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Stochastic Delay Equations and Inclusions with Mean Derivatives on Riemannian Manifolds. II

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Abstract

We find new existence of solution theorems to stochastic delay equations and inclusions with mean derivatives on a Riemannian manifold. The delays in both the equations and the inclusions are expressed in terms of stochastic Riemannian parallel translation.

Key words: Riemannian manifold; Riemannian parallel translation; mean derivative; quadratic mean derivative; equation with delay; inclusion with delay.

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Introduction

In [4] we considered the stochastic equations and inclusions with delay in terms of mean derivatives on stochastically complete Riemannian manifolds such that the right-hand side of the part with quadratic derivative was single valued and equal to I, the unit matrix. This paper is a continuation of [4]. Here we deal with the general case of the right-hand side of the part with quadratic derivative that may be set-valued and, generally speaking, is not constant. As well as in [4], the delay summands in right-hand sides are given in terms of Riemannian parallel translation. To avoid some technical difficulties we consider compact Riemannian manifolds where we specify the Levi-Civita connection.

We refer the reader to [4] for the definition of mean derivatives on manifolds and the other notions. Recall that here we use the defenitions of mean derivatives with respect to the past σ-algebras that is compatible with delay parallel translation.

1 Preliminaries on Itô equations on manifolds

Everywhere below we deal with a compact Riemannian manifold M.

Definition 1.1. *([2]) The couple $(a(t,m), A(t,m))$ where $a(t,m)$ is a vector field on M and $A(t,m)$ is a field of linear operators $A(t,m) : \mathbb{R}^k \to T_mM$ sending a certain Euclidean space $\mathbb{R}^k$ to the tangent spaces to M, is called an Itô vector field.*

Definition 1.2. *([2]) The forward stochastic differential*

$$(a(t,m)dt + A(t,m)dw(t))$$

at a point $m \in M$ given by an Itô vector field (a, A), is the class of stochastic processes in the tangent space T_mM that consists of solutions of all stochastic differential equations of the form

$$X(t+s) = \int_t^{t+s} \tilde{a}(\tau, X(\tau))d\tau + \int_t^{t+s} \tilde{A}(\tau, X(\tau))dw(\tau),$$

where $\tilde{a}(\tau, X)$ is a vector field on T_mM; $\tilde{A}(\tau, X) : \mathbb{R}^k \to T_mM$ is a linear operator depending on parameters $\tau \in R$ and $X \in T_mM$; and the following conditions are satisfied: $\tilde{a}(\tau, X)$ and $\tilde{A}(\tau, X)$ are Lipschitz continuous, are equal to zero outside a certain neighbourhood of origin in T_mM and such that for $\tau \geq t$ the equalities $\tilde{a}(\tau, 0) = a(t,m)$ and $\tilde{A}(\tau, 0) = A(t,m)$ hold.

Specify a connection H on M and denote by exp^H the exponential map of H.

Definition 1.3. *([2, 5]) We say that a process $\xi(t)$ satisfies the Itô equation in Belopolskaya-Daletskii form relative to H*

$$d\xi(t) = exp^H_{\xi(t)}(a(t,\xi(t))dt + A(t,\xi(t))dw(t)), \tag{1.1}$$

if for every point $\xi(t)$ there exists its neighbourhood in M such that before the exit of $\xi(t+s)$, $s \geq 0$ from this neighbourhood, $\xi(t+s)$ a.s. coincides with a certain process from the class $exp^H_{\xi(t)}(a(t,\xi(t))dt + A(t,\xi(t))dw(t))$.

Note that in a local chart equation (1.1) takes the form

$$d\xi(t) = a(t,\xi(t))dt - \frac{1}{2} tr\, \mathbf{\Gamma}^H_{\xi(t)}(A(t,\xi(t)), A(t,\xi(t)))dt + A(t,\xi(t))dw(t) \quad (1.2)$$

where $\mathbf{\Gamma}^H$ is the local connector of H in the chart. Equation (1.2) is called an Ito equation in Baxendale form. It is shown that under coordinate changes (1.2) is transformed according to Ito formula.

Lemma 1.1. *(see, e.g., [5]) Let $\xi(t)$ be a solution to equation* (1.1)*. Then: (i) $D\xi(t) = a(t,\xi(t))$ where the forward mean derivative $D\xi(t)$ is calculated with respect to H, the same connection that is in use in* (1.1)*; (ii) $D_2\xi(t) = (AA^*)(t,\xi(t))$ and it does not depend on the connection.*

On M we shall use the Levi-Civita connection both in equations of (1.1) type and in the calculation of mean derivatives. On some other manifolds (say, on the bundle of orthonormal frames OM) the connections will be introduced specially.

Let $\pi : OM \to M$ be the bundle of orthonormal frames on M. Note that the standard fiber of OM is the orthogonal group $O(n)$ where $n = \dim M$, that is compact. Since M is also compact, the total space of OM is compact as well.

Let H be the Levi-Cvitá connection on OM and V be the vertical distribution on OM. Recall (see, e.g., [5]) that the bundles V and H over OM are trivial: V is trivialazed by fundamental vector fields and H by basic vector fields $\mathsf{E}(x)$ where the vector $\mathsf{E}_b(x) \in \mathsf{H}_b$ for $b \in OM$ and $x \in \mathbb{R}^n$ is defined by equality $\mathsf{E}_b(x) = T\pi^{-1}(bx)_{|\mathsf{H}_b}$ (the frame b is considered here as a linear operator $b : \mathbb{R}^n \to T_{\pi b}M$, see, e.g., [5]). Thus the tangent bundle $TOM = \mathsf{H} \oplus \mathsf{V}$ is also trivial.

Definition 1.4 ([5])**.** *The Riemannian metric on OM, generated by the abovementioned trivialization of tangent bundle TOM is called induced.*

Denote by e the exponential mapping of Levi-Civitá connection of some induced metric on OM.

Lemma 1.2 ([5])**.** (i) *For all induced metrics the restrictions $\mathsf{e}_{|\mathsf{H}}$ coincide.*

(ii) *For every $Y \in \mathsf{H}$ the equality $\pi\mathsf{e}(Y) = exp(T\pi Y)$ holds where exp is the exponential mapping of Levi-Civitá connection on M.*

(iii) *For all induced metrics in every (specified) chart on OM the restrictions of local connectors $\mathbf{\Gamma}^{\mathsf{e}}(X,X)$ to H coincide as operators of $X \in \mathsf{H}$.*

2 Setting up the general problem

Let on M two vector fields $X(t,m)$ and $Y(t,m)$ and two $(2,0)$-tensor fields $\alpha(t,m)$ and $\beta(t,m)$ be given, $t \geq 0$. As well as in [4], we denote by $\Gamma_{t,s}$ the parallel translation along a smooth curve or a stochastic process of vectors or tensors from the time instant s to the time instant t. We shall also deal with set-valued vector fields $\boldsymbol{X}(t,m)$ and $\boldsymbol{Y}(t,m)$ and set-valued $(2,0)$-tensor fields $\boldsymbol{\alpha}(t,m)$ and $\boldsymbol{\beta}(t,m)$.

Specify $h > 0$. Consider the system

$$\begin{aligned} D\xi(t) &= X(t,\xi(t)) + \Gamma_{t,t-h}Y(t-h,\xi(t-h)) \\ D_2\xi(t) &= \alpha(t,\xi(t)) + \Gamma_{t,t-h}\beta(t-h,\xi(t-h)) \end{aligned} \tag{2.1}$$

that is called a stochastic differential equation with mean derivatives with delay. For set-valued vector and tensor fields the system

$$\begin{aligned} D\xi(t) &\in \boldsymbol{X}(t,\xi(t)) + \Gamma_{t,t-h}\boldsymbol{Y}(t-h,\xi(t-h)) \\ D_2\xi(t) &\in \boldsymbol{\alpha}(t,\xi(t)) + \Gamma_{t,t-h}\boldsymbol{\beta}(t-h,\xi(t-h)) \end{aligned} \tag{2.2}$$

is called a stochastic differential incclusion with mean derivatives with delay.

Specify a C^1-curve $\varphi\colon [-h,0] \to M$.

Definition 2.5. *We say that equation* (2.1) *(inclusion* (2.2)*, respectively) has a solution on the interval* $[-h,\varepsilon)$ *with initial condition* φ*, if there exists a probability space* $(\Omega,\mathcal{F},\mathsf{P})$ *and a stochastic process* $\xi(\cdot)\colon [-h,\varepsilon) \to M$*,* $\varepsilon > 0$ *given on* $(\Omega,\mathcal{F},\mathsf{P})$ *and taking values in* M *such that it coincides with* φ *on* $[-h,0]$*, and satisfies* (2.1) *(*(2.2)*, respectively) on* $[0,\varepsilon)$*.*

It is useful to first analyze simplified cases of (2.1) and (2.2), where the delayed summands depend on time only. These systems are given by the formulae

$$\begin{aligned} D\xi(t) &= X(t,\xi(t)) + \Gamma_{t,0}Y(t) \\ D_2\xi(t) &= \alpha(t,\xi(t)) + \Gamma_{t,0}\beta(t) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} D\xi(t) &\in \boldsymbol{X}(t,\xi(t)) + \Gamma_{t,0}\boldsymbol{Y}(t) \\ D_2\xi(t) &\in \boldsymbol{\alpha}(t,\xi(t)) + \Gamma_{t,0}\boldsymbol{\beta}(t). \end{aligned} \tag{2.4}$$

In this cases $Y(t)$, $\boldsymbol{Y}(t)$, $\beta(t)$ and $\boldsymbol{\beta}(t)$ take values in the tangent space to M at the initial point m_0. By Radon's mechanical interpretation of parallel translation (see, e.g., its presentation in [4, 5]) the physical meaning of (2.3) and (2.4) is that the second summands in right-hand sides are given a priori in the reference system that is a natural replacement of the constant system of coordinates in linear phase space.

Definition 2.6. *We say that equation* (2.3) *(inclusion* (2.4)*, respectively) has a solution on the interval* $[0, \varepsilon)$ *with initial condition* $m_0 \in M$*, if there exists a probability space* $(\Omega, \mathcal{F}, \mathsf{P})$ *and a stochastic process* $\xi(\cdot)\colon [0, \varepsilon) \to M$, $\varepsilon > 0$ *given on* $(\Omega, \mathcal{F}, \mathsf{P})$ *and taking values in* M *such that* $\xi(0) = m_0$ *and it satisfies* (2.3) *(*(2.4)*, respectively) on* $[0, \varepsilon)$.

Sometimes we shall consider $Y(t)$ and $\beta(t)$ as stochastic processes.

It is essential that (2.1) can be reduced to (2.3) and (2.2) to (2.4). This will be explained in the proof of Theorem 3.4.

3 Basic existence theorems for equations

Theorem 3.3. *Consider compact Riemannian manifold* M *as above and specify a certain point* $m_0 \in M$*. Let for* $m \in M$, $t \geq 0$ *the vectors* $X(t, m)$ *and* $Y(t) \subset T_{m_0}M$ *and the the tensors* $\alpha(t, m)$ *and* $\beta(t)$ *at* m_0 *be smooth and uniformly bounded. Then equation* (2.3) *has a solution for initial value* $\xi(0) = m_0$ *and that solution exists for all* $t > 0$.

Proof. Let OM be the orthonormal frame bundle over M and H the Levi-Civitá connection on OM. The tangent map of the natural projection $\pi\colon OM \to M$ induces the isomorphism $T\pi\colon \mathsf{H}_b \to T_{\pi b}M$ at every point $b \in OM$. Hence, at every $b \in OM$, we obtain the vector

$$X^T(t, b) = T\pi^{-1}X(t, \pi b) \in \mathsf{H}_b \subset T_bOM.$$

The vectors $X^T(t, b)$ form a horizontal (i.e., belonging to H) vector field on OM.

Let us specify an orthonormal frame $\mathcal{O}$ in $T_{m_0}M$. The frame $\mathcal{O}$ gives rise to the isomorphism $\mathcal{O}\colon \mathbb{R}^n \to T_{m_0}M$ where $n = \dim M$ and $\mathbb{R}^n$ is the arithmetic n-dimensional space of columns with n components. Thus we can construct the horizontal time-dependent basic vector field $Y^T(t, b) = E\big(\mathcal{O}^{-1}Y(t)\big)$ on OM where $\mathcal{O}^{-1}Y(t)$ denotes the column vector in arithmetic $\mathbb{R}^n$ consisting of the coordinates of Y with respect to the basis $\mathcal{O}$.

For an orthonormal basis b in a tangent space T_mM denote by b^* the dual basis in the cotangent space T^*_mM. We consider every basis b or b^* as a linear operator from the arithmetic space $\mathbb{R}^n$ to T_mM (T^*_mM, respectively) that sends a column of n real numbers to the vector having those coordinates with respect to b (1-form with those coordinates with respect b^*, respectively). Their inverse operators b^{-1} and b^{*-1} send the corresponding tangent and cotangent spaces to the arithmetic $\mathbb{R}^n$.

Now introduce on OM the $(2, 0)$-tensor field $\alpha^T(t, b)$ as follows. The pull back $T^*\pi : T^*_{\pi b}M \to T^*_bOM$ sends every 1-form $\zeta \in T^*_{\pi b}M$ to the set $Z_b \in T^*_bOM$ such that every $\theta \in Z_b$ on the vector $V \in T_bOM$ takes the value

$\theta(V) = \zeta(T\pi V)$. Obviously each 1-form θ in $T^*_b OM$ belongs to the pull back of some 1-form ζ in $T^*_{\pi b}M$ and the inverse $(T\pi^*)^{-1}$ sends θ to ζ. On every pair of 1-forms θ_1 and θ_2 in $T^*_b OM$ the tensor $\alpha^T(t, b)$ takes the value

$$\alpha^T(t, b)(\theta_1, \theta_2) = T\pi^{-1}\alpha(t, \pi b)((T\pi^*)^{-1}\theta_1, (T\pi^*)^{-1}\theta_2) \in H_b.$$

Introduce also the $(2, 0)$-tensor field $\beta^T(t, b)$ on OM as follows:

$$\beta^T(t, b)(\theta_1, \theta_2) = T\pi^{-1}b(\mathcal{O}^{-1}\beta(\mathcal{O}^* b^{*^{-1}} T\pi^{*^{-1}}\theta_1, \mathcal{O}^* b^{*^{-1}} T\pi^{*^{-1}}\theta_2)) \in H_b.$$

Note that all the fields X^T, Y^T, α^T and β^T are smooth by construction.

From [3, Corollary 9.2.4] it follows that there exists a Euclidean space $\mathbb{R}^K$ with K large enough and at least locally Lipschitz continuous field of linear operators $A^T(t, b) : \mathbb{R}^K \to H_b$ such that $A^T(t, b)A^{T^*}(t, b) = \alpha^T(t, b) + \beta^T(t, b)$.

Remark 3.1. *In fact $\mathbb{R}^K$ is the Euclidean space, in which OM with an induced metric can be isometrically embedded by Nash's theorem. Note that K depends only on the dimension of OM. If the field $\alpha^T(t, b) + \beta^T(t, b)$ is non-degenerate (i.e., positive definite), the field A is unique, smooth and for its construction the method described in [5] can be applied. Here we cannot guarantee that $\alpha^T(t, b) + \beta^T(t, b)$ is non-degenerate. Thus A is not unique and only its local Lipschitz continuity can be proved. See details in [3].*

Go on the proof of Theorem 3.3. Let $w(t)$ be a Wiener process in $\mathbb{R}^K$. Consider the following Itô equation in Belopolskaya-Daletskii form on OM:

$$d\zeta(t) = \mathbf{e}_{\zeta(t)}\big((X^T(t, \zeta(t)) + Y^T(t, \zeta(t)))dt + A^T(t, \zeta(t))dw(t)\big). \tag{3.1}$$

Since the coefficients of (3.1) are at least locally Lipschitz continuous and the manifold OM is compact, it has a unique strong solution for initial value $\zeta(0) = \mathcal{O}$ and this solution $\zeta(t)$ exists for all $t \geq 0$. Note that $\zeta(t)$ is a horizontal lift of the process $\xi(t) = \pi\zeta(t)$. Thus by construction and by Lemma 1.2 $\xi(t)$ satisfies (2.3) with initial condition $\xi(0) = m_0$ and it exists for all $t \geq 0$ since the manifold OM is compact. □

Theorem 3.4. *For a compact Riemannian manifold M as above, let for $m \in M$, $t \geq 0$ the vectors $X(t, m)$ and $Y(t, m)$ and the the tensors $\alpha(t, m)$ and $\beta(t, m)$ be smooth and uniformly bounded. Then equation* (2.1) *has a solution for every initial value $\varphi(t)$ as in Definition 2.5 and that solution exists for $t \in [-h, h]$.*

Proof. Here we use the notation from the proof of Theorem 3.3. Consider the following $T_{\varphi(0)}M$-valued functions of $t \in [0, h]$: $X(t) = \Gamma_{0,t-h}X\big(t-h, \varphi(t-h)\big)$ and $\beta(t) = \Gamma_{0,t-h}\beta\big(t-h, \varphi(t-h)\big)$. It is clear that the solution $\xi(t)$ of (2.3) with the introduced $X(t)$ and $\beta(t)$ that exists by Theorem 3.3, is a solution of (2.1) on $[-h, h]$. □

Remark 3.2. *The main difficulty for prolongation of solution $\xi(t)$ from Theorem 3.4 to $t \geq h$ is that for such t the delayed parts in the right-hand sides become random as follows.*

Consider the Banach manifold $\tilde{\Omega} = C^0([0,h], M)$ of continuous curves $m(\cdot) : [0,h] \to M$, the σ-algebra $\tilde{\mathcal{F}}$ in $\tilde{\Omega}$ generated by cylinder sets and the measure μ_ξ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ generated by $\xi(\cdot)$. Recall that on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mu_\xi)$ the process $\xi(t)$ is represented as the so called coordinate process $\xi(t)_{m(\cdot)} = m(t)$.

Specify a Borel measurable field $\mathcal{O}_m$ of orthonormal frames on M. For $t \in [h, 2h]$ introduce the random vector field on OM by the formula

$$Y^T_{m(\cdot)}(t,b) = E_b\big(\mathcal{O}^{-1}_{m(h)}\Gamma_{h,t-h}Y(t-h, m(t-h))\big) \tag{3.2}$$

where $m(\cdot)$ is considered as an elementary event from $\tilde{\Omega}$ and $\Gamma_{h,t}$ denotes the parallel translation along $\xi(t)$.

Introduce on OM also the random $(2,0)$-tensor field $\beta^T_{m(\cdot)}(t,b)$ for $t \in [h, 2h]$ by the relation

$$\beta^T_{m(\cdot)}(t,b)(\theta_1,\theta_2) =$$
$$T\pi^{-1}b(\mathcal{O}^{-1}_{m(h)}(\Gamma_{h,t-h}\beta)(\mathcal{O}^*_{m(h)}b^{*^{-1}}T\pi^{*-1}\theta_1, \mathcal{O}^*_{m(h)}b^{*^{-1}}T\pi^{*-1}\theta_2)) \in H_b.$$

Unfortunately we do not know results on the existence of square roots for $\alpha^T(t,b) + \beta^T_{m(\cdot)}(t,b)$ where $\alpha^T(t,b)$ is from the proof of Theorem 3.4, and so we cannot construct the corresponding Itô equation in Belopolskaya-Daletskii form on OM.

Theorem 3.5. *Let $X(t,m)$, $Y(t,m)$ and $\alpha(t,m)$ be like in Theorem 3.4. Then for every initial data $\varphi(t)$ as in Definition 2.5 there exists a solution $\xi(t)$ of equation*

$$\begin{aligned} D\xi(t) &= X(t,\xi(t)) + \Gamma_{t,t-h}Y(t-h,\xi(t-h)) \\ D_2\xi(t) &= \alpha(t,\xi(t)) \end{aligned} \tag{3.3}$$

that is well-defined for all $t \in [-h, \infty)$.

Proof. Here we use the probability space, the field $\mathcal{O}_m$ and the constructions from Remark 3.2. For $t \in [-h, h]$ the assertion of Theorem follows from Theorem 3.4. The prolongation to $t \geq h$ can be constructed step-by-step as follows. From [3, Corollary 9.2.4] it follows that there exist a Euclidean space $\mathbb{R}^K$ with K large enough and a locally Lipschitz continuous field of linear operators $A(t,m) : \mathbb{R}^K \to T_mM$ such that $A(t,m)A^*(t,m) = \alpha(t,m)$. Let $w(t)$ be a Wiener process in $\mathbb{R}^K$. Construct the field of linear operators $A^T(t,b)$: $\mathbb{R}^K \to H_b$ as $T\pi^{-1}A(t,\pi b)_{|H_b}$. On OM introduce the vector field $X^T(t,b)$ as in the proof of Theorem 3.3 and the random vector field $Y^T_{m(\cdot)}(t,b)$ by formula (3.2) and consider the following Itô equation in Belopolskaya-Daletskii form:

$$d\zeta(t) = \mathbf{e}_{\zeta(t)}\big((X^T(t,\zeta(t)) + Y^T_{m(\cdot)}(t,\zeta(t)))dt + A^T(t,\zeta(t))dw(t)\big). \tag{3.4}$$

Note that by construction $X^T(t,b)$ is smooth, $A^T(t,b)$ is locally Lipschitz continuous and $Y^T_{m(\cdot)}(t,b)$ is a.s. smooth jointly in t, b. Then there exist a solution $\zeta(t)$ of (3.4) with initial condition $\zeta(h) = \mathcal{O}_{m(h)}$. Since OM is compact, $\zeta(t)$ is well-defined for $t \in [h, 2h]$. Obviously $\xi(t) = \pi\zeta(t)$ is a prolongation of the solution to $[h, 2h]$. The next steps are quite analogous. □

Theorem 3.6. *Let $X(t,m)$, $Y(t,m)$ and $\beta(t,m)$ be like in Theorem 3.4. Then for every initial data $\varphi(t)$ as in Definition 2.5 there exists a solution $\xi(t)$ of equation*

$$
\begin{aligned}
D\xi(t) &= X(t,\xi(t)) + \Gamma_{t,t-h} Y(t-h, \xi(t-h)) \\
D_2\xi(t) &= \Gamma_{t,t-h}\beta(t,\xi(t))
\end{aligned} \tag{3.5}
$$

that is well-defined for all $t \in [-h, \infty)$.

Proof. The arguments here are analogous to those in the proof of Theorem 3.5 with the following modification. From [3, Corollary 9.2.4] it follows that there exist a Euclidean space $\mathbb{R}^K$ with K large enough and a locally Lipschitz continuous field of linear operators $B(t,m) : \mathbb{R}^K \to T_mM$ such that $B(t,m)B^*(t,m) =$
$beta(t,m)$. Instead of (3.4) we deal with

$$
d\zeta(t) = \mathbf{e}_{\zeta(t)}\big((X^T(t,\zeta(t)) + Y^T_{m(\cdot)}(t,\zeta(t)))dt + B^T_{m(\cdot)}(t,\zeta(t))dw(t)\big). \tag{3.6}
$$

where $B^T_{m(\cdot)} = T\pi^{-1} b\big(\mathcal{O}^{-1}_{m(h)}\Gamma_{h,t-h}B(t-h, m(t-h))\big)$. □

4 Generalizations for inclusions and for equations with continuous coefficients

Let E and G be metric spaces and $F : E \multimap G$ be a set-valued mapping. For completeness of presentation we recall the following classical definitions (see, e.g., [5]):

Definition 4.7. *For a given $\varepsilon > 0$ a continuous single-valued mapping $f_\varepsilon : E \to G$ is called an* ε-approximation *of a set-valued mapping $F : E \multimap G$ if the graph of f, as a set in $E \times G$, belongs to the ε-neighborhood of the graph of F.*

Definition 4.8. *A single-valued mapping $f : E \to G$ is called a selector of a set-valued mapping $F : E \to G$ if at every point $x \in E$ the inclusion $f(x) \in F(x)$ holds.*

In an n-dimesional linear space we denote $S(n)$ the linear space of symmetric $(2,0)$-tensors (i.e., having $n \times n$ matrices) that is a subspace in the space of all $(2,0)$-tensors. The symbol $S_+(n)$ denotes the set of positive definite symmetric $(2,0)$-tensors ($n \times n$ matrices) that is an open convex set in $S(n)$. Its closure, i.e., the set of positive semi-definite symmetric $(2,0)$-tensors ($n \times n$ matrices) is denoted by $\bar{S}_+(n)$.

Everywhere below for a set B in an arbitrary normed linear space we use the norm introduced by the formula $\|B\| = \sup_{y \in B} \|y\|$.

Condition 4.1. *Let $\mathbf{X}(t,m)$ and $\mathbf{Y}(t,m)$ be set-valued vector fields on M and $\boldsymbol{\alpha}(t,m)$ and $\boldsymbol{\beta}(t,m)$ be set-valued $(2,0)$-tensor fields taking values in $S_+(n)$ in linear space of $(2,0)$ tensors at every point $m \in M$. We suppose that the images of all $t \in \mathbb{R}$ and $m \in M$ for all those fields are closed convex sets in the corresponding spaces and that all those fields are jointly upper semi-continuous (see the definitions, e.g., in [5]).*

We shall also deal with set-valued mappings $\mathbf{Y}(t)$ from $[0,\infty)$ to the tangent space at a certain point $m_0 \in M$ and $\boldsymbol{\beta}(t)$ from $[0,\infty)$ to $S_+(n)$ in the space of symmetric $(2,0)$-tensors at m_0. Here we also suppose that the images of all points in $[0,\infty)$ are closed and convex and that those mappings are upper semi-continuous.

Theorem 4.7. *Let M be a compact Riemannian manifold as above. Specify a certain point $m_0 \in M$. Let for $m \in M$, $t \geq 0$ the set-valued vectors $\mathbf{X}(t,m)$ and $\mathbf{Y}(t) \subset T_{m_0}M$ and the set-valued $(2,0)$-tensors $\boldsymbol{\alpha}(t,m)$ and $\boldsymbol{\beta}(t)$ at m_0 satisfy Condition 4.1 and be uniformly bounded. Then inclusion (2.4) has a solution for initial value $\xi(0) = m_0$ and that solution exists for all $t > 0$.*

Proof. Specify a sequence of positive real numbers $\varepsilon_q \to 0$ as $q \to \infty$. By [5, Theorem 4.11] for upper semi-continuous set-valued mappings $\mathbf{X}(t,m)$ and $\mathbf{Y}(t)$ with closed convex images of points there exist a sequence of continuous ε_q approximations $X_q(t,m)$ and $Y_q(t)$, respectively such that $X_q(t,m)$ ($Y_q(t)$, respectively) point-wise tends to a Borel-measurable selector of $\mathbf{X}(t,m)$ (of $\mathbf{Y}(t)$, respectively). Analogous ε-approximations $\hat{\alpha}_q(t,m)$ and $\hat{\beta}_q(t)$ exist for set-valued $(2,0)$-tensor fields $\boldsymbol{\alpha}(t,m)$ and $\boldsymbol{\beta}(t)$, respectively, with an additional property: Since $S_+(n)$ is convex in $S(n)$, those approximations take values in $S_+(n)$ of the corresponding spaces of tensors.

Since continuous functions can be approximated by smooth ones up to an arbitrary $\varepsilon > 0$, without loss of generality one may suppose $X_q(t,m)$ and $Y_q(t)$ to be smooth. Introduce $\alpha_q(t,m) = \hat{\alpha}_q(t,m) + \frac{\varepsilon_q}{4}\tilde{g}(m)$ and $\beta_q(t) = \hat{\beta}_q(t) + \frac{\varepsilon_q}{4}\tilde{g}(m)$ where $g(m)$ is the $(2,0)$-metric tensor on M corresponding to the Riemannian metric (that is $(0,2)$-metric tensor) on M. Evidently $\alpha_q(t,m)$ ($\beta_q(t)$) tends point-wise to a Borel measurable selector of $\boldsymbol{\alpha}(t,m)$ ($\boldsymbol{\beta}(t)$, respectively) and those approximations belong to $\bar{S}_+(n)$ in the corresponding spaces. Thus one can $\frac{\varepsilon_q}{4}$ approximate them by smooth ones and so without loss of generality we may consider $\alpha_q(t,m)$ and $\beta_q(t)$ to be smooth.

Consider the equations

$$\begin{aligned} D\xi_q(t) &= X_q(t,\xi(t)) + \Gamma_{t,0} Y_q(t) \\ D_2\xi(t) &= \alpha_q(t,\xi(t)) + \Gamma_{t,0}\beta_q(t) \end{aligned} \tag{4.1}$$

that satisfy the hypothesis of Theorem 3.3. Denote by μ_q the measures on the spaces of sample paths corresponding to the solutions of (4.1) that exist by Theorem 3.3 on an arbitrary time interval $[0,T]$.

The rest of the proof is made as a simple modification of that for [1, Theorem 4] (it involves isometric embedding of the manifold into a Euclidean space of large enough dimension). It is shown that the set $\{\mu_q\}$ of measures is weakly compact so that one can select a weakly convergent subsequence. Then it is shown that the process corresponding to the limit measure satisfies (2.4). $\square$

Corollary 4.8. *The assertion of Theorem 3.3 is true if $X(t,m)$, $Y(t)$, $\alpha(t,m)$ and $\beta(t)$ are continuous.*

Indeed, a single-valued continuous object is a particular case of the set-valued upper semi-continuous one.

Obvious modifications of the constructions and arguments, used above, allow one to prove the next statements.

Theorem 4.9. *For a compact Riemannian manifold M as above, let for $m \in M$, $t \geq 0$ the set-valued vectors $\mathbf{X}(t,m)$ and $\mathbf{Y}(t,m)$ and the set-valued tensors $\boldsymbol{\alpha}(t,m)$ and $\boldsymbol{\beta}(t,m)$ satisfy Condition 4.1 and be uniformly bounded. Then inclusion (2.2) has a solution for every initial value $\varphi(t)$ as in Definition 2.5 and that solution exists for $t \in [-h,h]$.*

Corollary 4.10. *The assertion of Theorem 3.4 is true if $X(t,m)$, $Y(t,m)$, $\alpha(t,m)$ and $\beta(t,m)$ are continuous.*

Theorem 4.11. *Let $\mathbf{X}(t,m)$, $\mathbf{Y}(t,m)$ and $\boldsymbol{\alpha}(t,m)$ be like in Theorem 4.9. Then for every initial data $\varphi(t)$ as in Definition 2.5 there exists a solution $\xi(t)$ of inclusion*

$$\begin{aligned} D\xi(t) &\in \mathbf{X}(t,\xi(t)) + \Gamma_{t,t-h}\mathbf{Y}(t-h,\xi(t-h)) \\ D_2\xi(t) &\in \boldsymbol{\alpha}(t,\xi(t)) \end{aligned} \tag{4.2}$$

that is well-defined for all $t \in [-h,\infty)$.

Corollary 4.12. *The assertion of Theorem 3.5 is true if $X(t,m)$, $Y(t,m)$ and $\alpha(t,m)$ are continuous.*

Theorem 4.13. *Let* $\mathbf{X}(t,m)$*,* $\mathbf{Y}(t,m)$ *and* $\boldsymbol{\beta}(t,m)$ *be like in Theorem 4.9. Then for every initial data* $\varphi(t)$ *as in Definition 2.5 there exists a solution* $\xi(t)$ *of inclusion*

$$\begin{aligned} D\xi(t) &= \mathbf{X}(t,\xi(t)) + \Gamma_{t,t-h}\mathbf{Y}(t-h,\xi(t-h)) \\ D_2\xi(t) &= \Gamma_{t,t-h}\boldsymbol{\beta}(t,\xi(t)) \end{aligned} \tag{4.3}$$

that is well-defined for all $t \in [-h,\infty)$*.*

Corollary 4.14. *The assertion of Theorem 3.6 is true if* $X(t,m)$*,* $Y(t,m)$ *and* $\beta(t,m)$ *are continuous.*

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