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ANALYSIS OF MIXED PRIORITY RETRIAL QUEUEING SYSTEM WITH TWO WAY COMMUNICATION, COLLISIONS, WORKING BREAKDOWN, BERNOULLI VACATION, NEGATIVE ARRIVAL, REPAIR, IMMEDIATE FEEDBACK AND RENEGING

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Abstract. This paper deals with two way communication retrial queueing system with mixed priority services, where during server's idle time it can make two types of outgoing calls to the customers. The first type is the call in the orbit (outgoing retrial call) and the second one is to the calls outside (outgoing primary call) it. Durations of these calls follow two distinct exponential distributions. Calls (primary call) arrive at the server according to compound Poisson distribution. After completion of regular service for primary call, it is allowed for an immediate feedback with probability p. Otherwise the call may leave the system permanently. The primary call is considered as a high priority call and an outgoing call is considered as a low priority call. The server may takes Bernoulli type vacation. Two kind of break downs are allowed here while the server is busy. In addition to this collision, working breakdown, repair and customer's impatient behaviour are also discussed. Using the supplementary variable technique, the steady state distributions of the server state and the number of calls in the orbit are obtained.

1. Introduction

Retrial queues (orbit) are quite interesting because they have wide range of applications in various systems such as telephone switching systems, telecommunication, computer networks, call center network, etc. The basic characteristic of retrial queue (orbit) is the customers whose service cannot start immediately upon their arrival, they can join a virtual queue. After some time they try to get their service during the server's idle time.

Hiroyuki.S et al (2014) described about the two way communication retrial queues with multiple types of outgoing calls, where the server's idle time can make two types of outgoing calls. They are outgoing retrial call towards the orbit and outgoing primary call to outside the system.

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In most of the queueing systems on retrial queues, the server only serves arriving calls. After serving a call, the server waits for the next arrival. In various service systems such as a call center, the server not only serves arriving calls but also makes outgoing calls during its idle time. While the server is busy, the arriving calls cannot receive a service and calls join an orbit and try to get the service after some time independently of other calls. However, there exist many real life situations where the server can make outgoing calls. This queueing feature is known as models of two way communication. Artalejo et al.(2012, 2013) described about Markovian retrial queues with two way communication.

Bhulai et al. (2013) proposed a multiserver queueing model with infinite buffer for call centers where incoming and outgoing calls follow the same exponential distribution. Phung-Duc et al. (2014) developed retrial queues with balanced call blending analysis of single-server and multiserver model. One interesting feature which has been widely studied in queueing systems is the feedback of customers. After completing the service, if the customer does not satisfy with the service they may go to the server immediately to get the service again is known as immediate feedback (Re-service). Some authors like Rajadurai et al.(2016), have discussed the concept of immediate feedback. In this paper we use both priorities (pre-emptive and non-preemptive)which is known as mixed priority services. Dimitriou et al (2013) studied about the mixed priority queueing system with negative arrival.

The perfectly reliable servers are not possible in real world systems. In our paper, we have assumed that the server fails only in the operational state. The breakdown may cause due to normal as well as abnormal breakdown. The abnormal breakdown may cause due to negative arrival. It is otherwise known as G-queue. It was first introduced by Gelenbe (1989) in neural networks. Recently Bhagat et al (2016) extended the study about G-queues. Kalidass et.al (2012) introduced the working breakdown policy, in which the server works at a lower service rate rather than stopping service during the breakdown period. Tao Li et.al (2017) describes more about working breakdowns.

Dong et al (2017) studied the analysis of a finite capacity queueing system with working breakdown and customer's behaviour. The retrial queue with collision are appropriate for modeling the processes in tele-communication. Krishnakumar et al (2010) studied about the collision with various parameters.

In this paper, we consider a single server mixed priority retrial queueing system with two way communication, collision, working breakdown, negative arrival, immediate feedback, Bernoulli vacation and reneging. Here we use mixed priority services i.e. an incoming call may interrupt(pre-emptive) the service of an outgoing call or join in (non pre-emptive)an orbit. We assume a retrial queue with constant retrial rate for an arriving calls. If the server is idle, it starts making two types of outgoing calls, namely type 1 and type 2 call respectively, in different exponentially distributed times. Service times of these calls follow general distributions. An arriving call that finds the server being busy with incoming call it may collide with primary arrival or joins the orbit and retries to enter the service after some generally distributed time. The server takes vacation under Bernoulli vacation schedule. The server may become inactive when it is in operation due to normal as well as abnormal breakdown. If the server is inactive because of the negative arrival the repair process starts immediately. If the server is inactive due to the normal breakdown, then it will complete the remaining service at a lower service rate and after that the repair process starts. After service completion, vacation completion and repair completion the server is in idle state. We consider the arriving calls may balk and renege the orbit.

Further on, the structure of the paper is as follows. Section 1 is an introduction to mixed priority retrial queueing discipline with literature review. A detailed description of the model, notations used, mathematical formulation and governing equations of the model is given in Section 2. Section 3 gives the steady state solutions and the stationary joint distribution of the server state and orbit size. Section 4 demonstrates the performance measures of the model. In Section 5, the numerical results are computed and graphical studies are shown following which the conclusion is given.

2. Model description

We consider a single server retrial queue with two way communication and two types of customers(calls) namely, high priority(incoming call) and low-priority (outgoing call) calls. The basic operation of the model can be described as:

Two way communication: We consider single server retrial queueing system with two way communication. As described in previous section, there are two flows of calls: incoming and outgoing calls. The server make outgoing calls after some exponentially distributed idle time. Outgoing calls durations follow two distinct exponential distributions. If the server is idle it makes an outgoing retrial or primary call in an exponentially distributed time with mean $1/\pi_1$ and $1/\pi_2$, respectively.

Arrival process: Incoming calls arrive at the server according to compound Poisson process with arrival rate λ. Let $\lambda c_i\ dt$ $(i = 1, 2, 3, ...)$ be the first order probability that a batch of 'i' calls arrives at the system during a short interval of time $(t, t + dt)$, where for $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$. If the server is busy with an outgoing call then the arriving call may interrupt(pre-emptive) the service of an outgoing call with probability p_2 or join (non pre-emptive) the retrial queue (orbit) with probability $1 - p_2$. If the server is busy with primary or feedback call then the arriving call may collide with it and both are shifted to the orbit with probability p_1.

Retrial process: After joining the orbit the calls follow constant retrial policy that attempts to get the service. The retrial time is generally distributed with distribution function $I(s)$ and the density function $i(s)$. Let $\eta(x)dx$ be the conditional probability of completion of retrial during the interval $(x, x+dx]$ where x is the elapsed retrial time.

Service process: If the server is idle, then can make an outgoing retrial call or outgoing primary call which is exponentially distributed with mean $1/\pi_1$ and $1/\pi_2$. The service times of the arriving call, feedback call, outgoing retrial call and outgoing primary call are generally distributed with distribution functions $B_i(s)$ and the density functions $b_i(s)$, i = 1,2,3,4 respectively. Let $\mu_i(x)dx$ be the conditional probability of completion of the arriving call, feedback call, outgoing retrial call and outgoing primary calls service during the interval $(x, x+dx]$, where x is the elapsed service time.

Immediate feedback: After completing the primary call service, the customer who want the service again may to go to the server immediately with probability p or may leave the system with probability 1-p, which is known as immediate feedback (Re-service).

Bernoulli Vacation: After completing each and every service completion the server may take a vacation with probability θ. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$.
Let $\beta(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x+dx]$ where x is the elapsed vacation time.

Breakdown state: The server may become inactive during busy period due to normal(with rate α) as well as abnormal breakdown(with rate $\overline{\lambda}$). The abnormal breakdown causes due to negative arrival. The negative arrival will remove the call is in service and deactivate the server.

Working breakdown state: At the normal breakdown the call currently in service will get the remaining service at a lower service rate μ_5, which is exponentially distributed.

Repair Process: After completing the working breakdown period the server is sent for repair. If the server is broken down due to negative arrival then the repair process starts immediately so as to regain its functionality. Repair time is exponentially distributed with rate γ.

Reneging: If the server is busy or unavailable in the system, an arriving call may renege the orbit exponentially with rate ξ.

Balking: If the server is busy or unavailable in the system, an arriving call may balk the orbit with probability $1-b$.

Idle State: After service, vacation and repair completion the server is in idle state.

2.1. Definitions and notations. Let N(t) be the orbit size at time t, $B_i^0(t)$, i = 1,2,3,4, $V^0(t)$ and $I^0(t)$ be the elapsed service time of the primary call, feedback call, outgoing retrial call, outgoing primary call, vacation time and retrial time respectively at time t. Let S(t) denote the state of the server,

$$
\mathrm{S(t)} = \begin{cases}
\text{0, if the server is idle;} \\
\text{1, if the server is providing primary call service;} \\
\text{2, if the server is providing feedback call service;} \\
\text{3, if the server is providing outgoing retrial call service;} \\
\text{4, if the server is providing outgoing primary call service;} \\
\text{5, if the server is in vacation;} \\
\text{6, if the server is in working breakdown with primary call;} \\
\text{7, if the server is in working breakdown with feedback call;} \\
\text{8, if the server is in working breakdown with type1 call;} \\
\text{9, if the server is in working breakdown with type2 call;} \\
\text{10, if the server is in repair state.}
\end{cases}
$$

We have $I(x)$, $B_i(x)$, $V(x)$ and $M(x)$ is continuous at x = 0, and, $\eta(x)dx = \dfrac{dI(x)}{1 - I(x)}$, $\mu_i(x)dx = \dfrac{dB_i(x)}{1 - B_i(x)}$, i=1,2,3,4 $\beta(x)dx = \dfrac{dV(x)}{1 - V(x)}$ are the first order differential (hazard rate) functions of I(.), $B_i(.)$ and V(.) respectively.

2.2. Queue size distribution. Since the service time, vacation time and retrial time are not exponential, the process $\{S(t), N(t)\}$ is non Markovian. Here we introduce supplementary variables corresponding to elapsed times to make it Markovian [Cox(1955)]. Joint distributions of the server state and orbit size are defined as,

$$\overline{I}m(x,s,t)dx = Pr\{Y(t) = 0, x < I^0(t) \le x + dx, N(t) = m\}, m \ge 1$$

$$\overline{P}^{(1)}_{1,m}(x,s,t)dx = Pr\{Y(t) = 1, x < B^0_1(t) \le x + dx, N(t) = m\}, m \ge 0,$$

$$\overline{P}^{(1)}_{2,m}(x,s,t)dx = Pr\{Y(t) = 2, x < B^0_2(t) \le x + dx, N(t) = m\}, m \ge 0,$$

$$\overline{P}^{(2)}_{1,m}(x,s,t)dx = Pr\{Y(t) = 3, x < B^0_3(t) \le x + dx, N(t) = m\}, m \ge 0,$$

$$\overline{P}^{(2)}_{2,m}(x,s,t)dx = Pr\{Y(t) = 4, x < B^0_4(t) \le x + dx, N(t) = m\}, m \ge 0,$$

$$\overline{V}_m(x,s,t)dx = Pr\{Y(t) = 5, x < V^0(t) \le x + dx, N(t) = m, \}, m \ge 0$$

$$\overline{Q}^{(1)}_{1,m}(s,t) = Pr\{Y(t) = 6, N(t) = m\}, m \ge 0$$

$$\overline{Q}^{(1)}_{2,m}(s,t) = Pr\{Y(t) = 7, N(t) = m\}, m \ge 0$$

$$\overline{Q}^{(2)}_{1,m}(s,t) = Pr\{Y(t) = 8, N(t) = m\}, m \ge 0$$

$$\overline{Q}^{(2)}_{2,m}(s,t) = Pr\{Y(t)=9, N(t)=m\}, m \geq 0$$
$$\overline{R}_m(x,s,t) = Pr\{Y(t)=10, N(t)=m\}, m \geq 0$$

2.3. Equations Governing the System. The Kolmogorov forward equations which governs the model:

- The server is providing arriving primary call service:

$$\frac{\partial}{\partial t}P^{(1)}_{1,m}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{1,m}(x,t) = -(\lambda+\alpha+\overline{\lambda}+\xi+\mu_1(x))P^{(1)}_{1,m}(x,t) + (1-\delta_{m0}) \times \lambda(1-p_1)b\sum_{i=1}^{m} c_i P^{(1)}_{1,m-i}(x,t) + \lambda(1-b)P^{(1)}_{1,m}(x,t) + \xi P^{(1)}_{1,m+1}(x,t);\ m \geq 0, \tag{2.1}$$

- The server is providing feedback call service:

$$\frac{\partial}{\partial t}P^{(1)}_{2,m}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{2,m}(x,t) = -(\lambda+\alpha+\overline{\lambda}+\xi+\mu_2(x))P^{(1)}_{2,m}(x,t) + (1-\delta_{m0}) \lambda(1-p_1)\times b\sum_{i=1}^{m} c_i P^{(1)}_{2,m-i}(x,t) + \lambda(1-b)P^{(1)}_{2,m}(x,t) + \xi P^{(1)}_{2,m+1}(x,t);\ m \geq 0, \tag{2.2}$$

- The server is providing outgoing retrial call service:

$$\frac{\partial}{\partial t}P^{(2)}_{1,m}(x,t) + \frac{\partial}{\partial x}P^{(2)}_{1,m}(x,t) = -(\lambda+\alpha+\overline{\lambda}+\xi+\mu_3(x))P^{(2)}_{1,m}(x,t) + (1-\delta_{m0}) \times \lambda(1-p_2)b\sum_{i=1}^{m} c_i P^{(2)}_{1,m-i}(x,t) + \lambda(1-b)P^{(2)}_{1,m}(x,t) + \xi P^{(2)}_{1,m+1}(x,t);\ m \geq 0, \tag{2.3}$$

- The server is providing outgoing primary call service:

$$\frac{\partial}{\partial t}P^{(2)}_{2,m}(x,t) + \frac{\partial}{\partial x}P^{(2)}_{2,m}(x,t) = -(\lambda+\alpha+\overline{\lambda}+\xi+\mu_4(x))P^{(2)}_{2,m}(x,t) + (1-\delta_{m0}) \times \lambda(1-p_2)b\sum_{i=1}^{m} c_i P^{(2)}_{2,m-i}(x,t) + \lambda(1-b)P^{(2)}_{2,m}(x,t) + \xi P^{(2)}_{2,m+1}(x,t);\ m \geq 0, \tag{2.4}$$

- The server is on vacation:

$$\frac{\partial}{\partial t}V_m(x,t) + \frac{\partial}{\partial x}V_m(x,t) = -(\lambda+\xi+\beta(x))V_m(x,t) + (1-\delta_{m0}) \times \lambda b\sum_{i=1}^{m} c_i V_{m-i}(x,t) + \lambda(1-b)V_m(x,t) + \xi V_{m+1}(x,t); m \geq 0, \tag{2.5}$$

- The server is in working breakdown with primary call:

$$\frac{d}{dt}Q^{(1)}_{1,m}(t) = -(\lambda+\mu_5+\xi)Q^{(1)}_{1,m}(t) + (1-\delta_{m0})\lambda\sum_{i=1}^{m} c_i Q^{(1)}_{1,m-i}(t) + \lambda(1-b)Q^{(1)}_{1,m}(x,t) + \alpha\int_0^\infty P^{(1)}_{1,m}(x,t)dx + \xi Q^{(1)}_{1,m+1}(t); m \geq 1, \tag{2.6}$$

- The server is in working breakdown with feedback call:

$$\frac{d}{dt}Q^{(1)}_{2,m}(t) = -(\lambda+\mu_5+\xi)Q^{(1)}_{2,m}(t) + (1-\delta_{m0})\lambda\sum_{i=1}^{m} c_i Q^{(1)}_{2,m-i}(t) + \lambda(1-b)Q^{(1)}_{2,m}(x,t) + \alpha\int_0^\infty P^{(1)}_{2,m}(x,t)dx + \xi Q^{(1)}_{2,m+1}(t); m \geq 1, \tag{2.7}$$

- The server is in working breakdown with outgoing retrial call:

$$\frac{d}{dt}Q^{(2)}_{1,m}(t) = -(\lambda+\mu_5+\xi)Q^{(2)}_{1,m}(t) + (1-\delta_{m0})\lambda\sum_{i=1}^{m} c_i Q^{(2)}_{1,m-i}(t) + \lambda(1-b)Q^{(2)}_{1,m}(x,t) + \alpha\int_0^\infty P^{(2)}_{1,m}(x,t)dx + \xi Q^{(2)}_{1,m+1}(t); m \geq 1, \tag{2.8}$$

- The server is in working breakdown with outgoing primary call:

$$\frac{d}{dt}Q^{(2)}_{2,m}(t) = -(\lambda+\mu_5+\xi)Q^{(2)}_{2,m}(t) + (1-\delta_{m0})\lambda\sum_{i=1}^{m} c_i Q^{(2)}_{2,m-i}(t) + \lambda(1-b)Q^{(2)}_{2,m}(x,t) + \alpha\int_0^\infty P^{(2)}_{2,m}(x,t)dx + \xi Q^{(2)}_{2,m+1}(t); m \geq 1, \tag{2.9}$$

- The server is in repair process:

$$\frac{d}{dt}R_m(t) = -(\lambda+\gamma+\xi)R_m(t) + (1-\delta_{m0})\lambda b\sum_{i=1}^{m} c_i R_{m-i}(t) + \mu_5\ \Big\{\ \sum_{i=1}^{2} Q^{(1)}_{i,m}(t) + \sum_{i=1}^{2} Q^{(2)}_{i,m}(t)\ \Big\} + \bar{\lambda}\Big\{\sum_{i=1}^{2}\int_0^\infty P^{(1)}_{i,m}(x,t)dx + \sum_{i=1}^{2}\int_0^\infty P^{(2)}_{i,m}(x,t)dx\Big\} + \xi R_{m+1}(t) + \lambda(1-b)R_m(x,t);\ m \geq 0, \tag{2.10}$$

- The server is in retrial state

$$\frac{\partial}{\partial t}I_m(x,t) + \frac{\partial}{\partial x}I_m(x,t) = -(\lambda+\pi_1+\pi_2+\eta(x))I_m(x,t);\ m \geq 1, \tag{2.11}$$

- The server is in idle state

$$\frac{d}{dt}I_0(t) = -(\lambda+\pi_1+\pi_2)I_0(t) + (1-\theta)\ \Big\{\ (1-p)\int_0^\infty P_{1,0}^{(1)}(x,t)\mu_1(x)dx$$
$$+\int_0^\infty P_{2,0}^{(1)}(x,t)\mu_2(x)dx + \int_0^\infty P_{1,0}^{(2)}(x,t)\mu_3(x)dx + \int_0^\infty P_{2,0}^{(2)}(x,t)\mu_4(x)dx\ \Big\}$$
$$+\int_0^\infty V_0(x,t)\beta(x)dx + \gamma R_0(t), \tag{2.12}$$

The above set of equations are to be solved under the following boundary conditions at x=0.

$$I_m(0,t) = (1-\theta)\ \Big\{\ (1-p)\int_0^\infty P_{1,m}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{2,m}^{(1)}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty P_{1,m}^{(2)}(x,t)\mu_3(x)dx + \int_0^\infty P_{2,m}^{(2)}(x,t)\mu_4(x)dx\ \Big\} + \int_0^\infty V_m(x,t)\beta(x)dx$$
$$+\lambda b p_1 \sum_{j=1}^{2}\sum_{i=1}^{m} c_i \Big\{ \int_0^\infty P_{j,m-1-i}^{(1)}(x,t)dx \Big\} + \gamma R_m(t);\ m\geq 1, \tag{2.13}$$

$$P_{1,m}^{(1)}(0,t) = \int_0^\infty I_{m+1}(x,t)\eta(x)dx + \lambda c_{m+1}I_0(t) + \lambda b\sum_{i=1}^{m} c_i\int_0^\infty I_{m+1-i}(x,t)dx$$
$$+\lambda b p_2 \sum_{j=1}^{2}\sum_{i=1}^{m} c_i \Big\{ \int_0^\infty P_{j,m+1-i}^{(2)}(x,t)dx \Big\};\ m\geq 0, \tag{2.14}$$

$$P_{2,m}^{(1)}(0,t) = p\int_0^\infty P_{1,m}^{(1)}(x,t)dx;\ m\geq 0, \tag{2.15}$$

$$P_{1,m}^{(2)}(0,t) = \pi_1\int_0^\infty I_{m+1}(x,t)dx;\ m\geq 0, \tag{2.16}$$

$$P_{2,m}^{(2)}(0,t) = \pi_2\int_0^\infty I_{m}(x,t)dx;\ m\geq 0, \tag{2.17}$$

$$V_m(0,t) = \theta\ \Big\{\ (1-p)\int_0^\infty P_{1,m}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{2,m}^{(1)}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty P_{1,m}^{(2)}(x,t)\mu_3(x)dx + \int_0^\infty P_{2,m}^{(2)}(x,t)\mu_4(x)dx\ \Big\}\ \ m\geq 0. \tag{2.18}$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions are,

$$P_{1,m}^{(1)}(0) = P_{2,m}^{(1)}(0) = P_{1,m}^{(2)}(0) = P_{2,m}^{(2)}(0) = V_m(0) = Q_{1,m}^{(1)}(0) = Q_{2,m}^{(1)}(0)$$
$$= Q_{1,m}^{(2)}(0) = Q_{2,m}^{(2)}(0) = R_m(0) = I_m(0) = 0 \text{ and } I_0(0)=1; m\geq 0. \tag{2.19}$$

The Probability Generating Function(PGF) of this model:

$$A(x,z,t) = \sum_{m=1}^{\infty} z^m A_m(x,t),$$

where $A = I, V, P_i^{(1)}, P_i^{(2)}, Q_i^{(1)} Q_i^{(2)}, R, I$ where i=1,2.
By taking Laplace transforms from equation (1) to equation (18) and solving those equations, we get,

$$\overline{I}_0(x,s,z) = \overline{I}_0(0,s,z)[1-\overline{I}(\varphi(a,s))]e^{-\varphi(a,s)x}, \tag{2.20}$$

$$\overline{P}_i^{(1)}(x,s,z) = \overline{P}_i^{(1)}(0,s,z)[1-\overline{B}_i(\varphi_1(s,z))]e^{-\varphi_1(s,z)x},\ i=1,2 \tag{2.21}$$

$$\overline{P}_1^{(2)}(x,s,z) = \overline{P}_1^{(2)}(0,s,z)[1-\overline{B}_3(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}, \tag{2.22}$$

$$\overline{P}_2^{(2)}(x,s,z) = \overline{P}_2^{(2)}(0,s,z)[1-\overline{B}_4(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}, \tag{2.23}$$

$$\overline{V}(x,s,z) = \overline{V}(0,s,z)[1-\overline{V}(\varphi_3(s,z))]e^{-\varphi_3(s,z)x}, \tag{2.24}$$

$$\overline{Q}_i^{(1)}(s,z) = \frac{\alpha\overline{P}_i^{(1)}(0,s,z)[1-\overline{B}_i(\varphi_1(s,z))]e^{-\varphi_1(s,z)x}}{\varphi_4(s,z)},\ i=1,2 \tag{2.25}$$

$$\overline{Q}_1^{(2)}(s,z) = \frac{\alpha\overline{P}_1^{(2)}(0,s,z)[1-\overline{B}_3(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}}{\varphi_4(s,z)}, \tag{2.26}$$

$$\overline{Q}_2^{(2)}(s,z) = \frac{\alpha\overline{P}_2^{(2)}(0,s,z)[1-\overline{B}_4(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}}{\varphi_4(s,z)}, \tag{2.27}$$

$$\overline{R}(s,z) = \frac{\left\{\begin{array}{l} \overline{\lambda}\ \{\ \sum_{i=1}^{2}(\overline{P}_i^{(1)}(x,s,z)+\overline{P}_i^{(2)}(x,s,z))\ \} \\ +\mu_5\ \{\ \sum_{i=1}^{2}(\overline{Q}_i^{(1)}(s,z)+\overline{Q}_i^{(2)}(s,z))\ \} \end{array}\right\}}{\varphi_5(s,z)}. \tag{2.28}$$

where,
$\varphi(a,s) = s+\lambda+\pi_1+\pi_2$,
$\varphi_1(s,z) = s+\lambda b[1-(1-p_1)C(z)]+\overline{\lambda}+\xi[1-\frac{1}{z}]+\alpha$,
$\varphi_2(s,z) = s+\lambda b[1-(1-p_2)C(z)]+\overline{\lambda}+\xi[1-\frac{1}{z}]+\alpha$,
$\varphi_3(s,z) = s+\lambda b[1-C(z)]+\xi[1-\frac{1}{z}]$,
$\varphi_4(s,z) = s+\lambda b[1-C(z)]+\mu_5+\xi[1-\frac{1}{z}]$,
$\varphi_5(s,z) = s+\lambda b[1-C(z)]+\gamma+\xi[1-\frac{1}{z}]$.

By solving the above equations, we get,

$$\overline{P}_1^{(1)}(0,s,z) = \frac{\left\{\begin{array}{l} (1-(s+\lambda+\pi_1+\pi_2)\overline{\zeta}_4(s,z)\overline{I}_0(s)) \\ +\lambda bC(z)[1-\overline{\zeta}_3(s,z)]\overline{I}_0(s) \end{array}\right\}}{\{\ z[1-\overline{\zeta}_3(s,z)]-\overline{\zeta}_2(s,z)\overline{\zeta}_4(s,z)\quad\}}$$

$$\overline{I}(0,s,z) = \frac{\{\ z(1-(s+\lambda+\pi_1+\pi_2)\overline{I}_0(s))+\lambda bC(z)\overline{\zeta}_2(s,z)\overline{I}_0(s)\quad\}}{\{\ z[1-\overline{\zeta}_3(s,z)]-\overline{\zeta}_2(s,z)\overline{\zeta}_4(s,z)\quad\}}$$

where,

$$
\begin{aligned}
\overline{\zeta}_1(s,z) &= \frac{\gamma[\overline{\lambda}\varphi_4(s,z)+\alpha\mu_5]}{\varphi_5(s,z)}\\
\overline{\zeta}_2(s,z) &= [1-\theta+\theta\overline{V}(\varphi_3(s,z))](1-p)\overline{B}_1(\varphi_1(s,z))+[\frac{1-\overline{B}_1(\varphi_1(s,z))}{\varphi_1(s,z)}]\\
&\quad\times p\overline{B}_2(\varphi_1(s,z))[1+\overline{\zeta}_1(s,z)]+[\frac{1-\overline{B}_1(\varphi_1(s,z))}{\varphi_1(s,z)}][\lambda p_1 bC(z)+\overline{\zeta}_1(s,z)]\\
\overline{\zeta}_3(s,z) &= [\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\ \{\ \frac{\pi_1}{z}([1-\theta+\theta\overline{V}(\varphi_3(s,z))]\overline{B}_2(\varphi_1(s,z))+\overline{\zeta}_1(s,z)\\
&\quad\times[\frac{1-\overline{B}_3(\varphi_2(s,z))}{\varphi_2(s,z)}])+\pi_2([1-\theta+\theta\overline{V}(\varphi_3(s,z))]\overline{B}_4(\varphi_2(s,z))\\
&\quad+\overline{\zeta}_1(s,z)[\frac{1-\overline{B}_4(\varphi_2(s,z))}{\varphi_2(s,z)}])\ \},\\
\overline{\zeta}_4(s,z) &= \overline{I}(\varphi(a,s))+[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\lambda C(z)\ \{\ b+p_2\{\frac{\pi_1}{z}[\frac{1-\overline{B}_3(\varphi_2(s,z))}{\varphi_2(s,z)}]\\
&\quad+\pi_2[\frac{1-\overline{B}_4(\varphi_2(s,z))}{\varphi_2(s,z)}]\}\ \},,
\end{aligned}
$$

Theorem 2.1. *The inequality*

$$P^{(1)}(1)+P^{(2)}(1)+Q(1)=\rho<1,$$

is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state and orbit size distributions are given by,

$$
\begin{aligned}
\overline{I}(s,z) &= \overline{I}(0,s,z)[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}],\\
\overline{P}_i^{(1)}(s,z) &= \overline{P}_i^{(1)}(0,s,z)[\frac{1-\overline{B_i}(\varphi_1(s,z))}{\varphi_1(s,z)}]\ i=1,2,\\
\overline{P}_1^{(2)}(s,z) &= \overline{P}_1^{(2)}(0,s,z)[\frac{1-\overline{B_3}(\varphi_2(s,z))}{\varphi_2(s,z)}],\\
\overline{P}_2^{(2)}(s,z) &= \overline{P}_2^{(2)}(0,s,z)[\frac{1-\overline{B_4}(\varphi_2(s,z))}{\varphi_2(s,z)}],
\end{aligned}
$$

$$\begin{aligned}
\overline{V}(s,z) &= \theta\big\{(1-p)\overline{P}_1^{(1)}(0,s,z)\overline{B_1}(\varphi_1(s,z)) + \overline{P}_2^{(1)}(0,s,z)\overline{B_2}(\varphi_1(s,z)) \\
&\quad + \overline{P}_1^{(2)}(0,s,z)\overline{B_3}(\varphi_2(s,z)) + \overline{P}_2^{(2)}(0,s,z)\overline{B_4}(\varphi_2(s,z))\big\} \\
&\qquad \times [\frac{1-\overline{V}(\varphi_3(s,z))}{\varphi_3(s,z)}], \\
\overline{Q}_i^{(1)}(s,z) &= \frac{\alpha\overline{P}_i^{(1)}(0,s,z)[1-\overline{B}_i(\varphi_1(s,z))]e^{-\varphi_1(s,z)x}}{\varphi_4(s,z)},\ i=1,2 \\
\overline{Q}_1^{(2)}(s,z) &= \frac{\alpha\overline{P}_1^{(2)}(0,s,z)[1-\overline{B}_3(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}}{\varphi_4(s,z)}, \\
\overline{Q}_2^{(2)}(s,z) &= \frac{\alpha\overline{P}_2^{(2)}(0,s,z)[1-\overline{B}_4(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}}{\varphi_4(s,z)}, \\
\overline{R}(s,z) &= \frac{\left\{\begin{array}{l} \overline{\lambda}\ \{\ \sum_{i=1}^{2}(\overline{P}_i^{(1)}(x,s,z)+\overline{P}_i^{(2)}(x,s,z))\ \} \\ +\ \mu_5\ \{\ \sum_{i=1}^{2}(\overline{Q}_i^{(1)}(s,z)+\overline{Q}_i^{(2)}(s,z))\ \} \end{array}\right\}}{\varphi_5(s,z)}.
\end{aligned}$$

3. Steady state Analysis: Limiting Behaviour

Proper utilization of Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t),$$

helps to obtain the steady- state solutions of this model. The normalizing condition of this model is:

$$P_i^{(1)}(1) + P_i^{(2)}(1) + V(1) + Q_i^{(1)}(1) + Q_i^{(2)}(1) + R(1) + I(1) + I_0 = 1$$

i=1,2. Let, $P_q(z)$ be the probability generating function of the orbit size irrespective of the state of the system. And adding all the steady state equations, we obtain,

$$P_q(z) = P_i^{(1)}(z) + P_i^{(2)}(z) + V(z) + Q_i^1(z) + Q_i^2(z) + R(z) + I(z),\ i=1,2.$$

$$P_q(z) = \frac{\psi_1(z)+\psi_2(z)}{\varphi_3(z)},$$

where,

$$\psi_1(z) = I(0,z)[\frac{1-\overline{I}(\varphi(a))}{\varphi(a)}]\ \{\ \varphi_3(z) + (1-\overline{V}(\varphi_3(z)))[\frac{\pi_1}{z} + \pi_2] + \varphi_3(z)\zeta_5(z)$$
$$\times \{\frac{\pi_1}{z}[\frac{1-\overline{B}_3(\varphi_2(z))}{\varphi_2(z)}] + \pi_2[\frac{1-\overline{B}_4(\varphi_2(z))}{\varphi_2(z)}]\}\ \},$$

$$\psi_1(z) = P_1^{(1)}(0,z)\ \{\ (1-\overline{V}(\varphi_3(z)))[1-p+p\overline{B}_2(\varphi_1(z))] + \{[\frac{1-\overline{B}_1(\varphi_1(z))}{\varphi_1(z)}]$$
$$+ p\overline{B}_1(\varphi_1(z))[\frac{1-\overline{B}_2(\varphi_1(z))}{\varphi_1(z)}]\}\varphi_5(z)\zeta_5(z)\ \},.$$

The probability of idle time I_0, we use the normalizing condition, as

$$P_q^{(1)}(1) + I_0 = 1.$$

By using this we can have,

$$I_0 = \frac{\varphi_3'(1)}{\varphi_3'(1) + \psi_1'(1) + \psi_3'(1)}.$$

4. The Average Orbit Length and Waiting Time:

The Mean number of calls in the orbit under the steady state condition is,

$$L_q = \frac{d}{dz}P_q(z)|_{z=1}.$$

then,

$$L_q = \frac{\varphi_3'(1)\psi_1''(1) - \varphi_3''(1)\psi_1'(1)}{2(\varphi_3'(1))^2} + \frac{\varphi_3'(1)\psi_2''(1) - \varphi_3''(1)\psi_2'(1)}{2(\varphi_3'(1))^2},$$

By Little's law, the average waiting time of a call in the orbit is,

$$W_q = \frac{L_q}{\lambda},$$

5. Numerical Results

For the purpose of a numerical study, we assume that all distribution functions like retrial, service for priority and ordinary customers, vacation are exponentially distributed. All the parameter values are selected to satisfy the stability condition.

Table 1: $(\lambda, \mu_2, \mu_3, \mu_4, \mu_5, \eta, \theta, \alpha, \overline{\lambda}, \gamma, \beta, \xi, p, b, \pi_1, \pi_2, p_1, p_2) =$

$(0.4, 8, 5, 5, 3, 0.1, 0.1, 10, 10, 5, 0.1, 0.2, 0.5, 0.6, 0.1, 0.1, 0.3, 0.3)$

μ_1	I_0	ρ	L_q	W_q
5.1	0.0415	0.9585	0.7010	1.7526
5.2	0.0415	0.9585	0.7009	1.7523
5.3	0.0416	0.9584	0.7008	1.7519
5.4	0.0416	0.9584	0.7006	1.7516
5.5	0.0416	0.9584	0.7005	1.7513
5.6	0.0416	0.9584	0.7004	1.7509
5.7	0.0416	0.9584	0.7003	1.7506
5.8	0.0417	0.9583	0.7001	1.7503
5.9	0.0417	0.9583	0.7000	1.7500
6.0	0.0417	0.9583	0.6999	1.7497

Table 2: Let $(\lambda, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \theta, \alpha, \overline{\lambda}, \gamma, \beta, \xi, p, b, \pi_1, \pi_2, p_1, p_2) =$

$(0.8, 15, 5, 5, 5, 0.4, 0.1, 0.5, 0.5, 5, 0.5, 0.1, 0.4, 0.8, 1, 3, 0.5, 0.3)$

η	I_0	ρ	L_q	W_q
8.2	0.0466	0.9534	1.3780	1.7225
8.4	0.0465	0.9535	1.2445	1.5556
8.6	0.0463	0.9537	1.1181	1.3976
8.8	0.0462	0.9538	0.9981	1.2477
9.0	0.0461	0.9539	0.8843	1.1053
9.2	0.0459	0.9541	0.7760	0.9700
9.4	0.0458	0.9542	0.6730	0.8413
9.6	0.0457	0.9543	0.5749	0.7187
9.8	0.0456	0.9544	0.4814	0.6017
10.0	0.0455	0.9545	0.3921	0.4902

Table 3: Let $(\lambda, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \eta, \theta, \alpha, \gamma, \beta, \xi, p, b, \pi_1, \pi_2, p_1, p_2) =$

$(0.4, 10, 8, 5, 5, 3, 0.1, 0.1, 10, 5, 0.1, 0.2, 0.5, 0.6, 0.1, 0.1, 0.3, 0.3)$

$\overline{\lambda}$	I_0	ρ	L_q	W_q
3.5	0.0432	0.9568	0.6095	1.5238
4.0	0.0431	0.9569	0.6107	1.5268
4.5	0.0430	0.9570	0.6133	1.5331
5.0	0.0429	0.9571	0.6170	1.5425
5.5	0.0429	0.9571	0.6218	1.5546
6.0	0.0428	0.9572	0.6276	1.5690
6.5	0.0427	0.9573	0.6342	1.5855
7.0	0.0427	0.9573	0.6416	1.6039
7.8	0.0426	0.9574	0.6495	1.6238
8.0	0.0426	0.9574	0.6581	1.6451

Table 4: Let $(\lambda, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \eta, \theta, \overline{\lambda}, \gamma, \beta, \xi, p, b, \pi_1, \pi_2, p_1, p_2) =$

$(0.4, 10, 8, 5, 5, 3, 0.1, 0.75, 10, 5, 0.1, 0.2, 0.5, 0.6, 0.1, 0.1, 0.3, 0.3)$

α	I_0	ρ	L_q	W_q
1.5	0.0434	0.9566	0.4215	1.0539
2.0	0.0433	0.9567	0.4272	1.0679
2.5	0.0432	0.9568	0.4352	1.0879
3.0	0.0431	0.9569	0.4452	1.1130
3.5	0.0430	0.9570	0.4570	1.1426
4.0	0.0429	0.9571	0.4704	1.1761
4.5	0.0429	0.9571	0.4851	1.2128
5.0	0.0428	0.9572	0.5010	1.2524
5.5	0.0427	0.9573	0.5178	1.2945
6.0	0.0427	0.9573	0.5354	1.3386

Table 1 reveals that as long as the arriving call service rate increases the server's idle time increases, and the average orbit size for incoming call and the utilisation factor decreases. Table 2 shows that as long as the retrial rate increases the server's idle time and orbit size decreases and the utilisation factor increases. Table 3 and 4 explains that as long as the breakdown rate and negative arrival rate increases the server's idle time decreases and the utilisation factor, average orbit size are increases.

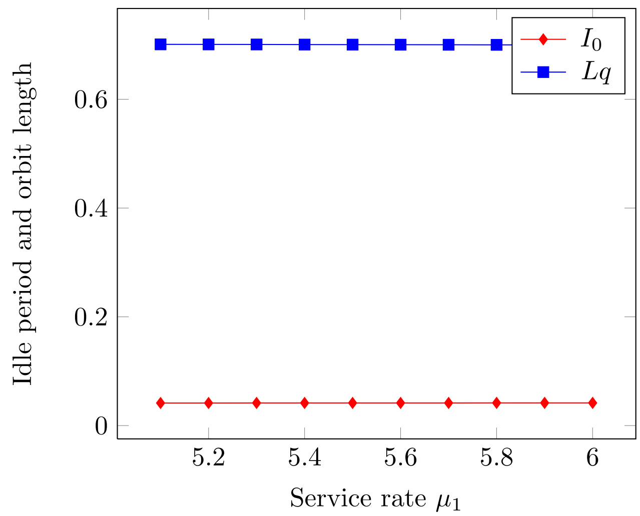

FIGURE 1. Average orbit size Vs Service rate μ_1

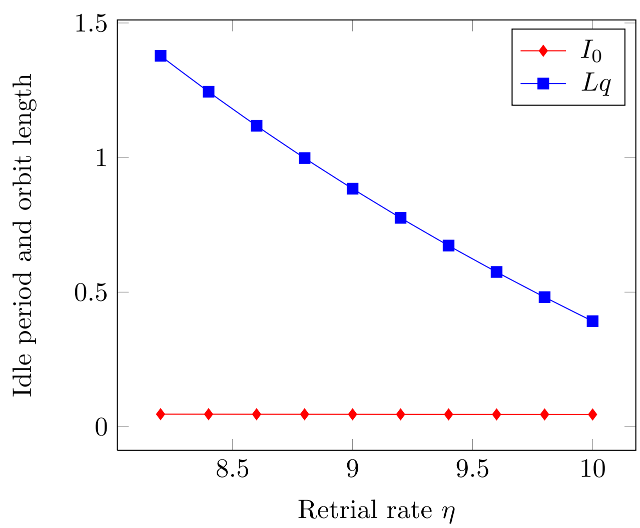

FIGURE 2. Average orbit size Vs Retrial rate η

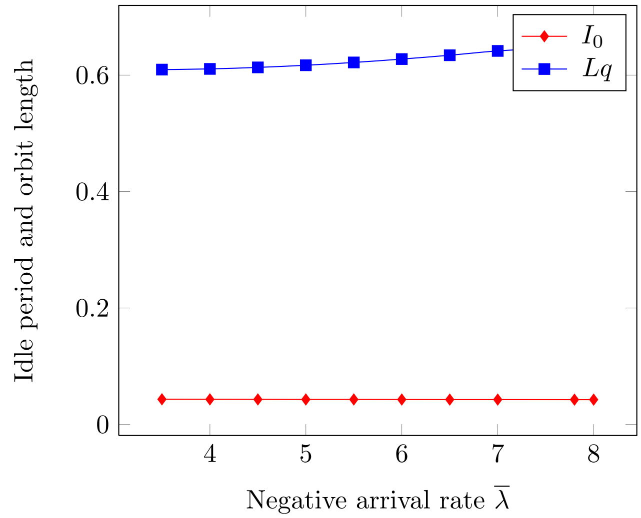

FIGURE 3. Average orbit size Vs Negative arrival rate $\overline{\lambda}$

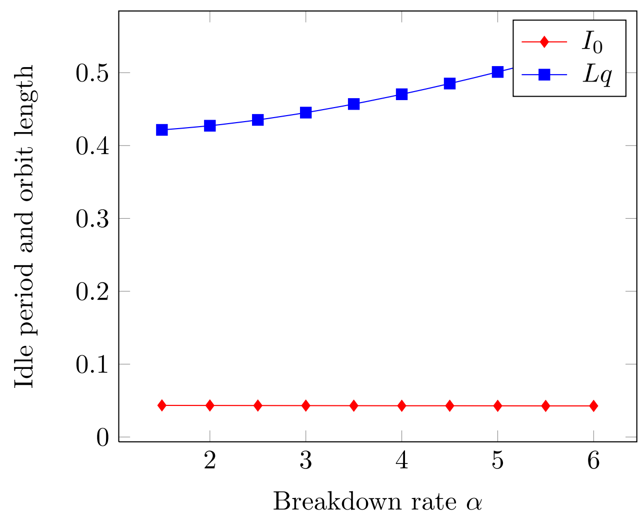

FIGURE 4. Average orbit size Vs Breakdown rate α

6. Conclusion

In this paper we analyze $M^X/G/1/1$ retrial queue with two way communication in which the server makes outgoing calls of two types i.e., outgoing retrial call towards the orbit and outgoing retrial call outside the orbit. The server may not know the number of calls are present in the orbit. As a result, the server can have the chance to make an outgoing call even when there are some more incoming calls present in the orbit. This shows that the arriving calls will nave some priority over the outgoing calls. Also we have analysed the mixed priority retrial queueing system under Bernoulli vacation subject to repair, collision, working breakdown, immediate feedback and negative arrival. In addition to this, the effect of impatient behaviour of the customer on a service system is investigated. Numerical examples have been carried out to observe the mean number of calls in the orbit for varying parametric values. This paper analyzes a single-server retrial queue with constant retrial policy. The novelty of this investigation is the discussion of the constant retrial policy, mixed priority service, two way communication with retrial queueing system.

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