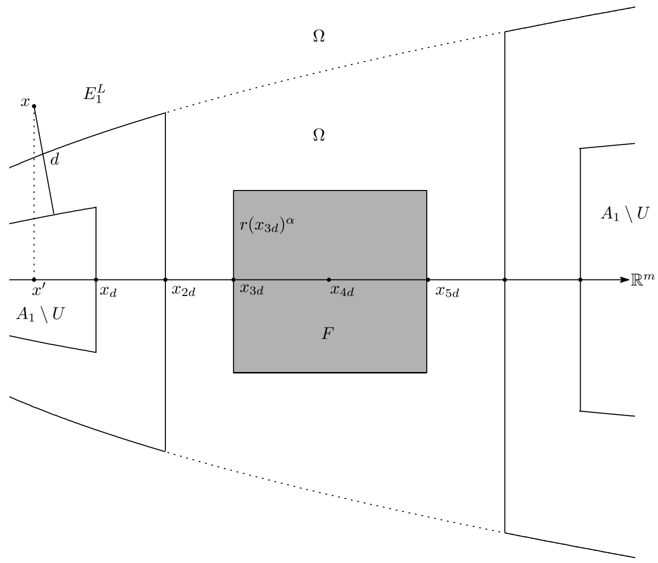

Figure 9: Good domain Ω with respect to $A_1 \setminus U$.

is a good domain with respect to $A_1 \setminus U$ containing F (see Figure 9).

Then Theorem 2.6 and Lemma 6.1 imply that

$$\begin{aligned} p(t,x,y) \geqslant & c\left(\left(\frac{r(x)^\alpha}{d}\right)^{n-m-2} + \frac{c}{d^{(1-\alpha)(n-m-2)}}\right) \exp\left(-B\frac{F(x)^2}{t}\right) \\ & \times \inf_{\substack{t/2\leqslant s\leqslant t \\ z\in\partial A_1\cap U}} p(s,z,y). \end{aligned} \tag{6.4}$$

Next we estimate

$$\inf_{\substack{t/2\leqslant s\leqslant t \\ z\in\partial A_1\cap U}} p(s,z,y).$$

Due to the quasi-isometry

$$f_2^L : \mathbb{R}^n \setminus A' \to M_{m,\alpha}^n \setminus E_2^L,$$

the Harnack inequality (7.3) holds on $M_{m,\alpha}^n \setminus E_2^L$ for all balls which do not intersect the boundary. For $z \in \partial A_1 \cap U$, set $\zeta = \left(f_2^L\right)^{-1}(z) \in \mathbb{R}^n \setminus A'$. Define a smooth curve $\gamma_z(t)$ by

$$\gamma_z(t) = f_2^L\left(\zeta' + (1-t)\left(\zeta - \zeta'\right)\right), \quad 0 \leqslant t \leqslant T, \tag{6.5}$$

where $T = T(\zeta)$ is the time so that $\zeta' + (1-T)(\zeta - \zeta')$ is on $\partial A'$. Set

$$w(z) = f_2^L\left(\zeta' + (1-T)\left(\zeta - \zeta'\right)\right) \in \partial E_2^L, \quad \mathcal{W} = \cup_{z\in\partial A_1\cap U} w(z).$$

Since the map f_2^L is quasi-isometric, $r(z) \asymp r(\zeta)$, $T \asymp r(\zeta)^\alpha$, and then

$$\ell_z \asymp r(z)^\alpha \asymp \rho_z \asymp d^\alpha.$$

Therefore ℓ^2/ρ^2 is uniformly bounded, and hence Lemma 6.2 implies that, for all $t \geqslant 2C^2 d^{2\alpha}$ we obtain

$$\inf_{\substack{t/2 \leqslant s \leqslant t \\ z \in \partial A_1 \cap U}} p(s, z, y) \geqslant c t^{-n/2} \exp\left(-B \frac{D_2(y)^2}{t}\right). \tag{6.6}$$

To finish the proof, we show that there exist $B_1, B_2 > 0$ such that

$$F(x) = \sup\{d_\Omega(x, z) \mid z \in F\} \leqslant B_1 d(x, y), \tag{6.7}$$

$$\mathcal{W}(y) = \sup\left\{d_{E_2^L}(y, w) \mid w \in \mathcal{W}\right\} \leqslant B_2 d(x, y). \tag{6.8}$$

From the definition of F,

$$F(x) \leqslant 9d + h(x) + r(x_{3d})^\alpha.$$

Since

$$\begin{aligned} h(x) \leqslant & Cd + r(x)^\alpha, \\ r(x_{3d})^\alpha \leqslant & r(x)^\alpha + 3d, \end{aligned}$$

by taking $L \geqslant L_0$ large enough, for all $x \in M_1$ satisfying $d \geqslant L r(x)^\alpha$, there exists $B_1 > 0$ such that

$$F(x) \leqslant B_1 d \leqslant B_1 d(x, y).$$

To prove (6.8), we introduce the following notation. For any $y \in \mathbb{R}^n$, we denote the coordinates of y by

$$(y', h(h), \eta(y)) \in \mathbb{R}^m \times \mathbb{R}^{n-m},$$

where $(h(y), \eta(y))$ is the polar coordinates of $y - y' \in \mathbb{R}^{n-m}$ with radius $h(y) \geqslant 0$ and angle $\eta(y) \in \mathbb{S}^{n-m-1}$. For $y \in E_2^L$ and $w = w(z) \in \mathcal{W}$, set

$$\begin{aligned} w_y = & (w', h(y), \eta(y)) \\ y_w = & (y', h(y) + h(w), \eta(y)) \end{aligned}$$

(see Figure 10).

Then we obtain

$$\begin{aligned} d_{E_2^L}(y, w) \leqslant & d(y, y_w) + d(y_w, w_y) + d_{E_2^L}(w_y, w) \\ \leqslant & h(w) + d(y_w, y) + d(y, x) + d(x, z) + d(z, w) \\ & + d(w, w_y) + d_{E_2^L}(w_y, w) \\ \leqslant & 2h(w) + d(y, x) + d(x, z) + d(z, w) + 2 d_{E_2^L}(w_y, w). \end{aligned}$$

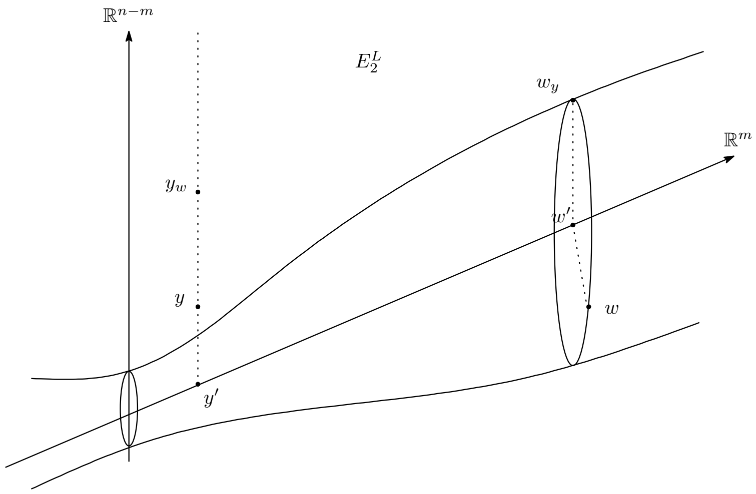

Figure 10: w_y and y_w

Taking $L \geqslant L_0$ large enough and $x \in E_1^L$, there exists $B_2' > 0$ such that

$$\begin{aligned} d_{E_2^L}(w_y, y) \leqslant& Ch(w), \\ h(w) \leqslant& 2d(z, w), \\ d(z, w) \leqslant& B_2' d \leqslant B_2' d(x, y), \\ d(x, z) \leqslant& B_2' d \leqslant B_2' d(x, y) \end{aligned}$$

whence we obtain (6.8).

Combining the above estimates (6.4), (6.6), (6.7) and (6.8), we obtain that the heat kernel $p(t, x, y)$ on $M^n_{m,\alpha}$ admits the following lower estimate:

$$p(t, x, y) \geqslant ct^{-n/2} \left(\left(\frac{r(x)^\alpha}{d} \right)^{n-m-2} + \frac{1}{d^{(1-\alpha)(n-m-2)}} \right) \exp\left(-B \frac{d(x, y)^2}{t} \right).$$

By the symmetry of $p(t, x, y)$ with respect to x and y, we conclude the theorem. $\square$

Finally we prove the rest of Theorem 1.1. Set $L \geqslant L_0$ as we have chosen in the above theorem. The lower bound of $p(t, x, y)$ for $x, y \in E_1^L$ has already proved in Theorem 2.6 and Theorem 5.1 because E_1^L is a good domain with respect to J. Let us set

$$C(L) = M^n_{m,\alpha} \setminus \left(E_1^L \cup E_2^L \right)$$

and consider the lower bound of the heat kernel $p(t, x, y)$ on $M^n_{m,\alpha}$ for $x, y \in C(L)$ or $x \in E_1^L$, $y \in C(L)$.

For $z \in C(L) \cup E_1^L$, let γ_z be a curve from z to E_1^L given by the same manner of (6.5) ($\gamma_z = \mathit{const}$ if $z \in E_1^L$). By using the argument in Lemma 6.2, for all $t \geqslant 2(\ell_x^2 + \ell_y^2)$, we have

$$p(t,x,y) \geqslant \exp\left(-H'\left(1 + \frac{\ell_x^2}{\rho_x^2} + \frac{\ell_y^2}{\rho_y^2}\right)\right) p_{M_1 \setminus A_1}(t - 2\left(\ell_x^2 + \ell_y^2\right), w(x), w(y)).$$

Since

$$\begin{aligned} \ell_x \leqslant & Cd(x, E_1^L), \quad \rho_x \geqslant cr(x)^\alpha \geqslant c'd(x, E_1^L), \\ \ell_y \leqslant & Cd(y, E_1^L), \quad \rho_y \geqslant cr(y)^\alpha \geqslant c'd(y, E_1^L), \end{aligned}$$

Theorem 5.1 implies that, for $t \geqslant T\left(d(x, E_1^L) + d(y, E_1^L)\right)^2$

$$p(t,x,y) \geqslant \frac{c}{V(x,\sqrt{t})} \exp\left(-B\frac{d(x,y)^2}{t}\right),$$

which completes the proof of Theorem 1.1.

7 Appendix

The following theorem is a combined result of [10], [13], [22].

Theorem 7.1. *For any geodesically complete non-compact Riemannian manifold M, the following three properties are equivalent:*

(i) The Li-Yau bound (1.2).

(ii) The Poincaré inequality: there exists $P > 0$ such that for all $x \in M$, $r > 0$ and all $f \in C^\infty(B(x, 2r))$,

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leqslant Pr^2 \int_{B(x,2r)} |\nabla f|^2 d\mu, \tag{7.1}$$

where

$$f_{B(x,r)} = \frac{1}{V(x,r)} \int_{B(x,r)} f d\mu,$$

and the volume doubling condition: there exists $D > 1$ such that for all $x \in M$, $r > 0$,

$$V(x, 2r) \leqslant DV(x, r). \tag{7.2}$$

(iii) The parabolic Harnack inequality: there exists $H > 0$ such that for all $x \in M$, $r > 0$ and for any positive solution u of the heat equation (1.1) on a cylinder $Q = (0, r^2) \times B(x, r)$, the following inequality holds

$$\sup_{Q_-} u \leqslant H \inf_{Q_+} u, \tag{7.3}$$

where

$$Q_+ = \left(\frac{3}{4}r^2, r^2\right) \times B\left(x, \frac{1}{2}r\right),$$
$$Q_+ = \left(\frac{1}{4}r^2, \frac{1}{2}r^2\right) \times B\left(x, \frac{1}{2}r\right).$$

Let us use this theorem to verify that the connected sum $M^n_{m,\alpha}$ does not satisfy the Li-Yau estimate (1.2). Of course, this follows from our main Theorem 1.1, but one can see directly the failure of the Poincaré inequality (7.1) on $M^n_{m,\alpha}$.

For any closed set $A \subset M^n_{m,\alpha}$, let $\Psi_A(z) = \mathbb{P}_z(\tau_A < \infty)$ be the hitting probability of A (see Section 2). For any $a \in J$ and $r > 0$, we write $B_r := B(a, r)$ and consider a function f on B_r given by

$$f(z) = \begin{cases} 1 - \Psi_{\overline{J \cap B_{2r}}}(z) & z \in (M_1 \setminus A_1) \cap B_{2r} \\ 0 & z \in J \cap B_{2r} \\ -c\left(1 - \Psi_{\overline{J \cap B_{2r}}}(z)\right) & z \in (M_2 \setminus A_2) \cap B_{2r} \end{cases},$$

where $c \in \mathbb{R}$ is chosen so that $f_{B_{2r}} = 0$. Since $\Psi_{\overline{J \cap B_{2r}}}$ is the equilibrium potential for $\operatorname{cap}(J \cap B_{2r})$ (cf. [11], [15]), we have

$$\int_{B_{2r}} |\nabla f|^2 d\mu \leqslant \operatorname{cap}(J \cap B_{2r}).$$

Moreover we have

$$\begin{aligned} \int_{B_{2r}} |f - f_{B_{2r}}|^2 d\mu \geqslant & \int_{(M_1 \setminus A_1) \cap B_{2r}} \left|1 - \Psi_{\overline{J \cap B_{2r}}}\right|^2 d\mu \\ \geqslant & (1-\epsilon)^2 \mu \left\{ z \in (M_1 \setminus A_1) \cap B_{2r} \, : \, \Psi_{\overline{J \cap B_{2r}}}(z) < \epsilon \right\} \end{aligned}$$

for all $0 < \epsilon < 1$. Then we obtain

$$\frac{\int_{B_{2r}} |\nabla f|^2 d\mu}{\int_{B_{2r}} |f - f_{B_{2r}}|^2 d\mu} \leqslant \frac{\operatorname{cap}(J \cap B_{2r})}{(1-\epsilon)^2 \mu\{z \in B_{2r} \cap M_1 : \Psi_{J \cap B_{2r}}(z) < \epsilon\}}.$$

By Theorem 2.6, for all $\epsilon > 0$, there exists $L > 0$ such that

$$\begin{aligned} \{z \in B_{2r} \cap M_1 : \Psi_{J \cap B_{2r}}(z) < \epsilon\} \supset & \{z \in B_{2r} \cap M_1 : \Psi_J(z) < \epsilon\} \\ \supset & B_{2r} \cap E_1^L. \end{aligned}$$

Since E_1^L is a good domain, there exists a positive constant $r_a > 0$ depending only on $a \in J$ such that for all $r \geqslant r_a$

$$\begin{aligned}\mu\{x \in B_{2r} \cap M_1 : \Psi_{J\cap B_{2r}}(x) < \epsilon\} \geqslant& \mu\left(B_{2r} \cap E_1^L\right) \\ \geqslant& cr^n.\end{aligned}$$

On the other hand, Lemma 2.5 implies that

$$\begin{aligned}\operatorname{cap}(J \cap B_{2r}) \leqslant& \operatorname{cap}\left(B_m(4r) \times B_{n-m}\left(\left(\sqrt{r^2+1}\right)^{\alpha}\right)\right) \\ \leqslant& Cr^{m+\alpha(n-m-2)}.\end{aligned}$$

Then we obtain

$$\frac{\int_{B_{2r}} |\nabla f|^2 d\mu}{\int_{B_{2r}} |f - f_{B_{2r}}|^2 d\mu} \leqslant \frac{C}{r^{2+(1-\alpha)(n-m-2)}}$$

for $r \geqslant r_a$, which shows that the Poincaré inequality (7.1) fails on $M^n_{m,\alpha}$.

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