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Global Invariant Manifolds in a Problem of Kalman-Bucy Filtering for Gyroscopic Systems

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Abstract

We use a geometric singular perturbations method for reducing the model order in optimal estimation problems for singularly perturbed stochastic differential systems. The method relies on the theory of integral manifolds, which essentially replaces the original system by another system on an integral manifold.

Keywords: filtering, integral manifolds, invariant manifolds, singular perturbations, stochastic differential systems, gyroscopic systems.

1 Introduction

The effect of random inputs on the movement of systems of solid bodies was investigated by many authors, see, for example, [1]. The paper deals with the analysis of the equations of gyroscopic systems under the influence of random forces. The possibility of the replacement of the equations of motion by the corresponding precessional equations is investigated. This approach is widespread in mechanics and gives suitable results in numerous cases. But there are a great number of examples when the substitution of the original equations by the precessional ones leads to inaccurate or qualitatively incorrect results.

In this respect, there have been a few works studying either the reasoning behind such a procedure, or the conditions under which it gives an appropriate result [9, 10].

This problem was solved by the method of integral manifolds [19]. The essence of this method is in the separation of the class of slow motions of the original system. The dimension of the system is reduced, but the system obtained, while of lower dimension, inherits the main features of its qualitative behavior. In this paper the equations of motion of the gyroscopic system of the form suggested by Merkin [10] are analyzed. It is shown that the method of integral manifolds can be applied to systems of this type.

Note that the equations of the flow along the integral manifold to the specified accuracy coincide with the corresponding precessional equations. In most applications the restrictions under which this slow integral manifold is stable are fulfilled. This means that any solution of the original equations, starting in the vicinity of the integral manifold, may be represented as a sum of some solution of the precessional equations and a small rapidly vanishing term. In this sense conversion to the precessional equations is permissible.

The main result of the paper is concerned with the possibility of conversion to the precessional equations in the presence of random terms. It is shown that the use of precessional equations as the basis for equations of the filtering error in the problem of optimal estimation may provide inadmissible errors.

2 Elements of the Integral Manifolds Method

2.1 Slow integral manifolds

It is common knowledge that a wide range of processes in various aspects of nature are characterized by extreme differences in the rates of change of variables, so singularly perturbed ordinary differential systems are used as models of such processes [13, 23, 24]).

Consider the ordinary differential system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, t, \varepsilon), \\ \varepsilon \frac{dy}{dt} &= g(x, y, t, \varepsilon), \end{aligned} \tag{2.1}$$

with vector variables x and y, and a small positive parameter ε. The usual approach in the qualitative study of (2.1) is to consider first the degenerate system

$$\frac{dx}{dt} = f(x, y, t, 0),$$
$$0 = g(x, y, t, 0),$$

and then to draw conclusions about the qualitative behavior of the full system (2.1) for sufficiently small ε. A special case of this approach is the quasi–steady state assumption. A mathematical justification of that method can be given by means of the theory of integral manifolds for singularly perturbed systems (2.1) (see e.g. [5, 11, 18, 23, 25, 26]). The integral manifolds method has been applied to a wide range of problems (see e. g. [2, 3, 6, 11, 12, 17, 19, 20, 22, 21]). In order to recall a basic result of the geometric theory of singularly perturbed systems we introduce the following terminology and assumptions.
The system of equations

$$\frac{dx}{dt} = f(x, y, t, \varepsilon) \tag{2.2}$$

represents the slow subsystem, and the system of equations

$$\varepsilon\frac{dy}{dt} = g(x, y, t, \varepsilon) \tag{2.3}$$

the fast subsystem, so it is natural to call (2.2) the slow subsystem and (2.3) the fast subsystem of system (2.1). In the present paper we use a method for the qualitative asymptotic analysis of differential equations with singular perturbations. The method relies on the theory of integral manifolds, which essentially replaces the original system by another system on an integral manifold with dimension equal to that of the slow subsystem. In the zero-epsilon approximation ($\varepsilon = 0$), this method leads to a modification of the quasi-steady-state approximation. Recall, that a smooth surface S in $R^m \times R^n \times R$ is called an integral manifold of the system (2.1) if any trajectory of the system that has at least one point in common with S lies entirely in S. Formally, if $(x(t_0), y(t_0), t_0) \in S$, then the trajectory $(x(t, \varepsilon), y(t, \varepsilon), t)$ lies entirely in S. An integral manifold of an autonomous system

$$\begin{aligned} \dot{x} &= f(x, y, \varepsilon), \\ \varepsilon\dot{y} &= g(x, y, \varepsilon) \end{aligned}$$

has the form $S_1 \times (-\infty, \infty)$, where S_1 is a surface in the phase space $R^m \times R^n$. The only integral manifolds of system (2.1) of relevance here are those of dimension m (the dimension of the slow variables) that can be represented as the graphs of vector-valued functions

$$y = h(x, t, \varepsilon).$$

We also stipulate that $h(x, t, 0) = h^{(0)}(x, t)$, where $h^{(0)}(x, t)$ is a function whose graph is a sheet of the slow surface, and we assume that $h(x, t, \varepsilon)$ is a sufficiently smooth function of ε. In autonomous systems the integral manifolds will be graphs of functions

$$y = h(x, \varepsilon).$$

Such integral manifolds are called manifolds of slow motions – the origin of this term lies in nonlinear mechanics. An integral manifold may be regarded as a surface on which the phase velocity has a local minimum, that is, a surface characterized by the most persistent phase changes (motions). Integral manifolds of slow motions constitute a refinement of the sheets of the slow surface, obtained by taking the small parameter ε into consideration.

The motion along an integral manifold is governed by the equation

$$\dot{x} = f(x, h(x, t, \varepsilon), t, \varepsilon).$$

If $x(t, \varepsilon)$ is a solution of this equation, then the pair $\Big(x(t, \varepsilon), y(t, \varepsilon)\Big)$, where $y(t, \varepsilon) = h(x(t, \varepsilon), t, \varepsilon)$, is a solution of the original system (2.1), since it defines a trajectory on the integral manifold.

Consider the *associated* subsystem, that is,

$$\frac{dy}{d\tau} = g(x, y, t, 0), \quad \tau = t/\varepsilon,$$

treating x and t as parameters. We shall assume that some of the steady states $y^0 = y^0(x, t)$ of this subsystem are asymptotically stable and that a trajectory starting at any point of the domain approaches one of these states as closely as desired as $t \to \infty$. This assumption will hold, for example, if the matrix

$$(\partial g/\partial y)(x, y^0(x, t), t, 0)$$

is stable for part of the stationary states and the domain can be represented as the union of the domains of attraction of the asymptotically stable steady states.

Let I_i be the interval $I_i := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon_i\}$, where $0 < \varepsilon_i \ll 1$, $i = 0, 1, \dots$.

($\mathbf{A_1}$). $f : R^m \times R^n \times R \times \overline{I_0} \to R^m$, $g : R^m \times R^n \times R \times \overline{I_0} \to R^n$ are sufficiently smooth and uniformly bounded together with their derivatives.

($\mathbf{A_2}$). There is some region $G \in R^m$ and a map $h : G \times R \to R^m$ of the same smoothness as g such that

$$g(x, h(x, t), t, 0) \equiv 0, \qquad \forall (x, t) \in G \times R.$$

($\mathbf{A_3}$). The spectrum of the Jacobian matrix $g_y(x, h(x, t), t, 0)$ is uniformly separated from the imaginary axis for all $(x, t) \in G \times R$.

Then the following result is valid (see e.g. [23, 25]):

Proposition 1.1. *Under the assumptions* $(\mathbf{A_1})$–$(\mathbf{A_3})$ *there is a sufficiently small positive* ε_1, $\varepsilon_1 \leq \varepsilon_0$, *such that, for* $\varepsilon \in \overline{I_1}$, *system* (2.1) *has a smooth integral manifold* $\mathcal{M}_\varepsilon$ *with the representation*

$$\mathcal{M}_\varepsilon := \{(x, y, t) \in R^{n+m+1} : y = \psi(x, t, \varepsilon),\ (x, t) \in G \times R\}.$$

Remark. The global boundedness assumption in $(\mathbf{A_1})$ with respect to (x, y) can be relaxed by modifying f and g outside some bounded region of $R^n \times R^m$.

2.2 Asymptotic representation of integral manifolds

When the method of integral manifolds is being used to solve a specific problem, a central question is the calculation of the function $h(x, t, \varepsilon)$ in terms of the manifold described. Exact calculation is generally impossible, and various approximations are necessary. One possibility is the asymptotic expansion of $h(x, t, \varepsilon)$ in integer powers of the small parameter:

$$h(x, t, \varepsilon) = h_0(x, t) + \varepsilon h_1(x, t) + \cdots + \varepsilon^k h_k(x, t) + \dots . \tag{2.4}$$

Substituting this formal expansion in equation (2.3) i.e.,

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial x} f(x, h(x, t, \varepsilon), t, \varepsilon) = g(x, h, \varepsilon),$$

we obtain the relationship

$$\begin{aligned} \varepsilon \sum_{k \geq 0} \varepsilon^k \frac{\partial h_k}{\partial t} + \varepsilon \sum_{k \geq 0} \varepsilon^k \frac{\partial h_k}{\partial x} f(x, \sum_{k \geq 0} \varepsilon^k h_k, t, \varepsilon) \\ = g(x, \sum_{k \geq 0} \varepsilon^k h_k, t, \varepsilon). \end{aligned} \tag{2.5}$$

We use the formal asymptotic representations

$$f(x, \sum_{k \geq 0} \varepsilon^k h_k, t, \varepsilon) = \sum_{k \geq 0} \varepsilon^k f^{(k)}(x, h_0, \dots, h_{k-1}, t),$$

and

$$g(x, \sum_{k \geq 0} \varepsilon^k h_k, t, \varepsilon) = B(x, t) \sum_{k \geq 1} \varepsilon^k h_k + \sum_{k \geq 1} \varepsilon^k g^{(k)}(x, h_0, ..., h_{k-1}, t),$$

where the matrix

$$B(x,t) \equiv (\partial g/\partial y)(x,h_0,t,0) \tag{2.6}$$

and where $g(x,h^{(0)}(x,t),t,0)=0$.

Substituting these formal expansions into (2.5) and equating powers of ε, we obtain

$$\frac{\partial h_{k-1}}{\partial t} + \sum_{0\le p\le k-1} \frac{\partial h_p}{\partial x} f^{(k-1-p)} = Bh_k + g^{(k)}.$$

Since B is invertible

$$h_k = B^{-1}\left[g^{(k)} - \frac{\partial h_{k-1}}{\partial t} - \sum_{0\le p\le k-1} \frac{\partial h_p}{\partial x} f^{(k-1-p)}\right]. \tag{2.7}$$

Note that asymptotic expansions of slow integral manifolds were first used in [19, 21, 22].

2.3 Stability of slow integral manifolds

In applications it is often assumed that the spectrum of the Jacobian matrix

$$g_y(x,h(x,t),t,0)$$

is located in the left half plane. Under this additional hypothesis the manifold $\mathcal{M}_\varepsilon$ is exponentially attracting for $\varepsilon \in I_1$. In this case, the solution $x = x(t,\varepsilon)$, $y = y(t,\varepsilon)$ of the original system that satisfied the initial condition $x(t_0,\varepsilon) = x^0$, $y(t_0,\varepsilon) = y^0$ can be represented as

$$\begin{aligned} x(t,\varepsilon) &= v(t,\varepsilon) + \varepsilon\varphi_1(t,\varepsilon), \\ y(t,\varepsilon) &= \bar{y}(t,\varepsilon) + \varphi_2(t,\varepsilon). \end{aligned} \tag{2.8}$$

There exists a point v^0 which is the initial value for a solution $v(t,\varepsilon)$ of the equation $\dot{v} = f(v,h(v,t,\varepsilon),t,\varepsilon)$; the functions $\varphi_1(t,\varepsilon)$, $\varphi_2(t,\varepsilon)$ are corrections that determine the degree to which trajectories passing near the manifold tend asymptotically to the corresponding trajectories on the manifold as t increases. They satisfy the following inequalities:

$$|\varphi_i(t,\varepsilon)| \le N|y^0 - h(x^0,t_0,\varepsilon)| \exp[-\beta(t-t_0)/\varepsilon], \quad i = 1,2, \tag{2.9}$$

for $t \ge t_0$.

From (2.8) and (2.9) we obtain the following *reduction principle* for a stable integral manifold defined by a function $y = h(x,t,\varepsilon)$: a solution $x = x(t,\varepsilon)$, $y = h(x(t,\varepsilon),t,\varepsilon)$ of the original system (2.1) is stable (asymptotically stable, unstable) if and only if the corresponding solution of the system of equations

$\dot{v} = F(v, t, \varepsilon) = f(x, h(x, t, \varepsilon), t, \varepsilon)$ on the integral manifold is stable (asymptotically stable, unstable). The Lyapunov reduction principle was extended to ordinary differential systems with Lipschitz right–hand sides by Pliss [14], and to singularly perturbed systems in [23]. Thanks to the reduction principle and the representation (2.8), the qualitative behavior of trajectories of the original system near the integral manifold may be investigated by analyzing the equation on the manifold.

The case in which the assumption ($\mathbf{A_3}$) is violated is called critical. We consider the following subcase: The Jacobian matrix $g_y(x, y, t, 0)$ has eigenvalues on the imaginary axis with nonvanishing imaginary parts. A similar case has been investigated in [16, 19, 23]. If this part of the eigenvalues is pure imaginary but, after taking into account the perturbations of higher order, they move to the complex left half-plane, then the system under consideration has stable slow integral manifolds. Such systems appear in modelling gyroscopic systems and double spin satellites [19, 21, 22, 23].

3 Optimal Estimation in Gyroscopic Systems

3.1 Gyroscopic systems

Let some of the eigenvalues have a pure imaginary part which moves to the complex left half plane at higher orders of the perturbations. In this case the system has a stable slow integral manifold. The general equations of gyroscopic systems on a fixed base may be represented in the form [23]:

$$\frac{dx}{dt} = y,$$

$$\varepsilon \frac{d}{dt}(Ay) = -(G + \varepsilon B)y + \varepsilon R + \varepsilon Q, \; R = \frac{1}{2}\left[\frac{\partial (Ay)}{\partial x}\right]^T y. \tag{3.1}$$

Here $x \in R^n$, $A(t, x)$ is a symmetric positive definite matrix, $G(x, t)$ is a skew-symmetric matrix of gyroscopic forces, and $B(x, t)$ is a symmetric positive definite matrix of dissipative forces, $Q(x, t)$ is a vector of generalized forces and $\varepsilon = H^{-1}$ is a small positive parameter.

The precessional equations take the form

$$(G + \varepsilon B)\frac{dx}{dt} = \varepsilon Q. \tag{3.2}$$

Equations (3.2) are obtained from (3.1) by neglecting some of the terms multiplied by the small parameter. All roots of the characteristic equation

$$\det(G - \lambda I) = 0$$

are situated on the imaginary axis, so the main assumption of the Tikhonov theorem [24] is violated. To justify the permissibility of the precessional equations we use the integral manifold method.

3.2 The equations of a Kalman-Bucy filter for gyroscopic systems

We derive the equations of the Kalman-Bucy filter for gyroscopic systems. Consider the equations of motion of gyroscopic system in the non-stationary case under the action of random forces in the form in [10]

$$\ddot{x} + [HG_0(t) + G_1(t)]\dot{x} + N(t)x = B(t)\dot{\omega}(t). \tag{3.3}$$

Here x is the $n-$dimensional vector of the system state, $G_0(t)$ is a skew-symmetric matrix of gyroscopic forces, and possessing a bounded inverse for all $t \geq 0$, $G_1(t)$ is a symmetric matrix of damping forces, $N(t)$ is the matrix of potential and non-conservative forces, H is a large parameter proportional to the angular velocity of the proper rotation of the gyroscope and which is much larger than the values of all the other system parameters for many gyroscopic systems.

Let the observation take place in the presence of Gaussian white noise described by the equation

$$z = C(t)x + \dot{\xi}(t), \tag{3.4}$$

where z is $m-$dimensional vector, $C(t)$ is $m \times n$ matrix. Let $\dot{\omega}(t)$ and $\dot{\xi}(t)$ be independent Gaussian white noise with zero expected values and correlation matrices $Q(t)\delta(t-s)$ and $R(t)\delta(t-s)$, respectively, where $Q(t)$ and $R(t)$ are symmetric positive semidefinite $m \times m-$matrices.

Introducing $\varepsilon = 1/H$, we rewrite (3.3) as a system

$$\begin{cases} \dot{x} &= y \\ \varepsilon\dot{y} &= -[G_0(t) + \varepsilon G_1(t)]y - \varepsilon N(t)x + \varepsilon B(t)\dot{\omega}. \end{cases} \tag{3.5}$$

For simplicity of presentation we assume that $x_0 = x(0)$ and $y_0 = y(0)$ are known vectors.

We are required to obtain an estimate $(\hat{x}(t), \hat{y}(t))^T$ of the state $(x(t), y(t))^T$, (^{T} stands for transposition), of system (3.5) in accordance with the vector-function $z(t)$ available for measurement at $t > 0$. The vector-function $x(t)$ is not available for measurement. The system which determines the vector $(\hat{x}(t), \hat{y}(t))^T$ is usually called the filter. We examine filters which are non-stationary linear systems of the form

$$\dot{\rho} = F(t)\rho + G(t)z,$$

where $\rho(t)$ is a $2n$ dimensional vector, $F(t)$ is a $2n \times 2n$ matrix, $G(t)$ is a $2n \times m$ matrix. It is known ([4]) that the filter which provides an unbiased estimate

$$e(t) = (x(t), y(t))^T - (\hat{x}(t), \hat{y}(t))^T$$

for the system

$$\dot{x} = A(t)x + B(t)\dot{w},$$

with the observation (3.4), is defined by the differential equation

$$\frac{d\rho}{dt} = [A(t) - G(t)C(t)]\rho + G(t)z(t), \tag{3.6}$$

and satisfies the initial condition

$$\rho(0) = E[(x(0), y(0))^T].$$

Here $E[\cdot]$ is an expected value. Those filters which satisfy equation (3.6) contain the matrix $G(t)$ as a parameter, and it should be chosen to minimize the variance of the error $e(t)$. To ensure that the estimate is unbiased, we require that

$$E[(x(t), y(t))^T] = E[\rho(t)],$$

at all $t > 0$, whence $E[e(t)] = 0$.

Consequently, the correlation matrix $P(t)$ of the error $e(t)$ has the form

$$P(t) = E[e(t)e^T(t)].$$

It is clear that $P(t)$ is a symmetric matrix satisfying the initial condition

$$P(0) = E[e(0)e^T(0)] = P_0,$$

and the differential equation

$$\frac{dP}{dt} =$$

$$[A(t) - G(t)C(t)]P + P[A(t) - G(t)C(t)]^T + B(t)Q(t)B^T(t) + G(t)R(t)G^T(t).$$

Note that matrix $G(t)$ is still unknown. Following [4] the filter is optimal if

$$G(t) = P(t)C^T(t)R^{-1}(t). \tag{3.7}$$

Taking (3.7) into consideration we obtain the equation for the correlation matrix of errors in the form of the Riccati equation

$$\frac{dP}{dt} = A(t)P + PA^T(t) - PC^TR^{-1}CP + BQB^T, \tag{3.8}$$

$$P(0) = P_0. \tag{3.9}$$

It was shown in [4] that, if P_0 is a positive definite matrix, equation (3.8) can be solved uniquely for the matrix $P(t)$, which exists for all $t \geq 0$. Then the equation for the optimal filter, on using (3.6) and (3.7), takes the form

$$\frac{d\rho}{dt} = [A(t) - P(t)C^T(t)R^{-1}(t)C(t)]\rho + P(t)C^T(t)R^{-1}(t)z(t),$$

$$\rho(0) = E[(x(0), y(0))^T],$$

where $P(t)$ is the solution of the differential Riccati equation (3.8) satisfying the initial conditions (3.9).

Let $m_1(t,\varepsilon)$ and $m_2(t,\varepsilon)$ be the mathematical expectations of the vectors $x(t)$ and $y(t)$ of system (3.5), i. e.,

$$m_1(t,\varepsilon) = E[x(t)], m_2(t,\varepsilon) = E[y(t)].$$

Then the vector $m(t,\varepsilon) = (m_1(t,\varepsilon), m_2(t,\varepsilon))^T$ satisfies the differential equation

$$\dot{m} = A(t)m + PC^T(t)R^{-1}(t)(z - C(t)m). \tag{3.10}$$

We apply the above results to system (3.5). $A(t)$ is the matrix of linear terms of the system (3.5) and is defined by

$$A(t) = \begin{pmatrix} 0 & I \\ -N(t) & -\frac{1}{\varepsilon}G_0(t) - G_1(t) \end{pmatrix}.$$

Let $B_1(t)$ and $C_1(t)$ denote the block matrices

$$B_1(t) = \begin{pmatrix} 0 \\ -B(t) \end{pmatrix}, \quad C_1(t) = \begin{pmatrix} C(t) & 0 \end{pmatrix}.$$

Then the Riccati equation for the correlation matrix $P(t,\varepsilon)$ of system (3.5) is

$$\frac{dP}{dt} = A(t)P + PA^T(t) - PC_1^T R^{-1} C_1 P + B_1 Q B_1^T. \tag{3.11}$$

We designate the $n \times n$ blocks of the matrix $P(t,\varepsilon)$ as follows:

$$P(t,\varepsilon) = \begin{pmatrix} P_1(t,\varepsilon) & P_2(t,\varepsilon) \\ P_2^T(t,\varepsilon) & P_3(t,\varepsilon) \end{pmatrix}.$$

Then equation (3.11) implies the system

$$\dot{P}_1 = P_2^T + P_2 - P_1 S P_1, \tag{3.12}$$

$$\varepsilon\dot{P}_2 = \varepsilon P_3 - \varepsilon P_1 N^T - P_2(G_0 + \varepsilon G_1)^T - \varepsilon P_1 S P_2, \tag{3.13}$$

$$\varepsilon\dot{P}_3 = -\varepsilon(NP_2 + P_2^T N^T) - P_3(G_0 + \varepsilon G_1)^T$$

$$-(G_0 + \varepsilon G_1)P_3 - \varepsilon P_2^T S P_2 + \varepsilon L, \tag{3.14}$$

where $S = C^T R^{-1} C, L = BQB^T$.

Equation (3.10) may also be rewritten as a system:

$$\begin{cases} \dot{m}_1 &= m_2 + P_1 C R^{-1}(z - Cm_1), \\ \varepsilon \dot{m}_2 &= -(G_0 + \varepsilon G_1)m_2 - \varepsilon N m_1 + \varepsilon P_2 C R^{-1}(z - Cm_1), \end{cases}$$

where $m_1(t, \varepsilon)$ and $m_2(t, \varepsilon)$ satisfy the initial conditions

$$m_1(0, \varepsilon) = x_0, \ m_2(0, \varepsilon) = y_0.$$

We now use some results of integral manifold theory. The existence of an attracting integral manifold permits us to reduce the singularly perturbed system to a system of lower dimension.

3.3 Precessional equations in the deterministic case

Let's analyze the equations of a gyroscopic system

$$\ddot{x} + (HG_0 + G_1)\dot{x} + Nx = 0,$$

in the deterministic case. The notation coincides with that introduced above. Having denoted $\varepsilon = 1/H$, we obtain

$$\varepsilon \ddot{x} + (G_0 + \varepsilon G_1)\dot{x} + \varepsilon N x = 0. \tag{3.15}$$

It is a commonly held view that equations (3.15) may be replaced by the corresponding precessional equations

$$(G_0 + \varepsilon G_1)\dot{x} + \varepsilon N x = 0. \tag{3.16}$$

Note that the dimension of (3.16) is half the dimension of (3.15). We shall apply the results of the theory of integral manifolds to prove the possibility of such a replacement. With that aim in view, we transform Equation (3.15) into the first order system

$$\dot{x} = y, \ \varepsilon \dot{y} = -(G_0 + \varepsilon G_1)y - \varepsilon N x. \tag{3.17}$$

In terms given in the preceding subsection, we have the equation $g(t, x, y, 0) = 0$ in the form $G_0 y = 0$. Hence, $y = h_0(t, x) = 0$, and the flow on the integral manifold is described by an equation

$$y = h(x, \varepsilon). \tag{3.18}$$

The function $h(x, \varepsilon)$ may be found as an asymptotic series

$$h(x, \varepsilon) = \sum_{i \geq 1} \varepsilon^i h_i(x) \tag{3.19}$$

from the equation

$$\varepsilon \frac{\partial h(x, \varepsilon)}{\partial x} = -(G_0 + \varepsilon G_1) h(x, \varepsilon) - \varepsilon N x. \tag{3.20}$$

Now, the usual technique of asymptotic analysis is applied. The expansion (3.19) is put into (3.20). Having equated the coefficients of powers of the small parameter ε, we compute the approximate solution of (3.20) in the form

$$h(x, \varepsilon) = -(G_0 + \varepsilon G_1)^{-1} \varepsilon N x + O(\varepsilon^2).$$

Thus, Equation (3.17) turns into

$$\dot{x} = -(G_0 + \varepsilon G_1)^{-1} \varepsilon N x + O(\varepsilon^3). \tag{3.21}$$

We compare equations (3.16) and (3.21). Evidently, they coincide to the accuracy of $O(\varepsilon^3)$. Consequently, the solutions of the system (3.17) and the solutions of the precessional equations (3.16) differ in the rapidly vanishing terms only, which correspond to the so-called nutational oscillations in the gyroscopic system. So it is quite correct to examine the precessional equation instead of the full equations of the gyroscopic system in the deterministic case.

Notice that the dimension of the slow integral manifold coincides with the dimension of vector of slow variables.

3.4 Optimal filtering in the precessional equations of gyroscopic systems

Let us now examine optimal filtering in gyroscopic systems described by the precessional equations.

We do not discuss here the physical aspects of obtaining the precessional equations. We remark only that such equations may be derived by neglecting the second derivative terms in (3.3). Consider precessional equations corresponding to (3.3) in the form

$$\dot{x} = -(G_0 + \varepsilon G_1)^{-1} \varepsilon N x + \varepsilon (G_0 + \varepsilon G_1)^{-1} B \dot{w}.$$

Denote the correlation matrix of the vector $x(t)$ by $\Phi(t)$. Then, according to (3.11), this matrix must satisfy the equation

$$(G_0 + \varepsilon G_1)\dot{\Phi} = -\varepsilon N \Phi - \varepsilon (G_0 + \varepsilon G_1) \Phi N^T ((G_0 + \varepsilon G_1)^{-1})^T$$

$$- (G_0 + \varepsilon G_1)\Phi C^T R^{1} C\Phi + \varepsilon^2 BQB^T((G_0 + \varepsilon G_1)^T)^{-1}. \tag{3.22}$$

Notice that at $\varepsilon = 0$ equation (3.22) has much in common with equation (3.12). Still this similarity is not sufficient to consider the precessional equations (3.3) to be acceptable as the basis for Kalman-Bucy filtering.

We examine this topic in detail. System (3.12)–(3.14) has a stable integral manifold of slow motions [23]. The flow along this manifold is governed by the regularly (not singularly) perturbed equations of this system. At first sight only equation (3.12) is regular, and (3.22), being quite similar to it, may replace the full system (3.12)–(3.14). But, in fact, there are more regular equations in the system (3.12)–(3.14). We require that the matrix $G_0(t)$ has no zero eigenvalues for all $t \in R$. But the linear operator

$$LY = YG - GY$$

has a nontrivial kernel, since differences $(\lambda_i(t) - \lambda_j(t))$, $i, j = 1, \ldots, n$, form its spectrum. That is why there are many regular scalar equations in (3.14), since this operator has many zero eigenvalues. Thus, the dimension of the slow integral manifold of (3.12)–(3.14) is greater than the dimension of the matrix $\Phi(t)$, and the use of precessional equation for filtering can give unacceptable results.

Next we consider one problem illustrating this result.

4 The plane gyroscopic pendulum

The gyroscopic pendulum is the simplest apparatus for indicating the proper vertical line direction in a moving ship or aeroplane.

Consider the equations of the plane gyroscopic pendulum with the horizontal axis of a gimbal. This pendulum is provided with a gyroscope which can turn near the axis of its housing. The turning of the gyroscope housing is limited by a spring. We investigate the movement of a plane gyroscopic pendulum under the rolling of a ship. Assume that the system is supplied with an apparatus for radial correction. The latter imposes the moment proportional to the rotation angle of the gyroscope housing round the axis of the pendulum oscillation. Then the equations of motion of the plane gyroscopic pendulum are of the form

$$\begin{aligned} I_1\ddot{\alpha} + H\dot{\beta} + lp\alpha + M\beta + n\dot{\alpha} + b\dot{w} &= 0, \\ I_2\ddot{\beta} - H\dot{\alpha} + E\dot{\beta} + \kappa\beta &= 0. \end{aligned} \tag{4.1}$$

Here α is the angle of the pendulum rotation around its axis; β is the angle of gyroscope rotation around its housing axis; I_1 and I_2 are the corresponding moments of inertia; H is a moment of momentum of the gyroscope; lp is

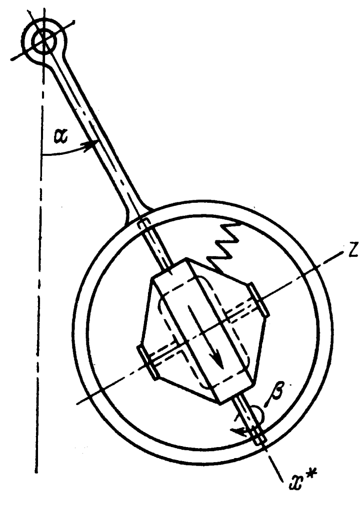

Figure 1: The plane gyroscopic pendulum

the static moment of the pendulum; M is the steepness of the moment of the radial correction; κ is the rigidity of the spring connecting the gyroscope housing with the pendulum; E and n are the coefficients of the viscous friction; $\dot{w}$ is a stationary random process corresponding to the angle of roll of the ship. Let $\dot{w}$ be a Gaussian white noise process with zero mean value and correlation function $q\delta(t-s)$.

Let the variable $z = \beta + \dot{\xi}$ be observed. At first, we consider the precessional equations for (4.1) neglecting the inertial terms $I_1\ddot{\alpha}$ and $I_2\ddot{\beta}$:

$$\begin{aligned} H\dot{\beta} + lp\alpha + n\dot{\alpha} + M\beta + b\dot{w} &= 0, \\ -H\dot{\alpha} + E\dot{\beta} + k\beta &= 0. \end{aligned} \tag{4.2}$$

Having divided both parts of the equations (4.2) by H and set $1/H = \varepsilon$, $(\alpha\,\beta)^T = \omega$ we obtain:

$$\dot{\omega} = \varepsilon \begin{pmatrix} -\varepsilon Elp & -\varepsilon EM + \kappa \\ -lp & -M - \varepsilon n\kappa \end{pmatrix} \omega - \varepsilon \begin{pmatrix} \varepsilon Eb \\ b \end{pmatrix} \dot{w} + O(\varepsilon^3). \tag{4.3}$$

Then the equations of the Kalman-Bucy filter for the correlation matrix P of the errors in the angles take the form

$$\dot{P} = \varepsilon \begin{pmatrix} \varepsilon Elp & -\varepsilon EM + \kappa \\ -lp & -M - \varepsilon n\kappa \end{pmatrix} P + \varepsilon P \begin{pmatrix} -\varepsilon Elp & -lp \\ -\varepsilon EM + \kappa & -M - \varepsilon n\kappa \end{pmatrix}$$

$$- P^T SP + \varepsilon^2 q \begin{pmatrix} \varepsilon^2 E^2 b^2 & b^2 E \\ \varepsilon E b^2 & b^2 \end{pmatrix} + O(\varepsilon^3), \tag{4.4}$$

where

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix}.$$

We seek a solution of (4.4) as a series:

$$P(\varepsilon) = D_0 + \varepsilon D_1 + O(\varepsilon^2).$$

From (4.4) we obtain that $D_0 = 0$, and D_1 satisfies the equation

$$\dot{D}_1 = \begin{pmatrix} 0 & \kappa \\ -lp & -M \end{pmatrix} D_1 + D_1 \begin{pmatrix} 0 & -lp \\ \kappa & -M \end{pmatrix} - D_1 S D_1 + \begin{pmatrix} 0 & 0 \\ 0 & qb^2 \end{pmatrix}.$$

It should be noted, that this mechanical system (plane gyroscopic pendulum) was examined in [15] by means of the precessional theory of gyroscopes, provided that $n = E = 0$. Under such assumptions, Equation (4.4) does not contain $O(\varepsilon^3)$ terms and, in coordinate form, is as follows:

$$\begin{aligned}
\dot{d}_1 &= 2\frac{\kappa}{H} d_2 - \frac{d_2^2}{r}, \\
\dot{d}_2 &= -\frac{lp}{H} d_1 - \frac{M}{H} d_2 + \frac{\kappa}{H} d_3 - \frac{d_2 d_3}{r}, \\
\dot{d}_3 &= -2\frac{lp}{H} d_2 - 2\frac{M}{H} d_3 - \frac{d_3^2}{r} + \frac{qb^2}{H^2}.
\end{aligned}$$

Here d_1, d_2 and d_3 denote the elements of the symmetric correlation matrix D. But we cannot compare these equations with those obtained on the basis of the theory of integral manifolds, since, for $n = E = 0$, the equations of motion of the plane gyroscopic pendulum may have no attracting integral manifold.

Next we consider the full equations (4.1) in the form

$$\begin{aligned}
\varepsilon\ddot{\alpha} + \frac{\beta}{I_1} + \varepsilon\frac{n}{I_1}\dot{\alpha} + \varepsilon\frac{lp}{I_1}\alpha + \varepsilon\frac{M}{I_1}\beta + \varepsilon\frac{b}{I_1}\dot{w} &= 0, \\
\varepsilon\ddot{\beta} - \frac{1}{I_2}\alpha + \varepsilon\frac{E}{I_2}\beta + \varepsilon\frac{\kappa}{I_2}\beta = 0,
\end{aligned}$$

or, in the more convenient form,

$$\begin{aligned}
&\varepsilon \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} + \begin{pmatrix} 0 & 1/I_1 \\ -1/I_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \varepsilon \begin{pmatrix} n/I_1 & 0 \\ 0 & E/I_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&\quad + \varepsilon \begin{pmatrix} lp/I_1 & M/I_1 \\ 0 & \kappa/I_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\varepsilon \begin{pmatrix} b/I_1 \\ 0 \end{pmatrix} \dot{w}. \qquad (4.5)
\end{aligned}$$

We use the following notation:

$$G_0 = \begin{pmatrix} 0 & 1/I_1 \\ -1/I_2 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} n/I_1 & 0 \\ 0 & E/I_2 \end{pmatrix},$$

$$N=\begin{pmatrix} lp/I_1 & M/I_1 \\ 0 & \kappa/I_2 \end{pmatrix},\quad B_2=\begin{pmatrix} b/I_1 \\ 0 \end{pmatrix}.$$

We designate the elements of the 2×2 matrices P_1, P_2, P_3 from (3.12)-(3.14) as follows:

$$P_1=\begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}, P_2=\begin{pmatrix} p_4 & p_7 \\ p_5 & p_8 \end{pmatrix}, P_3=\begin{pmatrix} p_6 & p_9 \\ p_9 & p_{10} \end{pmatrix}.$$

Then equation (3.14) for the plane gyroscopic pendulum may be transformed into a system of three scalar equations:

$$\begin{aligned}\varepsilon\dot p_6 &= -\frac{2}{I_1}p_9-2\varepsilon\frac{lp}{I_1}p_4-2\varepsilon\frac{M}{I_1}p_5-2\varepsilon\frac{n}{I_1}p_6+\varepsilon\frac{b^2}{I_1^2}q-\varepsilon\frac{p_5^2}{r},\\ \varepsilon\dot p_9 &= \frac{p_6}{I_2}-\frac{p_{10}}{I_1}-\varepsilon\left(\frac{\kappa}{I_2}p_5+\frac{M}{I_1}p_8+\frac{lp}{I_1}p_7+\left(\frac{E}{I_2}+\frac{n}{I_1}\right)p_9\right)-\varepsilon\frac{p_5p_8}{r},\\ \varepsilon\dot p_{10} &= 2\frac{1}{I_2}p_9-2\varepsilon\frac{\kappa}{I_2}p_8-2\varepsilon\frac{E}{I_2}p_{10}-\varepsilon\frac{p_8^2}{r}.\end{aligned}\tag{4.6}$$

Introduce the new variable as a linear combination of p_6 and p_{10}:

$$p_{11}=I_1p_6+I_2p_{10}.\tag{4.7}$$

Then for p_{10} we obtain the equation

$$\dot p_{11}=-2lpp_4-2Mp_5-\frac{2n}{I_1}p_{11}+\left(\frac{2nI_2}{I_1}-2E\right)p_{10}-2\kappa p_8-\frac{I_1}{r}p_5^2-\frac{I_2}{r}p_8^2+\frac{b^2q}{I_1}.$$

Thus, the variable p_{11} is slow. The equations for the correlation matrix $P(t,\varepsilon)$ may be written now in the following form:

$$\dot x=f(x,y),\tag{4.8}$$

$$\varepsilon\dot y=g_0(x,y)+\varepsilon g_1(x,y),\tag{4.9}$$

where $f(x,y)=$

$$\begin{pmatrix} 2y_1-\frac{1}{r}x_2^2 \\ y_2+y_3-\frac{1}{r}x_2x_3 \\ 2y_3-\frac{1}{r}x_3^2 \\ 2lpy_1-2My_2-\frac{2n}{I_1}x_4+\left(\frac{2nI_2}{I_1}-2E\right)y_6-2\kappa y_4-\frac{I_1}{r}y_2^2-\frac{I_2}{r}y_4^2+\frac{b^2q}{I_1} \end{pmatrix},$$

$$g_0(x,y) = \begin{pmatrix} -\frac{1}{I_1} y_3 \\ -\frac{1}{I_1} y_4 \\ \frac{1}{I_2} y_1 \\ \frac{1}{I_2} y_2 \\ -\frac{2}{I_2} y_6 + \frac{1}{I_1 I_2} x_4 \\ \frac{2}{I_2} y_5 \end{pmatrix},$$

$$g_1(x,y) = \begin{pmatrix} \frac{1}{I_1} x_4 - \frac{I_2}{I_1} y_6 - \frac{lp}{I_1} x_1 - \frac{M}{I_1} x_2 - \frac{n}{I_1} y_1 - \frac{1}{r} x_2 y_2 \\ y_5 - \frac{lp}{I_1} x_2 - \frac{M}{I_1} x_3 - \frac{n}{I_1} y_2 - \frac{1}{r} x_3 y_2 \\ y_5 - \frac{\kappa}{I_2} x_2 - \frac{E}{I_2} y_3 - \frac{1}{r} x_2 y_4 \\ y_6 - \frac{\kappa}{I_2} x_3 - \frac{E}{I_2} y_4 - \frac{1}{r} x_3 y_4 \\ -\frac{lp}{I_1} y_3 - \frac{M}{I_1} y_4 - \left(\frac{n}{I_1} + \frac{E}{I_2}\right) y_5 - \frac{\kappa}{I_2} y_2 - \frac{1}{r} y_2 y_4 \\ -2\frac{\kappa}{I_2} y_4 - 2\frac{E}{I_2} y_6 - \frac{1}{I_2} y_4^2 \end{pmatrix}.$$

The role of slow variable in (4.8)–(4.9) is played by vector x with coordinates (p_1, p_2, p_3, p_{11}) and the role of fast variable is played by vector y with coordinates $p_4, p_5, p_7 - p_{10}$

The system (4.8)–(4.9) possesses a four-dimensional stable slow invariant manifold. We search for this manifold as an asymptotic series according to (2.4). The matrix B in (2.6) takes on form

$$B = \begin{pmatrix} 0 & 0 & -1/I_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/I_1 & 0 & 0 \\ 1/I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/I_1 \\ 0 & 0 & 0 & 0 & 2/I_2 & 0 \end{pmatrix}$$

and it is invertible. Then we calculate the terms of asymptotic expansion h_0, h_1, h_2 and receive equation for slow variables on invariant manifold

$f(x,h(x,\varepsilon)) =$

$$\begin{pmatrix} -\frac{1}{r}x_2^2+2\varepsilon\kappa x_2-\varepsilon^2\left(\left(\frac{I_2 n}{I_1}+E\right)x_4-\frac{b^2qI_2}{I_1}-2Elpx_1-2EMx_2\right) \\ -\frac{1}{r}x_2x_3+\varepsilon\left(\kappa x_3-lpx_1-Mx_2\right)-\varepsilon^2\left(Elpx_2+EMx_2+n\kappa x_2-\frac{I_1}{r}x_2x_4\right) \\ -\frac{1}{r}x_3^2-\varepsilon\left(2lpx_2+2Mx_3\right)-\varepsilon^2\left(\frac{n}{2}x_4+\frac{EI_1}{2I_2}x_4+2n\kappa x_3-\frac{I_1}{r}x_3x_4-\frac{3b^2q}{2}\right) \\ -\frac{nI_2+EI_1}{I_1I_2}x_4+\frac{b^2q}{I_1}+O(\varepsilon) \end{pmatrix}$$

The latter differential system has the attractive invariant manifold

$$x_4=\frac{b^2qI_2}{nI_2+EI_1}+O(\varepsilon).$$

This constant is substituted into the equations for the first three slow variables. Thus, we receive the system of corrected differential equations of optimal filter.

$$\dot{x}_1=-\frac{1}{r}x_2^2+2\varepsilon(\kappa-EM\varepsilon)x_2-2\varepsilon^2Elpx_1,$$

$$\dot{x}_2=-\frac{1}{r}x_2x_3-\varepsilon lpx_1$$

$$-\varepsilon^2\left(Elp+M\varepsilon+n\kappa-\frac{b^2qI_1I_2}{2r(nI_2+EI_1)}\right)x_2+\varepsilon^2(\kappa\varepsilon-EM)x_3,$$

$$\dot{x}_3=-\frac{1}{r}x_3^2-2lp\varepsilon x_2-2\varepsilon^2\left(M\varepsilon+n\kappa-\frac{b^2qI_2}{2r(nI_2+EI_1)}\right)x_3+\varepsilon^2\frac{b^2q}{2}$$

Now we can correct the precessional equations, using on the received equations.

$$\dot{\omega}=\varepsilon\begin{pmatrix} -\varepsilon Elp & -\varepsilon EM+\kappa \\ -lp & -M-\varepsilon n\kappa+\varepsilon\frac{b^2I_1I_2}{r(nI_2+EI_1)} \end{pmatrix}\omega-\varepsilon\begin{pmatrix} 0 \\ b \end{pmatrix}\dot{w}+O\left(\varepsilon^3\right) \quad (4.10)$$

The approximations derived above permit us to follow how equation (4.3), derived on the basis of precessional equations, differs from the equations which describe the flow along the attracting invariant manifold of the system (4.1). We have received the corrected system of the same dimension that gives precessional theory. However the corrected system should yield more exact result during filtering process.

We compare the results obtained in this Example for the full equations of motion (4.1), and those got on the basis of the precessional equations (4.3) and on the corrected precessional equations (4.10). All the systems were solved numerically with the same initial conditions. The program for numerical modelling of Kalman-Bucy filters based on full, precessional and corrected equations has been written in MATLAB 7. Thus the result of solution modelling of the stochastic differential equations [8] of the movement of the plane gyroscopic pendulum was used as an input signal for filters.

By this is meant that the use of corrected system based on the stable invariant manifold of slow motions gives an accurate account of the behaviour of the original system, whereas the use of precessional equations, instead of the original ones, for calculating the filtering error may lead to an intolerable error if the motion is performed under the action of random forces of Gaussian white noise type.

5 Conclusion

We have examined the optimal estimation problem for a class of stochastic differential systems with slow and fast variables using the invariant manifolds theory. The gyroscopic systems are considered as an applications. It is stated that the use of the equation of the precessional theory may lead to an intolerable error in the case of random forces but in designing the Kalman-Bucy filter the full model can be replaced by the reduced order slow model.

Acknowledgements

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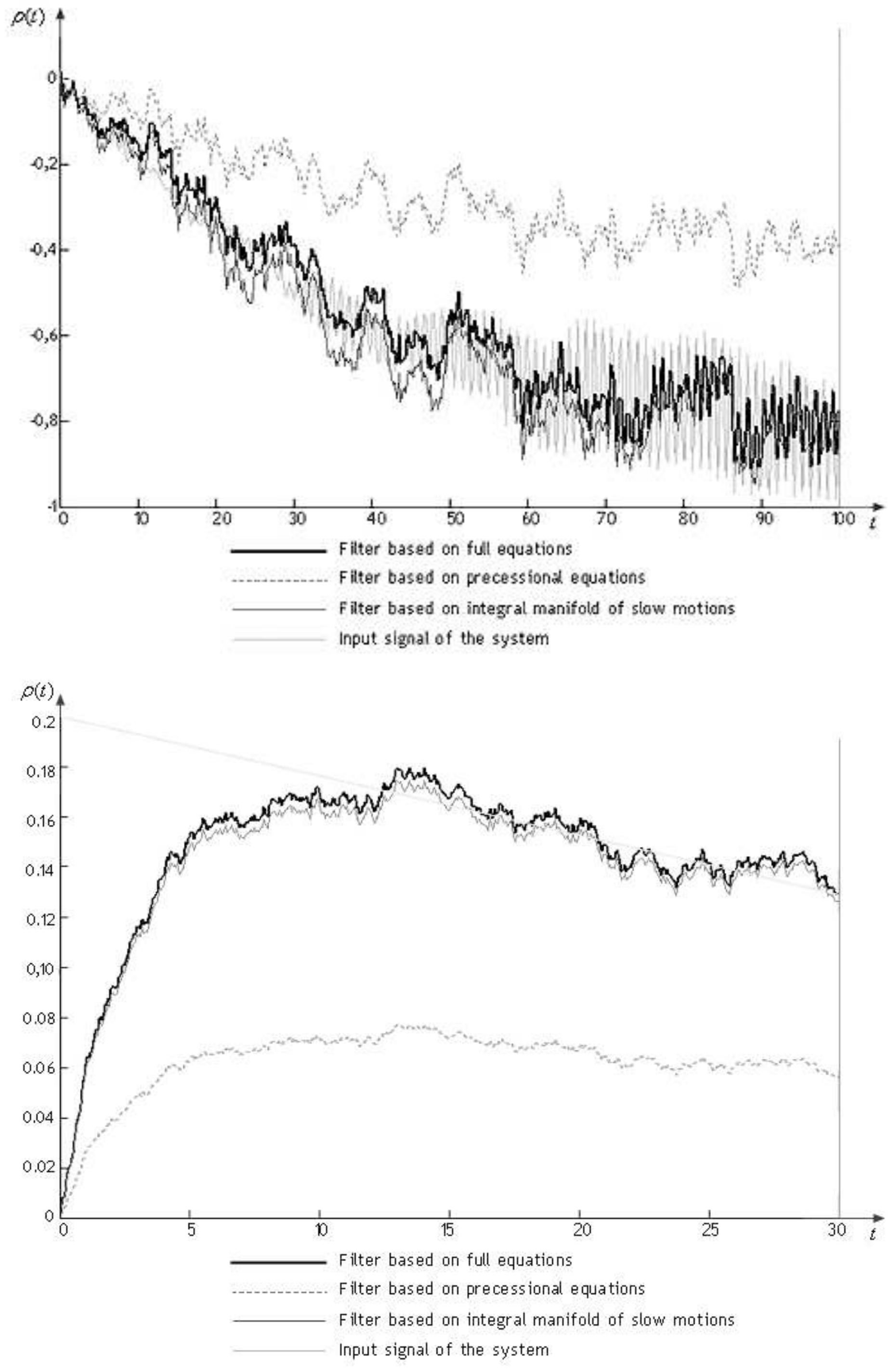

Figure 2: A numerical solutions for plane gyroscope pendulum

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