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PROBABILISTIC REPRESENTATION OF THE CAUCHY PROBLEM SOLUTIONS FOR SYSTEMS OF NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this paper we present some results concerning probabilistic approaches to construction of classical and generalized solutions to the Cauchy problem for systems of parabolic equations from two different classes and show key points where there arises a crucial difference between them.

Introduction

Among systems of parabolic equations that arise as mathematical models describing various physical, chemical and biological phenomena we consider two large classes, namely, systems with diagonal second order terms and nondiagonal terms of the first and zero order providing that all second order coefficients are equal and systems with nondiagonal second order terms. We are interested in probabilistic representations of solutions to the Cauchy problem for these systems. To be more precise we are interested in either classical or generalized solutions of the Cauchy problem. In addition it should be mentioned as well that we consider here both forward and backward Cauchy problem for systems of these types.

The investigation of systems of nonlinear parabolic equations of the first class via probabilistic approaches was started by Yu.Dalecky and Ya. Belopolskaya in [1], [2]. The fundamental results concerning the Cauchy problem solution for systems of this type one can find in a famous monograph by O. Ladyzenskaya, V.Solonnikov, N.Uraltzeva [3], where both classical and generalized solutions of such systems were investigated. The probabilistic approach developed in [1], [2] allows to reveal some peculiarities of this class of systems and in particular a possibility to treat a system from this class as a scalar equation of a special form defined on a new phase space. In addition it shows the way to reduce the Cauchy problem solution to solution of a certain stochastic system. A probabilistic approach to construction of generalized solutions of the Cauchy problem for systems of the first class was developed in [4] based on the Kunita results concerning probabilistic representation of generalized solutions for linear scalar equations [5], [6].

A construction of stochastic processes associated with parabolic systems of the second class appears to be the most tricky. This class of systems was studied by

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people working in the PDE theory started from seminal papers by Amann [7], see as well more recent review [8] and references therein. The probabilistic approach to the Cauchy problem for systems of the second class was developed in papers [9], [10].

In this paper we present some results concerning probabilistic approaches to construction of classical and generalized solutions to the Cauchy problem for systems of the first class and generalized solutions of the Cauchy problem for systems of the second class and show key points where there arises a crucial difference between them.

1. Stochastic approach to the first class systems

Consider a general system of the first class having the form

$$\frac{\partial u_m}{\partial s} + \mathcal{L}_0 u_m + \sum_{l=1}^{d_1} \mathcal{G}_{ml} u_l = 0, \quad u_m(T,x) = u_{0m}(x),\, m = 1,\ldots,d_1, \tag{1.1}$$

where $\mathcal{L}_0 u_m = \frac{1}{2}\sum_{i,j,k=1}^{d} A^u_{ik}(x)\nabla^2_{ji} u_m A^u_{jk}(x) + \sum_{i=1}^{d} a^u_i(x)\nabla_i u_m\rangle$ and $\sum_{l=1}^{d_1} \mathcal{G}_{ml} u_l = \sum_{l=1}^{d_1}\left[\sum_{j=1}^{d} B^{ml}_j(x,u)\nabla_j u_l + c^u_{ml}(x)u_l\right]$. We assume first that all coefficients depend on x, u, i.e. $a^u(x) = a(x,u)$. A stochastic system associated with (1.1) has the form

$$d\xi(\tau) = a^u(\xi(\tau))d\tau + A^u(\xi(\tau))dw(\tau), \quad \xi(s) = x \in R^d, \tag{1.2}$$

$$d\eta(\tau) = c^u(\xi(\tau))\eta(\tau)d\tau + C^u(\xi(\tau))(\eta(\tau), dw(\tau)), \quad \eta(s) = h \in R^{d_1}, \tag{1.3}$$

$$\langle h, u(s,x)\rangle = E_{s,x,h}\langle \eta(T), u_0(\xi(T))\rangle \tag{1.4}$$

where $B = C^*A$, $\langle h, u\rangle = \sum_{m=1}^{d_1} h_m u_m$ and $\langle Ch, u\rangle = \langle h, C^*u\rangle$.

Assume that condition **C.1** holds that is all coefficients and u_0 are $C^{k+\alpha}$-smooth functions, $k = 1, 2$, $\alpha \in (0,1)$ and have polynomial growth in u. Besides $a(x,u), A(x,u)$ have a sublinear growth in x uniformly in u while u_0, $c(x,u)$ and $C(x,u)$ are bounded in x.

The following assertions have been proved in [1]-[2].

Theorem 1.1. *Assume that* **C.1** *with $k = 2$ holds. Then there exists an interval $[T_1, T]$ such that for all $s \in [T_1, T]$ there exists a solution of the system (1.2)-(1.4). The length of the interval depends on coefficients a, A, c, C and u_0.*

Theorem 1.2. *Under assumptions of theorem 1 the function $u(s,x)$ defined by the system (1.2)-(1.4) is C^2-smooth and bounded on a possibly smaller interval $[T_2, T] \subset [T_1, T]$ and is a unique classical solution of the Cauchy problem (1.1).*

Detailed proofs of the above assertions can be found in [1], [2]. As a final remark concerning classical solutions of nonlinear parabolic systems from the first class let us mention that the above considerations can be extended to the case of quasilinear and fully nonlinear parabolic systems. Notice that in this case one has to construct a certain differential prolongation of the original system and consider a new semilinear system including the original one.

2. Stochastic approach to generalized solutions of the Cauchy problem for systems of parabolic equations

To construct a probabilistic approach to a generalized solution of a PDE or a system of PDEs we need a number of standard functional spaces, namely, the space $C^k(Y); R^{d_1})$ of k- times differentiable functions defined on a linear space Y and valued in R^{d_1}, the Schwartz space $C_0^\infty(Y;)R^{d_1}$ and Sobolev spaces $W^{k,q} \equiv W^{k,q}([0,T]\times R^d; R^{d_1})$.

A stochastic representation of a generalized solution to the forward Cauchy problem for a system from the first class was constructed in [4]. There we used a definition of a generalized solution from [3] and the generalized Ito formula was a crucial part in the construction.

Unfortunately it does not work when one considers a system from the second class. To obtain the required results we need a different though equivalent [11] definition of a generalized solution of the Cauchy problem for a system of parabolic equations and a notion of stochastic test function.

To illustrate the suggested approach we consider the Cauchy problem

$$\frac{\partial u^m}{\partial t} = \Delta(u^m[u^1+u^2]) + c_u^m u^m, \quad u^m(0,x) = u_0^m(x),\, m=1,2, \tag{2.1}$$

where $c_u^m = c_m - c_{m1}u^1 - c_{m2}u^2$ and $c_m, c_{mk}, m, k = 1,2$ are positive constants. We say that a pair of functions u^1, u^2 is a generalized solution of (2.1) if it has the following properties:

i) $u^1, u^2 \in L^\infty_{\rm loc}([0,\infty); L^\infty(R^d)) \cap \mathcal{W}$ and $u^1, u^2 \geq 0$ a.e. in $(0,\infty)\times R^d$;

ii) $\nabla u^m \in L^2_{\rm loc}((0,\infty)\times R^d)$,

iii) for any test function $h \in C_0^\infty([0,\infty)\times R^d)$ with compact support

$$\int_0^\infty \langle\langle u^m(\theta), [\frac{\partial h(\theta)}{\partial\theta} + [u^1(\theta)+u^2(\theta)]\Delta h(\theta)]d\theta \tag{2.2}$$

$$+\int_0^\infty \langle\langle u^m(\theta), [c_m - c_{m1}u^1(\theta) - c_{m2}u^2(\theta)]h(\theta)\rangle\rangle d\theta = -\langle\langle u_0^m, h(0)\rangle\rangle.$$

This version of definition allows to reveal a structure of a Markov process generator associated with (2.1).

Set

$$\frac{1}{2}M_u^2(x) = u^1(t,x) + u^2(t,x), \quad c_u^m(x) = c_m - c_{m1}u^1(t,x) - c_{m2}u^2(t,x) \tag{2.3}$$

and consider the Cauchy problem for parabolic equations

$$\frac{\partial h^m(s,y)}{\partial s} + \frac{1}{2}M_u^2(y)\Delta h^m(s,y) + c_u^m h^m(s,y) = 0, \quad h^m(t,y) = h(y), \quad 0\leq s\leq t, \tag{2.4}$$

Assume that $u^m(\theta,y)$ is a given bounded function twice differentiable in $y\in R^d$. Then from the previous section results we know that a probabilistic representation of a classical solution to (2.4) can be presented in the form

$$h^m(\theta,y) = E[\eta^m(t)h(\xi_{\theta,y}(t))], \quad 0\leq\theta\leq t, m=1,2, \tag{2.5}$$

where $\xi(t), \eta^m(t)$ are governed by SDEs

$$d\xi(\theta) = M_u(\xi(\theta))dw(\theta), \quad \xi(0) = y, 0\leq\theta\leq t, \tag{2.6}$$

$$d\eta^m(\theta) = c_u^m(\xi(\theta))\eta^m(\theta)d\theta, \quad \eta^m(0) = 1. \tag{2.7}$$

correspondingly.

We construct a probabilistic representation of a regular generalized solution $u^m(t,x), m = 1,2$ of (2.1) assuming that $u^m(t,x)$ exists and unique. Under this assumption we can prove that there exists a unique solution $\xi(t)$ to (2.6) and its time reversal $\hat{\xi}(\theta)$ satisfies the stochastic integral equation

$$\hat{\xi}_{0,x}(\theta) = x - \int_\theta^t [M_u \nabla M_u](\hat{\xi}_{0,x}(\tau))d\tau - \int_\theta^t M_u(\hat{\xi}_{0,x}(\tau))d\tilde{w}(\tau), \tag{2.8}$$

where $0 \le \theta \le \tau \le t$.

To derive the stochastic representation of $u^m(t,x)$ we have to modify (2.7) and apply some results from the Kunita stochastic flow theory [5]. Since now we cannot apply the generalized Ito formula immediately to the function u we introduce instead a notion of a stochastic test function.

Consider a stochastic process $\gamma^m(\theta) = \eta^m(\theta)h(\xi(\theta))J(\theta)$, where $\xi(\theta)$ satisfies (2.6) and the process $\eta^m(\theta)$ satisfies a linear SDE

$$d\eta^m(\theta) = \tilde{c}_u^m(\xi(\theta))\eta^m(\theta)d\theta + \eta^m(\theta)\langle C_u^m(\xi(\theta)), dw(\theta)\rangle, \quad \eta^m(0) = 1 \tag{2.9}$$

with coefficients $\tilde{c}_u^m$ and C_u^m to be specified below. In addition under the above assumptions on functions u^m there exists $J(\theta) = det\nabla\xi_{0,x}(\theta)$. To obtain an explicit expression for $d\gamma^m(\theta)$ we apply the Ito formula and note that as it is not difficult to verify that $dJ(\theta)$ has the form

$$dJ(\theta) = J(\theta)\langle \nabla M_u, dw(t)\rangle, \quad J(0) = 1. \tag{2.10}$$

As a result we get the following lemma.

Lemma 2.1. *Let coefficients $\tilde{c}_u^m$ and $\tilde{C}_u^m$ have the form*

$$\tilde{c}_u^m(\xi(\theta)) = c_u^m(\xi(\theta)) - \langle \nabla M_u(\xi(\theta)), \nabla M_u(\xi(\theta))\rangle, \quad C_u^m(\xi(\theta)) = -\nabla M_u(\xi(\theta)). \tag{2.11}$$

Then the processes $\gamma^m(\theta) = \eta^m(\theta)h(\xi_{0,y}(\theta))J(\theta), m = 1,2,$ have stochastic differentials of the form

$$d\gamma^m(\theta) = \left[\frac{1}{2}M_u^2\Delta h + c_u^m h\right](\xi(\theta))\eta^m(\theta)J(\theta)d\theta + \langle M_u\nabla h(\xi(\theta)), \eta^m(\theta)J(\theta)dw(\theta)\rangle. \tag{2.12}$$

By direct computation we can verify that the processes $\hat{\xi}(\theta), \hat{\eta}^m(\theta)$ which are time reversal with respect to processes $\xi(\theta), \eta^m(\theta)$ satisfying correspondingly to (2.6) and (2.9) allow to construct a probabilistic representation of a generalized solution to (2.1) in the form

$$u^m(t,x) = E[\hat{\eta}^m(t)u_0^m(\hat{\xi}_{0,x}(t))], \; m = 1,2. \tag{2.13}$$

Note that system describing $\hat{\xi}(\theta), \hat{\eta}^m(\theta), u^m(t,x)$ is not closed, hence though it gives a probabilistic representation of a generalized solution to (2.1) under a priori assumption of the existence of this solution but it still does not allow to reduce (2.1) to a closed stochastic problem. To reach this goal we have to add to the above stochastic system (2.6), (2.9) (2.13) some relations which allow to derive a stochastic representation to both u^m and ∇u^m.

To this end we apply some results of the previous section. Namely, by formal differentiation of (2.1) we get a PDE for $v_i^m = \nabla_i u^m$ with $v_i^m(0,x) = \nabla_i u_0^m(x)$ and

$$\frac{\partial v_i^m}{\partial t} = \Delta\{v_i^m(u^1+u^2) + u^m(v^1+v^2)\} + u^m\nabla_i c^m(u) + c^m(u)v_i^m. \quad (2.14)$$

In a similar way from

$$\frac{\partial h}{\partial\theta} + (u^1+u^2)\Delta h + c^m(u)h = 0, \quad h(t,y) = h(y), \quad (2.15)$$

we get a PDE for $g_i = \nabla_i h$

$$\frac{\partial g_i}{\partial\theta} + (u^1+u^2)\Delta g_i + (v_i^1+v_i^2)divg + \nabla_i c^m(u)h + c^m(u)g_i = 0, \quad g_i(0,y) = \nabla_i h(y). \quad (2.16)$$

In addition note that we can construct a stochastic representation of the solution to (2.15)-(2.16) in the form $\Gamma^m(\theta,y) = E[\eta^m(t)\Gamma_0(\xi_{\theta,y}(t))]$, where $\Gamma(t,y) = \begin{pmatrix} h(t,y) \\ \nabla h(t,y)\end{pmatrix}$ and stochastic processes $\xi(\tau)$ and $\eta_{ik}^m(\tau)$ satisfy SDEs

$$d\xi(\tau) = \sqrt{2[u^1(t,\xi(\tau)) + u^2(t,\xi(\tau))]}dw(\tau), \quad \xi(\theta) = y, 0 \le \theta \le \tau \le t,$$
$$d\beta^m(\tau) = n_u^m(\xi(t))\beta(\tau)d\tau + \langle N_u^m(\xi(\tau)), \beta^m(\tau)dw(\tau)\rangle.$$

Here for the Kronecker symbol δ with $\delta g = \sum_k \sum_j \delta_{jk}\nabla_j g_k = div\, g$ we denote

$$\beta^m(\tau) = \begin{pmatrix} \eta^m(\tau) \\ \nabla\eta^m(\tau)\end{pmatrix}, \quad n_u^m = \begin{pmatrix} c_u^m & 0 \\ \nabla c_u^m & c_u^m\end{pmatrix}, \quad N_u^m = \begin{pmatrix} 0 & 0 \\ 0 & \frac{[v^1+v^2]\delta}{\sqrt{2(u^1+u^2)}}\end{pmatrix}$$

and thus for $\Gamma_0(y) = \Gamma^m(0,y) = \begin{pmatrix} h(y) \\ \nabla h(y)\end{pmatrix}$ we set $\Gamma^m(\theta,y) =$

$$= E\left[\begin{pmatrix} \eta^m(t) & 0 \\ \nabla\eta^m(t) & \eta^m(t)\end{pmatrix}\begin{pmatrix} h(\xi(t)) \\ \nabla h(\xi(t))\end{pmatrix}\right] = \begin{pmatrix} E[\eta^m(t)h(\xi(t))] \\ E[\nabla\eta^m(t)h(\xi(t)) + \eta^m(t)\nabla h(\xi(t))]\end{pmatrix}.$$

To deduce the stochastic representation for the function $v_j^m = \nabla_j u^m$ given the PDE system (2.1),(2.14) we proceed as follows. We rewrite this system in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} u^m \\ v^m\end{pmatrix} = \mathcal{Z}^m\begin{pmatrix} u^m \\ v^m\end{pmatrix}, \quad m = 1,2, \quad \text{where} \quad (2.17)$$

$$\mathcal{Z}^m\begin{pmatrix} u^m \\ v^m\end{pmatrix} =$$

$$= \Delta\left[\begin{pmatrix} u^1+u^2 & 0 \\ 0 & u^1+u^2\end{pmatrix}\begin{pmatrix} u^m \\ v^m\end{pmatrix}\right] + \begin{pmatrix} 0 & 0 \\ 0 & [v^1+v^2]\end{pmatrix}\begin{pmatrix} u^m \\ v^m\end{pmatrix} + \begin{pmatrix} c_{11}^m & 0 \\ c_{21}^m & c_{22}^m\end{pmatrix}\begin{pmatrix} u^m \\ v^m\end{pmatrix}$$

then we consider a dual system derived from (2.17) as follows. Integrate over R^d a product of (2.17) and a vector test function $(h,g)^*$, where $g_j = \nabla_j h$, $j = 1,\dots,d$. As a result we obtain a system of the form

$$\left\langle\left\langle\begin{pmatrix} u^m \\ v^m\end{pmatrix}\left[\frac{\partial}{\partial t}\begin{pmatrix} h \\ g\end{pmatrix} + \mathcal{Q}^m\begin{pmatrix} h \\ g\end{pmatrix}\right]\right\rangle\right\rangle = 0, \quad \text{where} \quad (2.18)$$

$$\mathcal{Q}^m\begin{pmatrix} h \\ g\end{pmatrix} = \begin{pmatrix} u^1+u^2 & 0 \\ 0 & u^1+u^2\end{pmatrix}\Delta\begin{pmatrix} h \\ g\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & [v^1+v^2]\delta\end{pmatrix}\nabla\begin{pmatrix} h \\ g\end{pmatrix}$$

$$+\begin{pmatrix} c^m_{11} & 0 \\ c^m_{21} & c^m_{22} \end{pmatrix}\begin{pmatrix} h \\ g \end{pmatrix}.$$

Here and below we denote by

$$\left\langle\left\langle\begin{pmatrix} u^m \\ v^m_i \end{pmatrix}\begin{pmatrix} h \\ g_i \end{pmatrix}\right\rangle\right\rangle = \begin{pmatrix} \int_{R^d} u^m(x)h(x)dx \\ \int_{R^d} v^m_i(x)g_i(x)dx \end{pmatrix}.$$

Consider a stochastic equation of the form

$$d\eta^m(\theta) = [\tilde{c}^m]^*(\xi(\theta))\eta^m(\theta)d\theta + [\tilde{C}^m]^*(\xi(\theta))(\eta^m(\theta), dw(\theta)), \quad \eta^m(s) = \gamma^m \quad (2.19)$$

with respect to the two component process $\eta^m(\theta) = \begin{pmatrix} \eta^m_1(\theta) \\ \eta^m_2(\theta) \end{pmatrix}$ with coefficients $\tilde{c}^m$ and $\tilde{C}^m$ to be chosen below. Let $\zeta^m(t)$ maps γ^m to $\eta^m(\theta)$, that is

$$\zeta^m(\theta) = \begin{pmatrix} \zeta^m_{11}(\theta) & 0 \\ \zeta^m_{21}(\theta) & \zeta^m_{22}(\theta) \end{pmatrix}.$$

To simplify notation we omit index m and define a stochastic test function

$$\kappa(\theta) = \begin{pmatrix} \kappa_1(\theta) \\ \kappa_2(\theta) \end{pmatrix} = \begin{pmatrix} \zeta_{11}(\theta) & 0 \\ \zeta_{12}(\theta) & \zeta_{22}(\theta) \end{pmatrix}\begin{pmatrix} h(\xi(\theta)) \\ g(\xi(\theta)) \end{pmatrix} J(\theta). \quad (2.20)$$

The stochastic differential of the process $\kappa(\theta)$ has the form $d\kappa(\theta) = \begin{pmatrix} d\kappa_1(\theta) \\ d\kappa_2(\theta) \end{pmatrix}$ with

$$d\kappa_1(\theta) = [\tilde{c}_{11}h + M_u\Delta h + \langle C_{11}, [M_u\nabla h + \nabla M_u h](\xi(\theta))\rangle\zeta_{11}(\theta)J(\theta)]d\theta$$
$$+\langle M_u\nabla h(\xi(\theta)), \nabla M_u\rangle\zeta_{11}(\theta)J(\theta)d\theta + \langle N_1(\xi(\theta)), dw(\theta)\rangle,$$
$$d\kappa^i_2(\theta) = \big[[\tilde{c}_{21}h + M_u div g](\xi(\theta))\zeta^i_{21}(\theta) + \zeta_{22}(\theta)[\tilde{c}_{22}g_i + M_u\Delta g_i](\xi(\theta))\big]\, J(\theta)d\theta$$
$$+\{C_{21}\zeta^i_{21}(\theta)[M_u\nabla h + \nabla M_u h](\xi(\theta)) + C_{22}\zeta_{22}(\theta)[M_u\nabla g_i + g_i\nabla M_u](\xi(\theta))$$
$$+\zeta^i_{21}(\theta)M_u\langle\nabla h, \nabla M_u\rangle(\xi(\theta)) + \zeta_{22}(\theta)M_u\langle\nabla g_i, \nabla M_u\rangle(\xi(\theta))\}J(\theta)d\theta$$
$$+\langle[N_{21}(\xi(\theta))\zeta^i_{21}(\theta) + N^i_{22}(\xi(\theta))\zeta_{22}(\theta)], dw(\theta)\rangle J(\theta).$$

Let us specify coefficients $\tilde{c}^m$ and $\tilde{C}^m$. As it was done above we choose

$$\tilde{C}^m_{11} = -\nabla M_u, \quad \tilde{c}^m_{11} = c^m_u + \|\nabla M_u\|^2, \quad (2.21)$$

where I is the identity matrix. Next we choose

$$\tilde{C}^m_{21} = -\nabla M_u, \tilde{C}^m_{22} = \frac{(v^1+v^2)\delta}{M_u} - \nabla M_u,$$
$$[\tilde{c}^m_{21}]_i = \nabla_i c^m_u + \|\nabla M_u\|^2, \quad \tilde{c}^m_{22} = c^m_u + \|\nabla M_u\|^2. \quad (2.22)$$

We do not specify for the moment N^m_1 and N^m_2 since they do not take part in the probabilistic representation of u^m and v^m. Next we proceed as in the previous section.

To get a closed counterpart of the system (2.1) we state the following assertion.

Theorem 2.2. *Under assumptions of theorem 1.1 with $k=1$ both the functions $u^m(t,x)$ admit stochastic representations (2.13) and functions $v^m_j = \nabla_j u^m$ admit stochastic representations*

$$\begin{pmatrix} u^m(t,x) \\ \nabla_i u^m(t,x) \end{pmatrix} = E\left[\begin{pmatrix} \hat{\zeta}^m_{11}(t) & 0 \\ \hat{\zeta}^m_{21}(t) & \hat{\zeta}^m_{22}(\theta) \end{pmatrix}\begin{pmatrix} u^m_0(\hat{\xi}_{0,x}(t)) \\ v^m_i(\hat{\xi}_{0,x}(t)) \end{pmatrix}\right]. \quad (2.23)$$

Proof. To verify the last assertion of the theorem we note that we have the following matrix relations

$$\left\langle\left\langle\int_0^t \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} \begin{pmatrix} d\kappa_1^m(\theta) \\ d\kappa_2^m(\theta) \end{pmatrix}\right\rangle\right\rangle = \left\langle\left\langle \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} \begin{pmatrix} d\kappa_1^m(t) \\ d\kappa_2^m(t) \end{pmatrix}\right\rangle\right\rangle - \left\langle\left\langle \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} \begin{pmatrix} d\kappa_1^m(0) \\ d\kappa_2^m(0) \end{pmatrix}\right\rangle\right\rangle . \tag{2.24}$$

At the other hand from (2.20) we deduce

$$E\left[\left\langle\left\langle\int_0^t \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} \begin{pmatrix} d\kappa^1(\theta) \\ d\kappa^2(\theta) \end{pmatrix}\right\rangle\right\rangle\right]$$

$$= E\left[\int_0^t \left\langle\left\langle \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} d\left[\begin{pmatrix} \zeta_{11}^m(\theta) & 0 \\ \zeta_{21}^m(\theta) & \zeta_{22}^m(\theta) \end{pmatrix} \begin{pmatrix} h(\xi_{0,y}(\theta)) \\ g(\xi_{0,y}(\theta)) \end{pmatrix} J(\theta)\right]\right\rangle\right\rangle\right]$$

$$= E\left[\int_0^t \left\langle\left\langle \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} \begin{pmatrix} \zeta_{11}^m(\theta) & 0 \\ \zeta_{21}^m(\theta) & \zeta_{22}^m(\theta) \end{pmatrix} \mathcal{Q}^m \begin{pmatrix} h(\xi_{0,\cdot}(\theta) \\ g(\xi_{0,\cdot}(\theta)) \end{pmatrix} J(\theta)\right\rangle\right\rangle d\theta\right]. \tag{2.25}$$

By the change of variables $\xi_{0,y}(\theta) = x$ applying stochastic Fubini theorem we get

$$E\left[\left\langle\left\langle\int_0^t \begin{pmatrix} u_0^m \\ v_{i0}^m \end{pmatrix} \begin{pmatrix} d\kappa^1(\theta) \\ d\kappa^2(\theta) \end{pmatrix}\right\rangle\right\rangle\right] \tag{2.26}$$

$$= E\left[\int_0^t \left\langle\left\langle \begin{pmatrix} \hat\zeta_{11}^m(\theta) & 0 \\ \hat\zeta_{21}^m(\theta) & \hat\eta_{22}^m(\theta) \end{pmatrix} \begin{pmatrix} u_0^m(\hat\xi_{0,\cdot}(\theta)) \\ v_{i0}^m(\hat\xi_{0,\cdot}(\theta)) \end{pmatrix} \mathcal{Q}^m \begin{pmatrix} h \\ g \end{pmatrix}\right\rangle\right\rangle d\theta\right]$$

$$= \int_0^t \left\langle\left\langle E\left[\begin{pmatrix} \hat\zeta_{11}^m(\theta) & 0 \\ \hat\zeta_{21}^m(\theta) & \hat\zeta_{22}^m(\theta) \end{pmatrix} \begin{pmatrix} u_0^m(\hat\xi_{0,x}(\theta)) \\ v_{i0}^m(\hat\xi_{0,\cdot}(\theta)) \end{pmatrix}\right] \mathcal{Q}^m \begin{pmatrix} h \\ g \end{pmatrix}\right\rangle\right\rangle d\theta$$

$$= \int_0^t \left\langle\left\langle \mathcal{Z}^m E\left[\begin{pmatrix} \hat\zeta_{11}^m(\theta) & 0 \\ \hat\eta_{21}^m(\theta) & \hat\zeta_{22}^m(\theta) \end{pmatrix} \begin{pmatrix} u_0^m(\hat\xi_{0,\cdot}(\theta)) \\ v_{i0}^m(\hat\xi_{0,\cdot}(\theta)) \end{pmatrix}\right] \begin{pmatrix} h \\ g \end{pmatrix}\right\rangle\right\rangle d\theta.$$

Hence we derive that the functions

$$\begin{pmatrix} \lambda^m(t,x) \\ \nabla\lambda^m(t,x) \end{pmatrix} = E\left[\begin{pmatrix} \hat\zeta_{11}^m(\theta) & 0 \\ \hat\zeta_{21}^m(\theta) & \hat\eta_{22}^m(\theta) \end{pmatrix} \begin{pmatrix} u_0^m(\hat\xi_{0,x}(\theta)) \\ v_{i0}^m(\hat\xi_{0,x}(\theta)) \end{pmatrix}\right]$$

satisfy integral identities

$$\left\langle\left\langle \begin{pmatrix} \lambda^m(t) \\ \nabla\lambda^m(t) \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix}\right\rangle\right\rangle - \left\langle\left\langle \begin{pmatrix} \lambda^m(0)\ h \\ \nabla\lambda^m(0) g \end{pmatrix}\right\rangle\right\rangle = \left\langle\left\langle \mathcal{Q}^m \begin{pmatrix} \lambda^m(t) \\ \nabla\lambda^m(t) \end{pmatrix} \begin{pmatrix} h(x) \\ g(x) \end{pmatrix}\right\rangle\right\rangle$$

which results due to the assumed uniqueness of a solution to (1.2) that

$$\begin{pmatrix} \lambda^m(t,x) \\ \nabla\lambda^m(t,x) \end{pmatrix} = \begin{pmatrix} u^m(t,x) \\ \nabla u^m(t,x) \end{pmatrix}$$

and hence

$$\begin{pmatrix} u^m(t,x) \\ \nabla u^m(t,x) \end{pmatrix} = E\left[\begin{pmatrix} \hat\zeta_{11}^m(t) & 0 \\ \hat\eta_{21}^m(t) & \hat\zeta_{22}^m(t) \end{pmatrix} \begin{pmatrix} u_0^m(\hat\xi_{0,x}(t)) \\ v_{i0}^m(\hat\xi_{0,x}(t)) \end{pmatrix}\right].$$

Finally we deduce from the last equality that (2.19) holds and in addition

$$\nabla u_i^m(t,x) = E[\hat\zeta_{21}(t) u_0^m(\hat\xi_{0,x}(t)) + \hat\zeta_{22}^m(t) v_{i0}^m(\hat\xi_{0,x}(t))].$$

$\square$

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