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SHOCK WAVES IN INITIAL BOUNDARY VALUE PROBLEM FOR FILTRATION IN TWO-PHASE 2-DIMENSIONAL POROUS MEDIA

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Abstract. In this paper we study initial boundary value problem for mass transport in 2-dimensional porous media describing by the Buckley – Leverett's system of differential equations. We propose a method to construct radial invariant solutions of the initial boundary problem and show how to overcome possible singularities in solutions and shock waves.

1. The Buckley – Leverett model

We consider the Buckley – Leverett system of differential equations for filtration in two-phase 2-dimensional system, consisting of two incompressible and immiscible liquids (say, water and oil) in a porous media. The porous media is assumed to have rigid skeleton media.

The Buckley – Leverett system of differential equations, governing filtration consist of mass, momentum and energy conservations laws (see, for example, [1, 2, 7, 6]:

- Mass conservation law for each phase, in absence of sources and sinks, has the form:

$$m\frac{\partial(\rho_i s_i)}{\partial t} + \operatorname{div}(\rho_i U_i) = 0, \tag{1.1}$$

  where ρ_i, s_i, U_i are the densities, saturations and volumetric velocities of the phases and m is porosity, i.e. volume fraction occupied by the pores.
- Momentum conservation law or Darcy's law for each phase states:

$$U_i = -\frac{k}{\mu_i} f_i(\sigma)\ \mathrm{grad}_{p_i},$$

  where $f_i(\sigma)$ are the phase permeabilities, p_i are partial pressures, k is the hydraulic conductivity, μ_i are the liquid viscosities and $s_1 = \sigma, s_2 = 1 - \sigma$.

  We'll neglect capillary forces. Then the partial pressures coincide $p_1 = p_2 = p$ and the Darcy's law takes the form:

$$U_i = -\frac{k}{\mu_i} f_i(\sigma)\ \mathrm{grad}_{p}. \tag{1.2}$$

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Taking the sum of equations (1.1), we get

$$\operatorname{div} U = 0, \tag{1.3}$$

where $U = U_1 + U_2$ be the resulting velocity and the rest of (1.1) has the form

$$m\frac{\partial \sigma}{\partial t} + \operatorname{div}\left(F\left(\sigma\right) U\right) = 0, \tag{1.4}$$

where

$$F\left(\sigma, \mu\right) = \frac{f_1\left(\sigma\right)}{f_1\left(\sigma\right) + \mu f_2\left(\sigma\right)}$$

is the Buckley – Leverett function.

In terms of this function we get

$$U_1 = F\left(\sigma, \mu\right) U, \quad U_2 = \left(1 - F\left(\sigma, \mu\right)\right) U.$$

The sum of equations (1.2) gives us Darcy's law for the resulting velocity:

$$U = -k\left(\frac{f_1\left(\sigma\right)}{\mu_1} + \frac{f_2\left(\sigma\right)}{\mu_2}\right) \operatorname{grad}_p. \tag{1.5}$$

The resulting system

$$\begin{cases} m\dfrac{\partial \sigma}{\partial t} + U\left(F\left(\sigma\right)\right) = 0, \\ \\ \operatorname{div} U = 0, \\ \\ U = -f\left(\sigma\right) \operatorname{grad}_p. \end{cases} \tag{1.6}$$

2. Integrability of Cauchy problem

We consider the 2-dimensional model, assuming that saturation σ and pressure p are invariants of the rotation group.

Let

$$q = \frac{x^2 + y^2}{2}$$

and $p = p\left(t, q\right)$, $\sigma = \sigma\left(t, q\right)$. Then it is easy to check that the two last equations of (1.6) imply

$$U = \lambda\left(t\right) \frac{x\partial_x + y\partial_y}{q},$$

for some function $\lambda\left(t\right)$, and the first equation takes the form

$$m\frac{\partial \sigma}{\partial t} + \lambda\left(t\right) F_\sigma\left(\sigma\right) \frac{\partial \sigma}{\partial q} = 0.$$

Let's change parameter t and put $\sigma = \sigma\left(\tau\left(t\right), q\right)$, where $\tau' = \lambda\left(t\right)$, $\tau\left(0\right) = 0$.

Finally, the Buckley – Leverett system takes the following form

$$\begin{cases} m\dfrac{\partial \sigma}{\partial \tau} + F_\sigma\left(\sigma\right) \dfrac{\partial \sigma}{\partial q} = 0, \\ \\ p_q = -\dfrac{\lambda\left(t\right)}{f\left(\sigma\right) q^2}. \end{cases}$$

Solutions of the first equations could be easily found by the method of characteristics.

In our case the characteristics are solutions of the following system of ordinary differential equations :

$$\dot{\tau} = 1, \quad \dot{q} = m^{-1} F_\sigma(\sigma), \quad \dot{\sigma} = 0.$$

Therefore, the solution of the Cauchy problem

$$\sigma(0, q) = \sigma_0(q)$$

could be presented in the parametric form:

$$\begin{array}{rcl} q & = & \overline{q} + m^{-1} F_\sigma(\sigma)\,\tau, \\ \sigma & = & \sigma_0(\overline{q}), \end{array} \tag{2.1}$$

where $\overline{q} \geq 0$ is a parameter.

We consider this solution as surface

$$S = \left\{ (\tau, q, \sigma) \middle|\ \sigma - \sigma_0\left(q - m^{-1} F_\sigma(\sigma)\,\tau\right) = 0 \right\} \subset \mathbb{R}^3.$$

Then intersections

$$S_{\tau_0} = S \cap \{\tau = \tau_0\}$$

are graphs or profiles of the solution $\sigma(\tau, q)$, when $\tau = \tau_0$.

Geometrically, the solution $\sigma(\tau, q)$ could be obtained in the following way.

On the half-plane $\mathbb{R}^2_{(\overline{q}, y)}$, $\overline{q} \geq 0$ consider the graph Γ of the function

$$\phi(\overline{q}) = m^{-1} F'(\sigma_0(\overline{q}))$$

and straight lines

$$l_{(\tau, q)} = \frac{q - \overline{q}}{\tau}, \ \tau \neq 0.$$

Let $\overline{q}(\tau, x)$ be the intersection of Γ and $l_{(\tau,q)}$, then the value $\sigma(\tau, q)$ equals to $\sigma_0(\overline{q}(\tau, q))$. It shows that function $\sigma(\tau, q)$ is smooth, at least, for small values of τ.

In general, let's consider the restriction of the natural projection

$$\pi : \mathbb{R}^3_{(\tau, x, a)} \to \mathbb{R}^2_{(\tau, x)},$$

on the surface S, $\pi : S \to \mathbb{R}^2_{(\tau, x)}$, which is smooth. Then the intersection S_τ is a graph of a smooth function if and only if the differential $d\,\pi|_S$ is isomorphism at points S_τ.

A point $(\tau, q, \sigma) \in S$ is said to be *singular* (or *caustic*) if $\det(d\pi) = 0$ at this point. Denote by $\Sigma \subset S$ the set of all singular points.

In our case Σ is a smooth curve having the following parametric presentation:

$$\Sigma = \left\{ (\tau, q, \sigma) | \, \tau = -\frac{1}{\phi'(\overline{q})}, \quad q = \overline{q} \ - \frac{\phi(\overline{q})}{\phi'(\overline{q})}, \quad \sigma = \sigma_0(\overline{x}) \right\},$$

where $(\tau, \overline{q})$ are coordinates on the surface S.

Then the front of caustics $\pi(\Sigma) \subset \mathbb{R}^2_{(\tau, q)}$ has the following parametric presentation:

$$\tau = -\frac{1}{\phi'(\overline{q})}, \quad q = \overline{q} \frac{\phi(\overline{q})}{\phi'(\overline{q})},$$

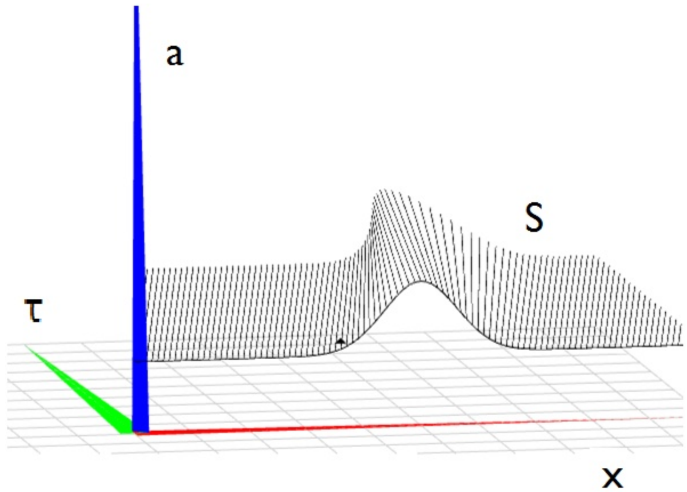

FIGURE 1. Whitney's cusp

where $\overline{q} \geq 0$ is a parameter.

Theorem 2.1. *Any smooth rotation invariant solution of the Buckley – Leverett system has the form*

$$\sigma(t,q) = \sigma_0(\tau(t), \overline{q}),$$

where $\sigma(0,q) = \sigma_0(q)$, $\overline{q}$ *is a solution of equation*

$$q = \overline{q} + m^{-1} F_\sigma(\sigma_0(\overline{q}))\, \tau(t),$$

and

$$\begin{aligned} U &= \tau'(t) \frac{x\partial_x + y\partial_y}{q}, \\ p(t,q) &= -\tau'(t) \int \frac{dq}{f(\sigma(t,q))\, q^2}. \end{aligned}$$

3. Shock waves and boundary value problem

Non smooth solutions and corresponding shock waves we'll analyze similar to [1]. The typical singularity of the projection $\pi : S \to \mathbb{R}^2$ is the Whitney cusp (see Fig. 1).

The intersection S_τ, for singular case, is showed on Fig. 2.

Remark that in the differential 1-form

$$\omega = \sigma dq - m^{-1} F(\sigma)\, d\tau$$

is the conservation law for saturation in the sense that the quantity

$$\int \sigma(\tau, q)\, dq$$

is constant in time.

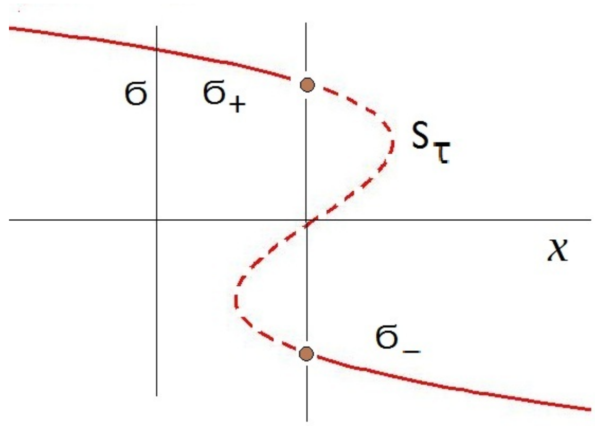

FIGURE 2. Maxwell rule

This means (see [4] for more details) that to satisfy this conservation law we have to use the Maxwell rule to cut the surface S on piecewise smooth components.

On the picture it is realized as the jump from the branch (σ_+) to the branch $(\sigma_-) : \sigma_+ \to \sigma_-$ and the jump points are chosen in such a way that shaded areas are equal.

To find function $\tau(t)$ we put boundary condition on saturation. Namely we'll assume that in addition to Cauchy problem the function $\sigma(t,0) = \gamma(t)$ is given.

The general formula for the solution of the Cauchy problem shows, that $\sigma(\tau,0) = q_0(\overline{x})$, where $\overline{q} = \overline{q}(\tau)$, is a solution of the equation:

$$\overline{q} + m^{-1} F_\sigma(\sigma_0(\overline{q}))\,\tau = 0.$$

Let

$$\alpha(\tau) = \sigma_0(\overline{q}(\tau)),$$

where we pick $\overline{q}(\tau)$ in correspondence to the branches in the Maxwell law separation.

To get solution of the initial boundary problem in both cases, we should find now a function $\tau(t), \quad \tau(0) = 0$, such that

$$\alpha(\tau(t)) = \gamma(t). \tag{3.1}$$

It was shown in [1] that there is time t_l, calling the living time, such that the following result valid.

Theorem 3.1. *The initial boundary value problem for the Buckley – Leverett system has unique and piecewise smooth solution $\sigma(t,q), p(t,q), U$ up to the living time.*

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