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Ergodic properties of Fractional Stochastic Burgers Equation

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Abstract

We prove the existence and uniqueness of invariant measures for the fractional stochastic Burgers equation (FSBE) driven by fractional power of the Laplacian and space-time white noise. We show also that the transition measures of the solution converge to the invariant measure in the norm of total variation. To this end we show first two results which are of independent interest: that the semigroup corresponding to the solution of the FSBE is strong Feller and irreducible.

Key words: stochastic fractional Burgers equation, ergodic properties, invariant measure, strong feller property, strongly mixing property, cylindrical Wiener noise.

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1. Introduction

Dynamics of many complex phenomena is driven by nonlocal interactions. Such problems are usually modeled by means of evolution equations including fractional powers of differential operators or more general pseudo-differential operators. For example, such equations arise in the theory of the quasi-geostrophic flows, the fast rotating fluids, the dynamic of the frontogenesis[1] in meteorology, the diffusions in fractal or disordered medium, the pollution problems, the mathematical finance and the transport problems, see for a short list e.g. [4, 10, 11, 12, 28, 29, 34, 41] and the references therein. In [29, 41] Kakutani and Sugimoto studied the wave propagation in complex solids, especially, viscoelastic materials (for example Polymers). They proved that the viscoelasticity affects the behavior of the wave. In particular, they showed that the relaxation function has the form $k(t) = ct^{-\nu},\ 0 < \nu < 1,\ \ c \in \mathbb{R}$, instead of the exponential form known in the standard models (Maxwell-Voigt and Voigt). This polynomial relaxation, called slow relaxation, is due to the non uniformity of the material. The main reason of this non uniformity is the accumulation of several relaxations in different scales. The far field is then described by a Burgers equation with the leading operator $(-\Delta)^{\frac{1+\nu}{2}}$ instead of the Laplacian:

$$\partial_t u = -(-\Delta)^{\frac{1+\nu}{2}} u + \partial_x u^2.$$

The above equation also describes the far-field evolution of acoustic waves propagating in a gas-filled tube with a boundary layer and has been used to study the acoustic waves in tunnels during the passage of the trains[41]. Indeed, in the last case the geometrical configurations can yield a memory effect and other types of resonance phenomena.

Frequently, lack of information about properties of the system makes it natural to introduce stochastic models. Moreover, the stochastic models are also powerful tools in the study of stability of deterministic systems under small perturbations. The stochastic character is visible when the initial data is random and/or when the coefficients are random.

Stochastic Partial Differential Equations play an essential role in the mathematical modeling of many physical phenomena. These equations are not only generalizations of the deterministic cases, but they lead to new and important phenomena as well. E.g. Crauel and Flandoli in [13] showed that the deterministic pitchfork bifurcation disappears as soon as an additive white noise of arbitrarily small intensity is incorporated the model. In [26], Hairer and Mattingly characterized the class of noises for which the 2 dimensional stochastic Navier-Stokes equation is ergodic. In recent series of papers and lectures, Flandoli and his Co-authors proved that for several examples of deterministic partial differential equations which are illposedness a suitable random noise can restore the illposedness see e.g. [3, 14, 22, 23].

In particular, the stochastic Burgers equation (briefly SBE) emerges in the

[1] The frontogenesis is the terminology used by atmosphere scientists for describing the formation in finite time of a discontinuous temperature front.

modeling of many phenomena, such as in hydrodynamics, in cosmology and in turbulence see for short list e.g. [2, 16, 32] and the references therein. While the physical models leading to Burgers equation are simple, the mathematical study is quite difficult and complex. The nonlinear term in this equation comes from kinematical considerations, hence it cannot be replaced by some simplifications or modifications. Let us also denote that recently some other generalizations of stochastic Burgers equation have also been investigated see e.g [37, 44].

The aim of this paper is to study the ergodic properties of the solution of the fractional stochastic Burgers equation. The ergodic properties of several stochastic partial differential equations have been extensively studied, e.g. [15, 17, 18, 20, 24, 25, 26, 36, 39]. These equations do not recover the FSBE. More precisely, we are interested in the fractional stochastic Burgers equation (FSBE) given by:

$$\partial_t u = -(-\Delta)^{\frac{\alpha}{2}} u + \partial_x u^2 + g(u)\partial^2_{tx} W_{tx}, \quad t > 0,\ x \in (0,1), \tag{1.1}$$

with the boundary conditions,

$$u(t,0) = u(t,1) = 0, \quad t > 0, \tag{1.2}$$

and the initial condition

$$u(0,x) = u_0(x), \quad x \in [0,1] \tag{1.3}$$

where $u_0 \in L^2(0,1)$. We will denote by A the negative Dirichlet Laplacian in the space $H = L^2(0,1)$, that is

$$A = -\Delta, \quad D(A) = H^{2,2}(0,1) \cap H^{1,2}_0(0,1).$$

Then the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ is defined as a fractional power of the operator A (see [35, pp. 72-73]):

$$(-\Delta)^{\frac{\alpha}{2}} := A_\alpha u := \frac{\sin\frac{\alpha\pi}{2}}{\pi} \int_0^\infty t^{\frac{\alpha}{2}-1} A(tI + A)^{-1} dt. \tag{1.4}$$

The diffusion coefficient g is a bounded and Lipschitz continuous map from $\mathbb{R}$ to $\mathbb{R}$ and $\partial^2_{tx} W_{tx}$ stands for the space-time white noise i.e. $\partial^2_{tx} W_{tx}$ is the distributional derivative of a mean zero Gaussian field $\{W(t,x), t \geq 0, x \in [0,1]\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, such that the covariance function is given by:

$$\mathbb{E}[W(t,x)W(s,y)] = (t \wedge s) x \wedge y, \quad t,s \geq 0,\ x,y \in [0,1]. \tag{1.5}$$

It is now customary, see for instance [19] (and [6]), to rewrite the problem (1.1-1.4) as a stochastic evolution problem in the Hilbert space $H = L^2(0,1)$ in the following way

$$\begin{cases} du(t) = (-A_\alpha u(t) + Bu^2(t))dt + g(u(t))\, dW_t,\ t > 0, \\ u(0) = u_0 \in L^2(0,1). \end{cases} \tag{1.6}$$

Here $B := \frac{\partial}{\partial x}$ and $W := \{W(t), t \geq 0\}$ is an H-cylindrical Wiener process given by (with $(e_j)_{j=1}^\infty$ is an ONB of H),

$$W(t) := \sum_{j=1}^\infty \beta_j(t) e_j, \tag{1.7}$$

where $(\beta_j)_{j=1}^\infty$ is a sequence of independent real Brownian motions.

Let us recall the definition of a solution we will use throughout the whole paper.

Definition 1.1. *Suppose that $1 < \alpha \leq 2$. An $\mathbb{F}$-adapted $L^2(0,1)$-valued continuous process $u = (u(t), t \geq 0)$ is said to be a mild solution of equation (1.6) if for some $p > \frac{2\alpha}{\alpha-1}$*

$$\mathbb{E} \sup_{t\in[0,T]} |u(t)|_{L^2}^p < \infty, \quad T > 0, \tag{1.8}$$

the function $(0,t) \ni s \to S_\alpha(t-s)Bu^2(s) \in L^2(0,1)$ is Bochner integrable and for every $t \geq 0$, the following identity holds $\mathbb{P}$-a.s. in $L^2(0,1)$

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)Bu^2(s)\, ds + \int_0^t S_\alpha(t-s) g(u(s))\, dW(s). \tag{1.9}$$

For the reader's convenience we recall the existence and uniqueness result for the case $u_0 \in L^2$. For the general case and proof see [6].

Theorem 1.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a bounded and Lipschitz continuous function and that $\frac{3}{2} < \alpha < 2$. Then for every $u_0 \in L^2(0,1)$ there exists a unique mild solution u of equation (1.6).*

In what follows the initial data $u_0 \in L^2(0,1)$ will usually be denoted by x and the unique mild solution u of equation (1.6) whose existence is guaranteed by Theorem 1.1 will be denote by $u(t,x)$, $t \geq 0$.

Using standard arguments, see for example [19], we can prove that the solution of equation (1.6) is an L^2-valued strong-Markov and Feller process. Let us define, the transition semigroup corresponding to the FSBE (1.6), i.e. a family $\mathbf{U} = \left(U_t\right)_{t\geq 0}$ of bounded linear operators on the space of bounded Borel functions $\mathcal{B}_b(L^2)$ by

$$(U_t\varphi)(x) = \mathbb{E}[\varphi(u(t,x)], \quad x \in L^2(0,1), \;\; t \geq 0. \tag{1.10}$$

Note that $\mathbf{U}$ is a semigroup (although not strongly continuous) on $\mathcal{B}_b(L^2(0,1))$. Let

$$\mu_t(x,B) = \mathbb{P}(u(t,x) \in B), \quad t \geq 0, x \in L^2(0,1), B \subset L^2(0,1), \text{ Borel},$$

be the family of transition measures of the process u. Then, clearly

$$U_t\phi(x) = \int_{L^2} \phi(y)\mu_t(x,dy), \quad \phi \in \mathcal{B}\left(L^2(0,1)\right).$$

The main aim of our article is to study the ergodic properties of the FSBE (1.6), or equivalently, the ergodic properties of the semigroup $\mathbf{U}$. Our main result is the following one.

Theorem 1.2. *(a) Assume that $g : \mathbb{R} \to \mathbb{R}$ is a bounded and Lipschitz continuous function. Suppose that $\frac{3}{2} < \alpha < 2$. Then there exists an invariant measure μ for the transition semigroup $\mathbf{U}$ corresponding to the fractional stochastic Burgers equation (1.6).*
(b) Assume additionally that

$$\inf_{x\in\mathbb{R}} |g(x)| > 0.$$

Then the invariant measure μ is unique and for every $x \in L^2(0,1)$

$$\lim_{t\to\infty} \|\mu_t(x,\cdot) - \mu(\cdot)\|_{TV} = 0, \tag{1.11}$$

where $\|\cdot\|_{TV}$ denotes the total variation norm.

We note that (1.11) trivially yields the strong mixing property, that is,

$$\lim_{t\to\infty} \int_{L^2} \left| U_t\phi - \int_{L^2} \phi d\mu \right|^2 d\mu = 0, \tag{1.12}$$

for every $\phi : L^2(0,1) \to \mathbb{R}$ such that

$$\int_{L^2} |\phi|^2 d\mu < \infty.$$

The paper is organized as follows. Section 2 contains some auxiliary facts and basic estimates that will be needed later. In the third and fourth sections, we will study the regularity of the semigroup corresponding to the solution of the FSBE. In particular, we will prove the strong Feller property and the irreducibility in sections three and four respectively. In the last section we will prove the existence of an invariant measure and, under stronger assumptions, its uniqueness and the strong mixing property (1.12).

2. Preliminaries and a priori estimates

Let $u = (u(t), t \geq 0)$ be a given process and let $\gamma \in \mathbb{R}_+$. We introduce the following stochastic process $z_\gamma = (z_\gamma(t), t \geq 0)$ defined by

$$z_\gamma(t) := \int_0^t e^{-(t-s)(A_\alpha+\gamma)} g(u(s))\, dW(s), \quad t \geq 0. \tag{2.1}$$

The process z_γ satisfies the following stochastic differential equation

$$\begin{cases} dz_\gamma(t) &= -(A_\alpha + \gamma) z_\gamma(t) dt + g(u(t))\, dW(t), \quad t \geq 0, \\ z_\gamma(0) &= 0. \end{cases}$$

Let us remark that if a process u is a solution of (1.6) then the stochastic process $v_\gamma := u - z_\gamma$ satisfies pathwise the following equation

$$\begin{cases} \frac{d}{dt} v_\gamma(t) &= -A_\alpha v_\gamma(t) + B(v_\gamma(t) + z_\gamma(t))^2 + \gamma z_\gamma(t), \quad t > 0, \\ v(0) &= u_0. \end{cases} \tag{2.3}$$

The following two results are the main tools allowing us to study the long time behaviour of the norm $|v_\gamma(t)|^2_{L^2}$. For $s \in \mathbb{R}^+ \setminus \mathbb{N}$ we will denote by $H^{s,p}$ the fractional order Sobolev space (called also Bessel potential spaces and sometimes Lebesgue Besov spaces), defined by the complex interpolation method, i.e.

$$H^{s,p}(0,1) = [H^{k,p}(0,1), H^{m,p}(0,1)]_\vartheta, \tag{2.4}$$

where $k, m \in \mathbb{N}, \vartheta \in (0,1)$, $k < m$, are chosen to satisfy

$$s = (1-\vartheta)k + \vartheta m. \tag{2.5}$$

These spaces coincide with the Sobolev spaces $W^{s,p}(0,1)$ for integer values of s if $1 < p < \infty$ and for all s when $p = 2$ see e.g. [1, p 219], [38] and [43, p 310]. Moreover, under condition (2.5), we have, see e.g. [43, p 103, p 196 & p 336]

$$H^{s,p}(0,1) = D((-\Delta)^{\frac{s}{2}}) = [L^p(0,1), H^{m,p}(0,1)]_\vartheta. \tag{2.6}$$

In what follows by $H^{s,p}_0(0,1)$, $s \geq 0$, $p \in (1,\infty)$, we will denote the closure of $C_0^\infty(0,1)$ in the Banach space $H^{s,p}(0,1)$. It is well known, see e.g. [30, Theorem 11.1] and [43, Theorem 1.4.3.2, p.317] that $H^{s,p}_0(0,1) = H^{s,p}(0,1)$ iff $s \leq \frac{1}{p}$. The norms in various Sobolev spaces $H^{s,p}(0,1)$ will be denoted by $|\cdot|_{H^{s,p}}$. Similarly, the norm in the $L^p(0,1)$ space will be denoted by $|\cdot|_{L^p}$.

Lemma 2.1. *[6, Lemma 3.4] There exist $\nu_1 > 0$, $q > 2$, $s \in (\frac{1}{q}, \frac{1}{2})$ and $C > 0$, such that*

$$\begin{aligned} \frac{d}{dt}|v_\gamma(t)|^2_{L^2} &\leq -\frac{\nu_1}{2}|v_\gamma(t)|^2_{H_0^{\frac{\alpha}{2},2}} + C|z_\gamma(t)|^{\frac{\alpha}{\alpha-1}}_{H^{s,q}}|v_\gamma(t)|^2_{L^2} \\ &+ C|z_\gamma(t)|^4_{H^{s,q}} + C|z_\gamma(t)|^4_{H^{1-\frac{\alpha}{2},2}} + \gamma^2 C|z_\gamma(t)|^2_{L^2}, \ t \geq 0. \end{aligned} \tag{2.7}$$

Proof. The proof is similar to the proof of [6, Proposition 3.3]. □

Lemma 2.2. *Let $(v_\gamma(t))_{t\geq 0}$ be the solution of the equation* (2.3). *Then there exist $\nu \geq 0$, $C > 0$, $s \in (\frac{1}{q}, \frac{1}{2})$, $q \in (2, \infty)$ and $s_0 \in (\frac{1}{2}, \frac{\alpha}{2}]$, such that*

$$\begin{aligned}\frac{d}{dt}|v_\gamma(t)|^2_{L^2} &+ \nu|v_\gamma(t)|^2_{H_0^{\frac{\alpha}{2},2}} \leq C|z_\gamma(t)|^{\frac{2\alpha}{\alpha-s_0}}_{H^{1-\frac{\alpha}{2},2}}\big|v_\gamma(t)\big|^2_{L^2} + C|z_\gamma(t)|^4_{H^{s,q}} \\ &+ C|z_\gamma(t)\big|^4_{H^{1-\frac{\alpha}{2},2}} + C\gamma^2\big|z_\gamma(t)\big|^2_{L^2},\ t \geq 0. \end{aligned} \tag{2.8}$$

Lemma 2.2 is an improved version of Lemma 2.1. The proofs are only slightly different from those in [6]. They are based on the Sobolev embedding theorems [38, Theorem 1, Tr 6, 3.3.1 section 2.4.4, p 82 & Proposition Tr 6, 2.3.5 section 2.1.2, p 14] as well as the following one which we recall for the readers convenience.

Theorem 2.3. *[38, Theorem 2, section 4.4.4, p.177 & Proposition Tr 6, 2.3.5 section 2.1.2, p.14] Assume that $0 < s_1 \leq s_2$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $s_1 + s_2 > \frac{n}{p_1} + \frac{n}{p_2} - n$ and if $s_1 < s_2$, $p \geq p_1$. Assume also that*

$$\frac{n}{p} - s_1 = \begin{cases} (\frac{n}{p_1} - s_1)_+ + (\frac{n}{p_2} - s_2)_+, & \textit{if} \ \ \max_i(\frac{n}{p_i} - s_i) > 0, \\ \max_i(\frac{n}{p_i} - s_i) > 0, & \textit{otherwise} \end{cases} \tag{2.9}$$

Then, provided that $\{i \in \{1, 2\} : s_i = \frac{n}{p_i}$ and $p_i > 1\} = \emptyset$,

$$H^{s_1,p_1}(0,1) \cdot H^{s_2,p_2}(0,1) \hookrightarrow H^{s_1,p}(0,1).$$

Proposition 2.4. *[6, Proposition 3.3] Assume that $\beta \in (\frac{1}{2}, 1)$. Then there exists a constant $C > 0$ such that for each $u \in H_0^{\beta,2}(0,1)$ and each $v \in H^{1-\beta,2}(0,1)$ the following inequality is satisfied*

$$\left|\int_0^1 u(x)Dv(x)\,dx\right| \leq C|u|_{H_0^{\beta,2}}|v|_{H^{1-\beta,2}}. \tag{2.10}$$

The following result is a generalization of [18, Lemma 5.1]. In order to formulate and prove it we need to recall some basic facts about the Itô integration in martingale type-2 Banach spaces. For more information, see e.g.[7, 5, 9, 45]. For brevity reason and to focus on the stochastic integral required in this paper, we restrict ourselves to the case when the separable Banach space E is $L^q(O, \mathcal{O}, \nu) = L^q(O)$, where $(O, \mathcal{O}, \nu)$ is a $\sigma-$finite measure space. In this case and for $2 \leq q < \infty$ the space $E = L^q(O)$ is both a UMD and type-2 (and hence, see [5] a martingale type 2). The construction of an Itô integral for such spaces is described in the paper [5] by the first named author.

Definition 2.1. *Let H be a separable Hilbert space and let E be a separable Banach space. A bounded linear operator $L : H \to E$, is called $\gamma-$radonifying operator if and only if for some (or equivalently for every) orthonormal basis (ONB) $(h_j)_{j=1}^{\infty}$ of H and some (equivalently every) sequence $(\gamma_j)_{j=1}^{\infty}$ of iid $N(0,1)$ random variables defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. we have*[2]

$$\|L\|^2_{R_\gamma(H,E)} := \mathbb{E}' |\sum_{j=1}^{\infty} \gamma_j L h_j|_E^2 < \infty. \tag{2.11}$$

The space of all γ-radonifying operators from H to E will be denoted by $R_\gamma(H,E)$.

Let us mention here that for certain classes of the target Banach spaces, a complete characterization of the γ-radonifying operators can be given in a purely non-probabilistic terms. For example, if E is a Hilbert space, then a bounded linear operator $L \in R_\gamma(H,E)$ if and only if it is a Hilbert-Schmidt operator. In this case $\|L\|_{R_\gamma(H,E)} = \|L\|_{HS}$. If, see [9], $E = L^q(O, \mathcal{O}, \nu)$, as described above, then $L \in R_\gamma(H,E)$ if and only if there exists a function $\kappa \in L^q(O, \mathcal{O}, \nu; H) = L^q(O, H)$ such that for every $h \in H$, $(Lh)(x) = \langle \kappa(x), h \rangle$ for ν-a.a. $x \in O$. In particular, see also [45, Proposition 13.7], if $1 < q < \infty$ and $(h_j)_{j \in J}$ is a fixed ONB of H then $L \in R_\gamma(H,E)$ if and only if $(\sum_{j \in J} |Lh_j|^2)^{\frac{1}{2}}$ is summable in $L^q(O)$. In the former, respectively the latter case, $\|L\|_{R_\gamma(H,L^q)}$ is equivalent to $|\kappa|_{L^q(O,\mathcal{O},\nu;H)}$, resp. $|(\sum_{j \in J} |Lh_j|^2)^{\frac{1}{2}}|_{L^q}$.
Let us now recall the fundamental property of such an integral resembling the Burhkolder inequality for local martingales, see [5, 21, 33]. There exists a constant $C = C_{p,q} > 0$ such that for every progressively measurable process $\Phi \in L^p(\Omega; L^2(0,T; R_\gamma(L^2, L^q))$, where $2 \le q < \infty$ and $1 < p < \infty$, we have

$$\mathbb{E} | \sup_{t \in [0,T]} \int_0^t \Phi(s) \, dW(s)|_{L^q}^p \le C_{p,q} \mathbb{E} \big(\int_0^T \|\Phi(t)\|^2_{R_\gamma(L^2, L^q)} \, dt \big)^{\frac{p}{2}}. \tag{2.12}$$

Finally, let us recall that A is a selfadjoint operator with compact inverse A^{-1} and that for every $\alpha \in \mathbb{R}$

$$A^{\frac{\alpha}{2}} e_j = \lambda_j^{\frac{\alpha}{2}} e_j, \quad j \ge 1,$$

where

$$e_j(\xi) = \sqrt{2} \sin \pi j \xi, \quad \lambda_j = \pi^2 j^2.$$

Lemma 2.5. *For all numbers $\varepsilon > 0$, $\sigma < \frac{\alpha - 1}{2}$, $p \ge 1$ and $q > 1$ we can find a positive number γ_0 such that for any $\gamma > \gamma_0$,*

$$\mathbb{E} \big| z_\gamma(t) \big|^p_{H^{\sigma,q}} \le \varepsilon, \ \ t \ge 0. \tag{2.13}$$

[2] Actually, in view of the Itô-Nisio Theorem, the class $R_\gamma(H,E)$, the exponent 2 below can be replaced by any $p \in (1, \infty)$.

Proof. By inequality (2.12) there exists a constant $C := C_{p,q} > 0$, such that for every $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\big|(-A)^{\sigma/2} z_\gamma(t)\big|^p_{L^q} && (2.14)\\
&\quad = \mathbb{E}\big|(-A)^{\sigma/2} \int_0^t e^{-(t-s)(A_\alpha+\gamma)} g(u(s))\, dW(s)\big|^p_{L^q}\\
&\quad \leq C_{p,q}\mathbb{E}\Big(\int_0^t \|(-A)^{\frac{\sigma}{2}} e^{-(t-s)(A_\alpha+\gamma)} g(u(s))\|^2_{R_\gamma(L^2,L^q)} ds\Big)^{\frac{p}{2}}.
\end{aligned}$$

Using the inequality (called the Ideal property), see e.g. [45, Theorem 6.2] $\|L_1L_2\|_{R_\gamma(L^2,L^q)} \leq \|L_1\|_{R_\gamma(L^2,L^q)}\|L_2\|_{\mathcal{L}(L^2)}$ and thanks to [9], see also [45, Proposition 13.7], there exists a positive constant $C := C_q > 0$, (we keep the same notation for all constants), such that for every $t \geq 0$,

$$\begin{aligned}
\mathbb{E}\big|(-A)^{\sigma/2} z_\gamma(t)\big|^p_{L^q} &\leq b_0^p C_{p,q}\Big(\int_0^t \|(-A)^{\frac{\sigma}{2}} e^{-s(A_\alpha+\gamma)}\|^2_{R_\gamma(L^2,L^q)} ds\Big)^{\frac{p}{2}}\\
&\leq C_{p,q,b_0}\Big(\int_0^t |(\sum_{j=1}^\infty |(-A)^{\frac{\sigma}{2}} e^{-s(A_\alpha+\gamma)} e_j|^2)^{\frac{1}{2}}|^2_{L^q} ds\Big)^{\frac{p}{2}}.
\end{aligned} \tag{2.15}$$

By arguing as in [6] around inequality (C.15), in particular using Proposition C.1.6 and Lemma 2.4, we infer that

$$|(\sum_{j=1}^\infty |(-A)^{\frac{\sigma}{2}} e^{-s(-A_\alpha+\gamma)} e_j|^2)^{\frac{1}{2}}|_{L^q} \leq (\sum_{k=1}^\infty \lambda_k^\sigma e^{-2s(\lambda_k^{\frac{\alpha}{2}}+\gamma)})^{\frac{1}{2}},\ s > 0, \tag{2.16}$$

By inserting the above inequality in (2.15) and replacing λ_k by $(k\pi)^{\frac{1}{2}}$, we get that for every $t > 0$,

$$\mathbb{E}\big|(-A)^{\sigma/2} z_\gamma(t)\big|^p_{L^q} \leq C_{p,q,b_0}\big(\sum_{k=1}^{+\infty} \frac{(k\pi)^{2\sigma}}{(k\pi)^\alpha+\gamma}\big)^{\frac{p}{2}} \tag{2.17}$$

The series on the RHS of (2.17) converges provided $\sigma \in [0, \frac{\alpha-1}{2})$. In order to get the inequality (2.13), it is sufficient to observe that in view of the Lebesgue dominated convergence theorem

$$\lim_{\gamma\to\infty} \sum_{k=1}^{+\infty} \frac{(k\pi)^{2\sigma}}{(k\pi)^\alpha+\gamma} = 0.$$

This completes the proof of the Lemma. $\square$

Proof of Lemma 2.2

We note first that for every $T > 0$ and each $z_\gamma \in L^\infty([0,T], H^{s,q}(0,1))$, $q > 2$ and where $s \in (\frac{1}{q}, \frac{1}{2})$, the deterministic problem (2.3) has a unique solution $v_\gamma \in C(0,T; L^2(0,1)) \cap L^2(0,T; H_0^{\frac{\alpha}{2},2}(0,1))$. Multiplying the both sides of (2.3) by the solution $v_\gamma(t)$ and applying Lemma III.1.2 from [42], then we get

$$\begin{aligned} 2\frac{d}{dt}|v_\gamma(t)|^2_{L^2} &= -\langle A_\alpha v_\gamma(t), v_\gamma(t)\rangle + \langle B(v_\gamma^2(t)), v_\gamma(t)\rangle \\ &+ 2\langle B(z_\gamma(t) v_\gamma(t)), v_\gamma(t)\rangle + \langle B z_\gamma^2(t), v_\gamma(t)\rangle + \gamma\langle z_\gamma(t), v_\gamma(t)\rangle. \end{aligned} \tag{2.18}$$

In what follows, the time argument is, for simplicity of notations, omitted. It is easy to see that $\langle B(v_\gamma^2), v_\gamma\rangle = 0$ and that

$$\gamma\big|\langle z_\gamma, v_\gamma\rangle\big| \;\le\; \gamma^2\big|z_\gamma\big|^2_{L^2} + \nu_1\big|v_\gamma\big|^2_{H_0^{\frac{\alpha}{2},2}}. \tag{2.19}$$

Moreover, since $\frac{\alpha}{4} \in (\frac{3}{8}, \frac{1}{2})$ and $\lambda_1 = \pi^2$, we obtain

$$\langle A_\alpha v_\gamma, v_\gamma\rangle = \langle A^{\frac{\alpha}{2}} v_\gamma, v_\gamma\rangle = |A^{\frac{\alpha}{4}} v_\gamma|^2_{L^2} \ge \pi^{2\alpha}|v_\gamma|^2_{H_0^{\frac{\alpha}{2},2}},\; v_\gamma \in D(A_\alpha). \tag{2.20}$$

Applying Proposition 2.4, with $\beta = \alpha/2$, we infer that

$$|\langle B(z_\gamma v_\gamma), v_\gamma\rangle| \;\le\; C|v_\gamma|_{H_0^{\frac{\alpha}{2},2}}|z_\gamma v_\gamma|_{H^{1-\frac{\alpha}{2},2}}.$$

By Theorems 2.3, then there exist $C > 0$, $q' > 1$ and $s' > \max\{\frac{1}{q'}, 1-\frac{\alpha}{2}\}$, such that

$$|z_\gamma v_\gamma|_{H^{1-\frac{\alpha}{2},2}} \le C|z_\gamma|_{H^{1-\frac{\alpha}{2},2}}|v_\gamma|_{H^{s',q'}}.$$

Furthermore, using [38, Theorem 1, Tr 6, 3.3.1 section 2.4.4, p 82 & Proposition Tr 6, 2.3.5 section 2.1.2, p 14], the following embedding

$$H^{s_0,2}(0,1) \hookrightarrow H^{s',q'}(0,1)$$

holds for all $s_0 \ge s' + \frac{1}{2} - \frac{1}{q'}$ and $s_0 > s'$. Hence, there exists a constant $C > 0$, such that

$$\big|v_\gamma\big|_{H^{s',q'}} \le C\big|v_\gamma\big|_{H^{s_0,2}}.$$

Now we choose $s_0 \le \frac{\alpha}{2}$. This choice is possible thanks to the conditions $\alpha > \frac{3}{2}$ and $q' > 1$. In fact, thanks to the above conditions, the following inequalities hold

$$\max\{\frac{1}{q'}, 1-\frac{\alpha}{2}\} < s' < s_0 \le \frac{\alpha}{2} - \frac{1}{2} + \frac{1}{q'}$$

and

$$\max\{\frac{1}{q'}, 1-\frac{\alpha}{2}\} < s' < \frac{\alpha}{2} - \frac{1}{2} + \frac{1}{q'}.$$

By interpolation, there exists a constant $C > 0$, such that

$$\big|v_\gamma\big|_{H^{s_0,2}} \leq C\big|v_\gamma\big|_{L^2}^{1-2\frac{s_0}{\alpha}} \big|v_\gamma\big|^{2\frac{s_0}{\alpha}}_{H_0^{\frac{\alpha}{2},2}},$$

and thereby

$$|\langle B(z_\gamma v_\gamma), v_\gamma\rangle| \;\; \leq \;\; C|v_\gamma|^{2\frac{s_0}{\alpha}}_{H_0^{\frac{\alpha}{2},2}} |z_\gamma|_{H^{1-\frac{\alpha}{2},2}} \big|v_\gamma\big|_{L^2}^{1-2\frac{s_0}{\alpha}}.$$

Invoking the classical Young inequality we find that for some generic constant $C > 0$

$$|\langle B(z_\gamma v_\gamma), v_\gamma\rangle| \;\; \leq \;\; \nu_2 |v_\gamma|^2_{H_0^{\frac{\alpha}{2},2}} + C|z_\gamma|^{\frac{2\alpha}{\alpha - s_0}}_{H^{1-\frac{\alpha}{2},2}} \big|v_\gamma\big|^2_{L^2}. \tag{2.21}$$

To estimate the term $|\langle B(z_\gamma^2), v_\gamma\rangle|$, we follow the same calculation from [6]. Applying again Proposition 2.4 with $\beta = \alpha/2$ we infer that

$$|\langle B(z_\gamma^2), v_\gamma\rangle| \;\; \leq \;\; C|v_\gamma|_{H_0^{\frac{\alpha}{2},2}} |z_\gamma^2|_{H^{1-\frac{\alpha}{2},2}}$$

Arguing as before with the choice $\frac{1}{q} < s < \frac{1}{2}, \; q > 2$, we get

$$\begin{aligned} |\langle B(z_\gamma^2), v_\gamma\rangle| \;\; &\leq \;\; C|v_\gamma|_{H_0^{\frac{\alpha}{2},2}} |z_\gamma|_{H^{s,q}} |z_\gamma|_{H^{1-\frac{\alpha}{2},2}} \\ &\leq \;\; \nu_3 |v_\gamma|^2_{H_0^{\frac{\alpha}{2},2}} + C|z_\gamma|^4_{H^{s,q}} + C|z_\gamma|^4_{H^{1-\frac{\alpha}{2},2}}. \end{aligned} \tag{2.22}$$

Combining equality (2.18), inequalities (2.19) and (2.20) with inequalities (2.21) and (2.22) and choose ν_1, ν_2 and ν_3, such that $-\frac{\pi^{2\alpha}}{2} + \frac{\nu_1}{2} + \nu_2 + \frac{\nu_3}{2} = -\nu$, where $\nu > 0$, we get (2.8).

3. Strong Feller Property

We recall first the Feller property of the transition semigroup associated to the solution of equation (1.6).

Proposition 3.1. *The transition semigroup* $\mathbf{U}$ *corresponding to the fractional stochastic Burgers equation* (1.6) *is a Markov Feller semigroup. i.e.*

$$U_t\phi \in C_b\left(L^2(0,1)\right), \quad \phi \in \mathcal{C}_b\left(L^2(0,1)\right). \tag{3.1}$$

Proof. Omitted. □

In order to show the uniqueness of the invariant measure we will need stronger properties of the transition semigroup. Below, for the reader's convenience we recall the concept of the strong Feller property and topological irreducibility.

Definition 3.1. *A Markov semigroup* $\mathbf{U} = (U_t)_{t\geq 0}$ *on a Polish space* X *is said to be (topologically) irreducible if for any* $t > 0$,

$$U_t 1_\Gamma(x) > 0, \quad x \in X, \tag{3.2}$$

for arbitrary non empty open set $\Gamma \subset X$. *It is said to have the strong Feller property if for any* $t > 0$

$$U_t\phi \in C_b(X), \quad \phi \in \mathcal{B}_b(X), \tag{3.3}$$

where $\mathcal{B}_b(X)$ *denotes the space of bounded Borel functions on* X.

In order to prove the strong Feller property of the Markov semigroup $\mathbf{U}$ corresponding to the fractional stochastic Burgers equation (1.6) we introduce a family of truncated systems. As in [6], for each natural number n we define a map $\pi_n : L^2(0,1) \to L^2(0,1)$ by

$$\pi_n(v) = \begin{cases} v & \text{if} \quad |v|_{L^2} \leq n, \\ \frac{n}{|v|_{L^2}} & \text{if} \quad |v|_{L^2} > n. \end{cases} \quad v \in L^2(0,1)$$

Then for each $n \geq 1$ we can consider an equation

$$\begin{cases} du_n(t) & = \left(-A_\alpha u_n(t) + B\left(\pi_n\left(u_n(t)\right)\right)^2\right) dt + g\left(u_n(t)\right)\, dW(t), \\ u_n(0) & = u_0, \end{cases} \tag{3.4}$$

By [6], for each $u_0 = x \in L^2(0,1)$, equation (3.4) admits a unique mild solution $(u_n(t,x), t \geq 0)$. Let $\mathbf{U}^n = \left(U_t^n\right)_{t\geq 0}$ be transition semigroup associated to the solution of equation (3.4), i.e. defined by

$$(U_t^n \varphi)(x) = \mathbb{E}[\varphi(u_n(t,x)], \quad \phi \in \mathcal{B}_b(L^2(0,1)), \quad x \in L^2(0,1). \tag{3.5}$$

Lemma 3.2. *for every* $n \geq 1$ *the semigroup* $\mathbf{U}^n$ *is strongly Feller on* $H = L^2(0,1)$. *More precisely, for every* $R > 0$, $t > 0$ *there exists* $C(R,t) > 0$ *such that for any* $\phi \in \mathcal{B}_b(H)$

$$|(U_t^n\phi)(x) - (U_t^n\phi)(y)| \leq C(R,t)|x-y|_{L^2}, \quad |x|_{L^2} \leq R, \ |y|_{L^2} \leq R.$$

Proof. The proof is a straightforward modification of the proof of analogous statement in [18], hence omitted. □

Theorem 3.3. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that there exist $a_0, b_0 > 0$ such that*

$$|g(x)| \in [a_0, b_0], \quad x \in \mathbb{R}. \tag{3.6}$$

Then the Markov semigroup $\mathbf{U}$ corresponding to the fractional stochastic Burgers equation (1.6) *is strong Feller.*

Proof. For $x \in L^2(0,1)$, let $(u(t,x), t \geq 0)$, respectively $(u_n(t,x), t \geq 0)$, be the unique mild solution of equation (1.6), respectively (3.4), with the initial condition $u(0,x) = x$. For $n \in \mathbb{N}$ and $x \in L^2(0,1)$ let $\tau_n(x)$ be an $\mathbb{F}$-stopping time defined by

$$\tau_n(x) := \inf\{t \geq 0 : |u(t,x)|_{L^2} \geq n\}.$$

Let us fix $t > 0$. It follows from [6, inequality (3.18)] that one can find a constant $C(t) > 0$ and a sequence $(q_n)_{n=1}^{\infty}$, independent of t, such that $q_k \nearrow \infty$ and

$$\mathbb{P}(\tau_n(x) < t) \leq \frac{C(t)}{q_n}(1 + \ln^+ |x|_{L^2}), \quad n \in \mathbb{N}. \tag{3.7}$$

On the other hand, due to the uniqueness property of the solution u proved in [6, section 3], $u(t,x) = u_n(t,x)$ for $t < \tau_n(x)$. Therefore, for all $x \in L^2(0,1)$ and $\phi \in \mathcal{B}_b(L^2(0,1))$

$$\begin{aligned}
|(U_t\phi)(x) - (U_t^n\phi)(x)| &= |\mathbb{E}\phi\big(u(t,x)\big) - \mathbb{E}\phi\big(u_n(t,x)\big)| \\
&= |\mathbb{E}[\phi\big(u(t,x)\big) - \phi\big(u_n(t,x)\big)]1_{\tau_n(x)<t}| \\
&\leq 2|\phi|_{L^\infty} P(\tau_n(x) < t).
\end{aligned} \tag{3.8}$$

Hence, in view of inequality (3.7), we infer that $U_t^n\phi$ converges to $U_t\phi$ uniformly on balls in $L^2(0,1)$. Hence, by applying Lemma 3.2, the proof of Theorem 3.3 is complete. □

4. Irreducibility

The main result of this section is as follows.

Theorem 4.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that condition* (3.6) *holds. Then the Markov semigroup $\mathbf{U}$ corresponding to the fractional stochastic Burgers equation* (1.6) *is irreducible.*

Proof. For $x \in L^2(0,1)$, let $(u(t,x), t \geq 0)$, respectively $(u_n(t,x), t \geq 0)$, be the unique mild solution of equation (1.9), respectively (3.4), with the initial condition $u(0,x) = x$. Let us fix $\varepsilon > 0$, $T > 0$, $t \in [0,T]$ and $x, y \in L^2(0,1)$. To prove the topological irreducibility, it is enough to show that

$$\mathbb{P}(u(t,x) \in B(y,\varepsilon)) > 0. \tag{4.1}$$

Let us first observe that because $D(A_\alpha)$ is a dense subspace of $L^2(0,1)$, it is enough to prove (4.1) for $y \in D(A_\alpha)$. In what follows we make this additional assumption.

For $n \geq 1$ and $0 < \tau < t$, let us define an $\mathbb{F}$-progressively measurable process

$$f^{\tau,n}(s,y) := \begin{cases} \left[\frac{1}{t-\tau} e^{-(s-\tau)A_\alpha}(y - u(\tau,x)) - A_\alpha y\right] 1_{B(0,n)}(u(\tau,x)), & \text{if } s > \tau, \\ 0, \text{ if } s \leq \tau. \end{cases} \tag{4.2}$$

where $B(0,n)$ is the closed ball in $L^2(0,1)$ of radius n and center 0. Later on we will choose τ to be sufficiently close to t. Let us consider the following stochastic evolution equation:

$$\begin{cases} du^{\tau,n}(s) = (-A_\alpha u^{\tau,n}(s) + \frac{\partial}{\partial x}(u^{\tau,n}(s))^2 + f^{\tau,n}(s,y))ds + g(u^{\tau,n}(s))\, dW(s), \\ u^{\tau,n}(0) = x, \end{cases} \tag{4.3}$$

Repeating the argument from [6] we can prove that the problem (4.3) has a unique mild solution $(u^{\tau,n}(s,x))$, $s \geq 0$, such that

$$M := \mathbb{E} \sup_{s \in [0,T]} |u^{\tau,n}(s)|^2_{L^2} < \infty. \tag{4.4}$$

Next we define an $\mathbb{F}$–progressively measurable process $(\beta^{\tau,n}(s))_{s \geq 0}$ by

$$\beta^{\tau,n}(s) := (g(u^{\tau,n}(s,x)))^{-1} f^{\tau,n}(s,y),$$

where for $u \in L^2(0,1)$, $g(u)$ is the multiplication operator on $L^2(0,1)$. Since the range of $|g|$ is a subset of $[a_0, b_0]$, $g(u)$ is a linear isomorphism of $L^2(0,1)$. By employing the Girsanov Theorem we will prove that the laws of the processes $u(\cdot,x)$ and $u^{\tau,n}(\cdot,x)$ are equivalent on $C([0,T], L^2(0,1))$. Indeed, since

$$\mathbb{E} e^{\frac{1}{2}\int_0^t |\beta^{\tau,n}(s)|^2_{L^2}\, ds} < \infty, \tag{4.5}$$

[19, Lemmata 10.14 and 10.15], the process $\hat{W}$ defined by

$$\hat{W}(t) := W(t) + \int_0^t \beta^{\tau,n}(s) ds, \quad t \in [0,T] \tag{4.6}$$

is an L^2-cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P}^*_T)$, where $\mathbb{F}_T := \{\mathcal{F}_t\}_{t \in [0,T]}$ and the probability measure $\mathbb{P}^*_T$ is equivalent to $\mathbb{P}$ with the Radon-Nikodym derivative $\frac{d\mathbb{P}^*_T}{d\mathbb{P}}$ equal to

$$Z_T := \exp\Big(\int_0^T \langle \beta^{\tau,n}(s), dW(s) \rangle - \frac{1}{2} \int_0^T |\beta^{\tau,n}(s)|^2_{L^2} ds \Big). \tag{4.7}$$

Therefore, using standard arguments we find that the process $u^{\tau,n}(\cdot,x)$ solves the equation on $[0,T]$,

$$\begin{aligned} du^{\tau,n}(s,x) &= (-A_\alpha u^{\tau,n}(s,x) + \frac{\partial}{\partial x}(u^{\tau,n}(s,x))^2 + f^{\tau,n}(s,y))\,ds \\ &+ g(u^{\tau,n}(s,x))d\hat{W}(s) - g(u^{\tau,n}(s,x))(g(u^{\tau,n}(s,x)))^{-1}[f^{\tau,n}(s,y)]\,ds \\ &= (-A_\alpha u^{\tau,n}(s,x) + \frac{\partial}{\partial x}(u^{\tau,n}(s,x))^2)ds + g(u^{\tau,n}(s,x))\,d\hat{W}(s). \end{aligned}$$

It follows from the uniqueness in law of the solution of the (1.6) that the laws on $C([0,T],L^2(0,1))$ of $u(\cdot,x)$ under $\mathbb{P}$ and of $u^{\tau,n}(\cdot,x)$ under $\mathbb{P}^*_T$ are the same. Hence,

$$\mathbb{P}(u(t,x)\in B(y,\varepsilon)) = \mathbb{P}^*_T(u^{\tau,n}(t,x)\in B(y,\varepsilon)). \tag{4.8}$$

Therefore, in order to prove that (4.1) holds, it is enough to show that we can choose n and τ such that $\mathbb{P}^*_T(u^{\tau,n}(t,x)\in B(y,\varepsilon))>0$.

Since $u^{\tau,n}(\cdot,x)$ is a mild solution to (4.3), we infer that

$$\begin{aligned} u^{\tau,n}(t,x) &= e^{-(t-\tau)A_\alpha}u(\tau,x) + \int_\tau^t e^{-(t-s)A_\alpha}\big(\frac{\partial}{\partial x}(u^{\tau,n}(s,x))^2 + f^{\tau,n}(s,y))\big)ds \\ &+ \int_\tau^t e^{-(t-s)A_\alpha} g(u^{\tau,n}(s,x))\,dW(s),\ \ t>\tau. \end{aligned}$$

Direct calculations based on the semigroup property of (e^{-tA_α}) and equality (4.2), yield, see also [36],

$$e^{-(t-\tau)A_\alpha}u(\tau) + \int_\tau^t e^{-(t-s)A_\alpha} f^{\tau,n}(s,y))ds = y, \quad t>\tau.$$

Consequently,

$$u^{\tau,n}(t,x) = y + \int_\tau^t e^{-(t-s)A_\alpha}\frac{\partial}{\partial x}(u^{\tau,n}(s,x))^2 ds + \int_\tau^t e^{-(t-s)A_\alpha} g(u^{\tau,n}(s,x))\,dW(s).$$

Hence

$$\begin{aligned} \mathbb{P}\big(u^{\tau,n}(t,x)\in B(y,\varepsilon)\big) &\geq 1 - \mathbb{P}\Big(\Big|\int_\tau^t e^{-(t-s)A_\alpha}\frac{\partial}{\partial x}(u^{\tau,n}(s))^2 ds\Big|_{L^2} > \frac{\varepsilon}{2}\Big) \\ &- \mathbb{P}\Big(\Big|\int_\tau^t e^{-(t-s)A_\alpha} g(u^{\tau,n}(s))\,dW(s)\Big|_{L^2} > \frac{\varepsilon}{2}\Big). \end{aligned} \tag{4.9}$$

Invoking the Chebyshev inequality, inequality (2.12), the boundedness of the operator g and the inequality $||AB||_{HS} \leq ||A||_{HS}|B|_{\mathcal{L}(H)}$, we infer that

$$\mathbb{P}\big(|\int_\tau^t e^{-(t-s)A_\alpha} g(u^{\tau,n}(s))\, dW(s)|_{L^2} > \frac{\varepsilon}{2}\big)$$
$$\leq \frac{2^p}{\varepsilon^p}\mathbb{E}|\int_\tau^t e^{-(t-s)A_\alpha} g(u^{\tau,n}(s))\, dW(s)|_{L^2}^p$$
$$\leq \frac{2^p}{\varepsilon^p}\mathbb{E}\big(\int_\tau^t ||e^{-(t-s)A_\alpha} g(u^{\tau,n}(s))||_{HS}^2 ds\big)^{\frac{p}{2}} \leq \frac{2^p}{\varepsilon^p} b_0^p \big(\int_0^{t-\tau} ||e^{-rA_\alpha}||_{HS}^2 ds\big)^{\frac{p}{2}}.$$

Since $\int_0^T ||e^{-rA_\alpha}||_{HS}^2\, ds < \infty$, we can choose $\tau < t$ such that

$$\mathbb{P}\big(|\int_\tau^t e^{-(t-s)A_\alpha} g(u^{\tau,n}(s))\, dW(s)|_{L^2} > \frac{\varepsilon}{2}\big) \quad \leq \quad \frac{1}{4}. \tag{4.10}$$

Furthermore, by the Chebyshev inequality, [6, Lemma 2.11] and (4.4), we obtain

$$\begin{aligned} &\mathbb{P}\left(\left|\int_\tau^t e^{-(t-s)A_\alpha} \tfrac{\partial}{\partial x}(u^{\tau,n}(s))^2 ds\right|_{L^2} > \tfrac{\varepsilon}{2}\right) \\ &\leq \tfrac{4}{\varepsilon^2}\mathbb{E}\left|\int_\tau^t e^{-(t-s)A_\alpha} \tfrac{\partial}{\partial x}(u^{\tau,n}(s))^2 ds\right|_{L^2}^2 \\ &\leq \tfrac{4C_\alpha}{\varepsilon^2}(t-\tau)^{1-\frac{3}{2\alpha}}\mathbb{E}\sup_{s\leq t}|u^{\tau,n}(s)|_{L^2}^2 \\ &\leq \tfrac{4MC_\alpha}{\varepsilon^2}(t-\tau)^{1-\frac{3}{2\alpha}}. \end{aligned} \tag{4.11}$$

Therefore, there exists $\tau < t$ such that

$$\mathbb{P}\left(\left|\int_\tau^t e^{-(t-s)A_\alpha} \frac{\partial}{\partial x}(u^{\tau,n}(s))^2 ds\right|_{L^2} > \frac{\varepsilon}{2}\right) \leq \frac{1}{4}$$

and combining this result with (4.10) we find that

$$\mathbb{P}\left(u^{\tau,n}(t,x) \in B(y,\varepsilon)\right) \geq \frac{1}{2}.$$

Thanks to the equivalence of $\mathbb{P}$ and $\mathbb{P}_T^*$, we obtain

$$\mathbb{P}_T^*\big(u^{\tau,n}(t,x) \in B(y,\varepsilon)\big) > 0.$$

and invoking identity (4.8), we find that

$$\mathbb{P}\big(u(t,x) \in B(y,\varepsilon)\big) = \mathbb{P}_T^*\big(u^{\tau,n}(t,x) \in B(y,\varepsilon)\big) > 0.$$

This proves (4.1) for $y \in D(A_\alpha)$. As we observed at the beginning of the proof this implies the general case and therefore the proof of Theorem 4.1 is complete. □

5. Existence and properties of the invariant measure

The main task in this section is to prove the existence of an invariant measure. Our main tool is the Krylov-Bogoliubov Theorem. Our result here is

Theorem 5.1. *There exists at least one invariant measure for the transition semigroup* $\mathbf{U}$*, corresponding to the fractional stochastic Burgers equation* (1.6).

To prove this theorem we start with some auxiliary lemmas.

Lemma 5.2. *There exists* γ_1 *such that for every* $\gamma > \gamma_1$ *and every* $\varepsilon > 0$ *we can find* $M > 0$ *such that*

$$\frac{1}{t}\int_0^t \mathbb{P}\Big(|v_\gamma(s)|^2_{L^2} \geq M\Big)\, ds < \varepsilon, \quad t > 0. \tag{5.1}$$

Proof. Let us fix $\varepsilon > 0$. Let C denote a constant greater than the third power of the maximum of the embedding constant of $H^{1-\frac{\alpha}{2},2}(0,1) \hookrightarrow L^2(0,1)$, the embedding constant of $H^{\frac{\alpha}{2}-\frac{1}{2},2}(0,1) \hookrightarrow H^{1-\frac{\alpha}{2},2}(0,1)$ and the constant C from (2.8). By Lemma 2.5, we can find $\gamma_0 > 0$ such that, for all $\gamma_1 \geq \gamma_0$

$$\begin{aligned}\max\Big[\sup_{[0,+\infty)} \mathbb{E}\big|z_{\gamma_1}(t)\big|^{\frac{2\alpha}{\alpha-s_0}}_{H^{\frac{\alpha}{2}-\frac{1}{2},2}}, \sup_{[0,+\infty)} \mathbb{E}\big|z_{\gamma_1}(t)\big|^4_{H^{\frac{\alpha}{2}-\frac{1}{2},2}},\\ \sup_{[0,+\infty)} \mathbb{E}\big|z_{\gamma_1}(t)\big|^2_{H^{\frac{\alpha}{2}-\frac{1}{2},2}}\Big] \leq \varepsilon\frac{\nu}{4C}\end{aligned} \tag{5.2}$$

Let $M > 1$. We define a process ζ_{γ_1} by $\zeta_{\gamma_1}(t) = \log\big(\big|v_{\gamma_1}(t)\big|^2_{L^2} \vee M\big)$, $t \geq 0$. Using (2.8) and the weak derivative of $\zeta_{\gamma_1}(t)$, we get

$$\begin{aligned}\zeta'_{\gamma_1}(t) &= \frac{1}{|v_{\gamma_1}(t)|^2_{L^2}}\mathbf{1}_{\{|v_{\gamma_1}(t)|^2_{L^2}\geq M\}}\frac{d}{dt}|v_{\gamma_1}(t)|^2_{L^2}\\ &\leq \frac{1}{|v_{\gamma_1}(t)|^2_{L^2}}\mathbf{1}_{\{|v_{\gamma_1}(t)|^2_{L^2}\geq M\}}\Big(-\nu\big|v_{\gamma_1}(t)\big|^2_{L^2} + C|z_{\gamma_1}(t)\big|^{\frac{2\alpha}{\alpha-s_0}}_{H^{1-\frac{\alpha}{2},2}}\big|v_{\gamma_1}(t)\big|^2_{L^2}\\ &+ C|z_{\gamma_1}(t)|^4_{H^{s,q}} + C|z_{\gamma_1}(t)|^4_{H^{1-\frac{\alpha}{2},2}} + \gamma_1^2|z_{\gamma_1}(t)|^2_{L^2}\Big).\end{aligned} \tag{5.3}$$

Taking expectation of the both sides, we get

$$
\begin{aligned}
\Big(\mathbb{E}\zeta_{\gamma_1}(t)-\mathbb{E}\zeta_{\gamma_1}(0)\Big) \quad &+ \quad \nu\int_0^t \mathbb{P}\Big(|v_{\gamma_1}(s)|_{L^2}^2 \geq M\Big)ds \qquad (5.4)\\
&\leq \quad \int_0^t \Big(C\mathbb{E}|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}}^{\frac{2\alpha}{\alpha-s_0}} + \frac{C}{M}\mathbb{E}|z_{\gamma_1}(s)|_{H^{s,q}}^4\\
&+ \quad \frac{C}{M}\mathbb{E}|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}}^4 + \frac{C{\gamma_1}^2}{M}\mathbb{E}|z_{\gamma_1}(s)|_{L^2}^2\Big)ds.\\
&\leq \quad t\Big(C\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}}^{\frac{2\alpha}{\alpha-s_0}} + \frac{C}{M}\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{H^{s,q}}^4\\
&+ \quad \frac{C}{M}\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}}^4 + \frac{C{\gamma_1}^2}{M}\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{L^2}^2\Big).
\end{aligned}
$$

It is easy to see that if $M \geq |v(0)|_{L^2}^2$, then

$$
\mathbb{E}\zeta_{\gamma_1}(t)-\mathbb{E}\zeta_{\gamma_1}(0) \geq 0, \quad t \geq 0.
$$

Hence

$$
\begin{aligned}
\frac{1}{t}\int_0^t \mathbb{P}\Big(|v_{\gamma_1}(s)|_{L^2}^2 \geq M\Big)ds \quad &\leq \quad \frac{C}{M\nu}\Big[M\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}}^{\frac{2\alpha}{\alpha-s_0}} \qquad (5.5)\\
+\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{H^{s,q}}^4 \quad &+ \quad \sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}}^4 + \gamma_1^2\sup_{(0,\infty)}\mathbb{E}|z_{\gamma_1}(s)|_{L^2}^2\Big].
\end{aligned}
$$

Let us remark that thanks to the condition $\alpha > \frac{3}{2}$, we have $H^{\frac{\alpha}{2}-\frac{1}{2},2}(0,1) \hookrightarrow H^{1-\frac{\alpha}{2},2}(0,1)$, hence $|z_{\gamma_1}(s)|_{H^{1-\frac{\alpha}{2},2}} \leq C|z_{\gamma_1}(s)|_{H^{\frac{\alpha}{2}-\frac{1}{2},2}}$. Consequently, from the formulae (5.2) and for the choice $\frac{1}{q} < s < \frac{\alpha}{2}-\frac{1}{2} < \frac{1}{2}$, $q > \frac{2}{\alpha-1}$, Lemma 2.5 and thanks to the condition $M > \gamma^2$, then we get

$$
\frac{1}{t}\int_0^t \mathbb{P}\Big(|v_{\gamma}(s)|_{L^2}^2 \geq M\Big)ds < \varepsilon. \qquad (5.6)
$$

□

The following Lemma is a slight generalization of [6, Lemma 2.11].

Lemma 5.3. *For every $T > 0$ there exist $C > 0$ and $\theta > 0$ such that for all $v \in C([0,T];L^1(0,1))$,*

$$
\int_0^t \big|(-A)^{\frac{2\alpha-3}{4}}e^{-(t-s)A_\alpha}Bv(s)\big|_{L^2}ds \leq Ct^\theta|v|_{L^\infty([0,t];L^1(0,1))}^2, \quad t \geq 0. \qquad (5.7)
$$

Proof. Let us choose (and fix) $\beta > 0$ and $v \in C([0,T]; L^1(0,1))$. Then we have

$$\begin{aligned}\left|(-A)^\beta e^{-(t-s)A_\alpha} Bv(s)\right|_{L^2}^2 &= \sum_{k=1}^{+\infty} \langle (-A)^\beta e^{-(t-s)A_\alpha} Bv(s), e_k\rangle^2 \\ &= \sum_{k=1}^{+\infty} \langle v(s), Be^{-(t-s)A_\alpha}(-A)^\beta e_k\rangle^2 \;\;\leq\;\; \sum_{k=1}^{+\infty} (k\pi)^2 \lambda_k^{2\beta} e^{-\lambda_k^{\frac{\alpha}{2}}(t-s)} \langle v(s), f_k\rangle^2,\end{aligned}$$

where $f_k(\cdot) := \cos(k\pi\cdot)$. Thanks to the inequality $|\langle v, f_k\rangle| \leq |v|_{L^1}|f_k|_{L^\infty}$, we get

$$\begin{aligned}&\int_0^t \left|(-A)^\beta e^{-(t-s)A_\alpha} Bv(s)\right|_{L^2} ds \\ &\leq \int_0^t \Big(\sum_{k=1}^{+\infty} (k\pi)^{4\beta+2} e^{-(k\pi)^\alpha (t-s)}\Big)^{\frac{1}{2}} |v(s)|_{L^1(0,1)} ds \\ &\leq \sup_{[0,t]} |v(s)|_{L^1(0,1)} \int_0^t \Big(\sum_{k=1}^{+\infty} (k\pi)^{4\beta+2} e^{-(k\pi)^\alpha s}\Big)^{\frac{1}{2}} ds.\end{aligned}$$

Since $e^{-(k\pi)^\alpha s} \leq C_\sigma s^{-\sigma}(k\pi)^{-\sigma\alpha}$ for all $s \geq 0$ and a certain constant $C_\sigma > 0$, we get

$$\int_0^t \Big(\sum_{k=1}^{+\infty} (k\pi)^{4\beta+2} e^{-(k\pi)^\alpha s}\Big)^{\frac{1}{2}} ds \leq C_N \Big(\sum_{k=1}^{+\infty} (k\pi)^{4\beta+2-\sigma\alpha}\Big)^{\frac{1}{2}} \int_0^t s^{-\frac{\sigma}{2}}\, ds. \tag{5.8}$$

The series and the integral in the RHS of (5.8) converge provided $\frac{4\beta+3}{\alpha} < \sigma < 2$. Such a value of σ exists provided $\beta < \frac{2\alpha-3}{4}$. Moreover, if $\alpha > \frac{3}{2}$, then we can choose $0 \leq \beta < \frac{2\alpha-3}{4}$, such that

$$\int_0^t \left|(-A)^\beta e^{-(t-s)A_\alpha} Bv(s)\right|_{L^2} ds \leq C t^{1-\frac{\sigma}{2}} \sup_{[0,t]} |v(s)|_{L^1}. \tag{5.9}$$

To get the inequality (5.7), we take $\theta := 1 - \frac{\sigma}{2}$ and $\beta = \frac{2\alpha-3}{4}$. □

Proof of Theorem 5.1

We use the Krylov-Bogoliubov method, in particular, we follow the proof in [16, 18]. Let us remark that for all $\theta > 0$ the embedding $H^{\theta,2}(0,1) \hookrightarrow L^2(0,1)$ is compact, see e.g. [16] and [20, p273]. Hence the ball

$$B_{H^{\theta,2}}(0, M) := \{v \in H^{\theta,2}(0,1), |v|_{H^{\theta,2}} \leq M\}$$

is a compact subset of $L^2(0,1)$.

In the first step, let us prove that for all $\beta \in (0, \frac{2\alpha-3}{4})$ and $\varepsilon > 0$ there exists $M > 0$, large enough such that $\frac{1}{t}\int_0^t P\Big(|(-A)^\beta v_\gamma(s)|_{L^2}^2 \geq M\Big) ds \leq \varepsilon$, where v_γ is the solution of (2.3). We have,

$$\begin{aligned} v_\gamma(t+1) = e^{-A_\alpha} v_\gamma(t) \quad &+ \quad \int_t^{t+1} e^{-A_\alpha(t+1-s)} B(v_\gamma(s) + z_\gamma(s))^2 ds \\ &+ \quad \gamma \int_t^{t+1} e^{-A_\alpha(t+1-s)} z_\gamma(s) ds. \;\; a.s. \end{aligned} \tag{5.10}$$

Let $0 < \beta < \frac{2\alpha-3}{4}$. Then

$$\begin{aligned} \big|(-A)^\beta v_\gamma(t+1)\big|_{L^2} \;\; &\leq \;\; \big|A_\alpha^{2\frac{\beta}{\alpha}} e^{-A_\alpha} v_\gamma(t)\big|_{L^2} \\ &+ \;\; \int_t^{t+1} \big|(-A)^\beta e^{-A_\alpha(t+1-s)} B(v_\gamma(s) + z_\gamma(s))^2\big|_{L^2} ds \\ &+ \;\; \gamma \int_t^{t+1} \big|(-A)^\beta e^{-A_\alpha(t+1-s)} z_\gamma(s)\big|_{L^2} ds \;\; a.s. \end{aligned} \tag{5.11}$$

We denote the three terms in the RHS of (5.11) by I_1, I_2, I_3 respectively. It is easy to see, using the properties of the heat semigroup that

$$I_1 := \big|A_\alpha^{2\frac{\beta}{\alpha}} e^{-A_\alpha} v_\gamma(t)\big|_{L^2} \leq C\big|v_\gamma(t)\big|_{L^2} \quad \text{and} \quad I_3 \leq \gamma \int_t^{t+1} \big|z_\gamma(s)\big|_{H^{2\beta,2}} ds. \tag{5.12}$$

For the second term, arguing as in the proof of Lemma 5.3, we get for some $0 < \eta < 1$,

$$\begin{aligned} I_2 \;\; &\leq \;\; \int_t^{t+1} \big|(-A)^\beta e^{-A_\alpha(t+1-s)} B(v_\gamma(s))^2\big|_{L^2} ds \\ &+ \;\; 2\int_t^{t+1} \big|(-A)^\beta e^{-A_\alpha(t+1-s)} B(v_\gamma(s) z_\gamma(s))\big|_{L^2} ds \\ &+ \;\; \int_t^{t+1} \big|(-A)^\beta e^{-A_\alpha(t+1-s)} B(z_\gamma(s))^2\big|_{L^2} ds \\ &\leq \;\; \sup_{0\leq r\leq 1} \big|v_\gamma(t+r)\big|_{L^2}^2 + 2\int_t^{t+1} (t+1-s)^{-\eta} \big|v_\gamma(s)\big|_{L^2} \big|z_\gamma(s))\big|_{L^2} ds \\ &+ \;\; \int_t^{t+1} (t+1-s)^{-\eta} \big|z_\gamma(s)\big|_{L^2}^2 ds \\ &\leq \;\; c_\eta \sup_{0\leq r\leq 1} \big|v_\gamma(t+r)\big|_{L^2}^2 + 2\int_t^{t+1} (t+1-s)^{-\eta} \big|z_\gamma(s)\big|_{L^2}^2 ds. \end{aligned} \tag{5.13}$$

By subsisting (5.12)- (5.13) on (5.11), we get

$$\begin{aligned}\big|(-A)^{\beta} v_{\gamma}(t+1)\big|_{L^2} &\leq C\big|v_{\gamma}(t)\big|_{L^2} + c_{\eta}\sup_{0\leq r\leq 1}\big|v_{\gamma}(t+r)\big|_{L^2}^2\\ +2\int_t^{t+1}(t+1-s)^{-\eta}\big|z_{\gamma}(s)\big|_{L^2}^2 ds &+ \gamma\int_t^{t+1}\big|z_{\gamma}(s)\big|_{H^{2\beta,2}} ds. \ \ a.s.\end{aligned} \tag{5.14}$$

Hence

$$\begin{aligned}\frac{1}{T}\int_0^T \mathbb{P}\Big(\quad | \quad &(-A)^{\beta} v_{\gamma}(t+1)|_{L^2}^2 > M\Big)dt \leq \frac{1}{T}\int_0^T \mathbb{P}\Big(\big|v_{\gamma}(t)\big|_{L^2}^2 > \sqrt{\frac{M}{4C}}\Big)dt\\ + \quad &\frac{1}{T}\int_0^T \mathbb{P}\Big(\sup_{0\leq r\leq 1}\big|v_{\gamma}(t+r)\big|_{L^2}^4 > \frac{M}{4c_{\eta}}\Big)dt\\ + \quad &\frac{1}{T}\int_0^T \mathbb{P}\Big(\int_t^{t+1}(t+1-s)^{-\eta}\big|z_{\gamma}(s)\big|_{L^2}^2 ds > \frac{M}{8C}\Big)dt\\ + \quad &\frac{1}{T}\int_0^T \mathbb{P}\Big(\int_t^{t+1}\big|z_{\gamma}(s)\big|_{H^{2\beta,2}} ds\Big) > \frac{M}{4\gamma}\Big)dt.\end{aligned} \tag{5.15}$$

From Lemma 5.2, there exists $M_1 > 0$, such that $\forall M > M_1$, we have

$$\frac{1}{T}\int_0^T \mathbb{P}\Big(\big|v_{\gamma}(t)\big|_{L^2}^2 > \sqrt{\frac{M}{4C}}\Big)dt < \frac{\varepsilon}{4}. \tag{5.16}$$

By Chebyshev inequality and Lemma 2.5 and considering the choice $M > \max\{M_1, 16\gamma, 32C\}$ and the condition $\alpha < 2$ which guaranty that $\beta < \frac{2\alpha-3}{4} < \frac{\alpha-1}{4}$, we infer

$$\begin{aligned}&\frac{1}{T}\int_0^T \mathbb{P}\Big(\int_t^{t+1}(t+1-s)^{-\eta}\big|z_{\gamma}(s)\big|_{L^2}^2 ds > \frac{M}{8C}\Big)dt\\ \leq &\frac{8C}{M}\frac{1}{T}\int_0^T \Big(\int_t^{t+1}(t+1-s)^{-\eta}\mathbb{E}\big|z_{\gamma}(s)\big|_{L^2}^2 ds\Big)dt < \frac{\varepsilon}{4}\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}&\frac{1}{T}\int_0^T \mathbb{P}\Big(\int_t^{t+1}\big|z_{\gamma}(s)\big|_{H^{2\beta,2}} ds\Big) > \frac{M}{4\gamma}\Big)dt\\ \leq &\frac{4\gamma}{M}\frac{1}{T}\int_0^T \Big(\int_t^{t+1}\mathbb{E}\big|z_{\gamma}(s)\big|_{H^{2\beta,2}} ds\Big)dt < \frac{\varepsilon}{4}.\end{aligned} \tag{5.18}$$

To estimate the second term in RHS of the inequality (5.15), we apply Gronwall Lemma to the formulae (2.7), then we get

$$
\begin{aligned}
|v_\gamma(t+s)|_{L^2}^2 &\le |v_\gamma(t)|_{L^2}^2 e^{C\int_0^s |z_\gamma(t+r)|_{H^{s,q}}^{\frac{\alpha}{\alpha-1}} dr} + \int_0^s \Big[C|z_\gamma(t+r)|_{H^{s,q}}^4 \\
&+ C|z_\gamma(t+r)|_{H^{1-\frac{\alpha}{2},2}}^4 + \gamma^2 C|z_\gamma(t+r)|_{L^2}^2 \Big] e^{C\int_r^s |z_\gamma(t+\xi)|_{H^{s,q}}^{\frac{\alpha}{\alpha-1}} d\xi} dr \\
&\le e^{C\int_0^1 |z_\gamma(t+r)|_{H^{s,q}}^{\frac{\alpha}{\alpha-1}} dr} \Big[|v_\gamma(t)|_{L^2}^2 + \int_0^1 C|z_\gamma(t+r)|_{H^{s,q}}^4 dr \\
&+ \int_0^1 C|z_\gamma(t+r)|_{H^{1-\frac{\alpha}{2},2}}^4 dr + \gamma^2 C \int_0^1 |z_\gamma(t+r)|_{L^2}^2 dr \Big].
\end{aligned} \tag{5.19}
$$

Hence,

$$
\begin{aligned}
\frac{1}{T}\int_0^T \mathbb{P} &\Big(\sup_{0\le r\le 1} \big|v_\gamma(t+r)\big|_{L^2}^4 > \frac{M}{4c_\eta}\Big) dt \le \frac{1}{T}\int_0^T \mathbb{P}\Big(e^{C\int_0^1 |z_\gamma(t+r)|_{H^{s,q}}^{\frac{\alpha}{\alpha-1}} dr} \\
&> \sqrt{\frac{M}{4c_\eta}}\Big) dt + \frac{1}{T}\int_0^T \mathbb{P}\Big(|v_\gamma(t)|_{L^2}^2 > \sqrt{\frac{M}{4^3 c_\eta}}\Big) dt \\
&+ \frac{1}{T}\int_0^T \mathbb{P}\Big(\int_0^1 |z_\gamma(t+r)|_{H^{s,q}}^4 dr > \sqrt{\frac{M}{4^3 c_\eta C^2}}\Big) dt \\
&+ \frac{1}{T}\int_0^T \mathbb{P}\Big(\int_0^1 |z_\gamma(t+r)|_{H^{1-\frac{\alpha}{2},2}}^4 dr > \sqrt{\frac{M}{4^3 c_\eta C^2}}\Big) dt \\
&+ \frac{1}{T}\int_0^T \mathbb{P}\Big(\int_0^1 |z_\gamma(t+r)|_{L^2}^2 dr > \sqrt{\frac{M}{4^3 c_\eta \gamma^4 C^2}}\Big) dt.
\end{aligned} \tag{5.20}
$$

Arguing as above and use Lemmata 2.5, 5.2 and Chebychev inequality, then for enough large M, we get

$$
\frac{1}{T}\int_0^T \mathbb{P}\Big(\sup_{0\le r\le 1} \big|v_\gamma(t+r)\big|_{L^2}^4 > \frac{M}{4c_\eta}\Big) dt < \frac{\varepsilon}{4}. \tag{5.21}
$$

Finally, from (5.16)–(5.18) and (5.21), we get

$$
\frac{1}{T}\int_0^T \mathbb{P}\Big(|(-A)^\beta v_\gamma(t+1)|_{L^2}^2 > M\Big) dt < \varepsilon. \tag{5.22}
$$

Let us recall that $u = v_\gamma + z_\gamma$, hence using (5.21) and Lemma 2.5, we prove that

$$
\frac{1}{T}\int_0^T \mathbb{P}\Big(|(-A)^\beta u(t+1)|_{L^2}^2 > M\Big) dt < \varepsilon. \tag{5.23}
$$

Hence the family of probability measures $\mu_T(\cdot) := \frac{1}{T}\int_0^T (\mathcal{L}(X_{t+1}))(\cdot)dt$ is tight. By Porokhorov's Theorem, we can subtract a weak convergent sequence $\mu_{T_{n_k}}(\cdot), T_{n_k} \uparrow +\infty$. Using Krylov-Bogoliubov existence Theorem, see e.g. [20], we confirm the existence of at least one invariant measure.

Proof of the Theorem 1.2

The proof of the main Theorem 1.2 follows, in a standard way, as a conclusion of the above results. In fact, thanks to Theorem 5.1, there exists an invariant measure. Since the semigroup $\mathbf{U}$ is strong Feller and irreducible, by the the Khas'minskii Theorem, see for instance [20, Proposition 4.1.1], it is regular. Hence, by using the Doob Theorem [20, Theorem 4.2.1], we conclude that the invariant measure is unique. The convergence in (1.11) follows immediately from [40, Theorem 1] and [20, Proposition 4.2.1], see also [31]. Finally, the ergodicity of the invariant measure follows from the uniqueness, see e.g. [20, Theorem 3.2.6].

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