Journal of Mathematical Control Science and Applications (JMCSA)
Vol. 3 No. 1 (January-June, 2017), ISSN : 0974-0570

On the Dynamic Behavior for a Kind of Fifth-order Nonlinear Difference Equation

Dongsheng Li*, Pingping Li** & Mei Sun***

Abstract: In this paper we consider the fifth-order rational difference equation

$$x_{n+1} = \frac{F(x_n, x_{n-2}, x_{n-3}, x_{n-4})}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \cdots,$$

where $F(x, y, z, w) = xy + xz + xw + yz + yw + zw + xyzw + 1 + a$, $G(x, y, z, w) = x + y + z + w + \mathrm{xyz} + xyw + yzw + a$, $a \in [0, \infty)$ and the initial values $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, +\infty)$.

It is found that, with change of the initial values, the rule for the lengths of positive and negative semi-cycles for nontrivial solutions of this equation to successively occur is

$$\cdots, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-,$$
$$4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, \cdots.$$

By the use of the rule, we proved that the positive equilibrium point of the equation is globally asymptotically stable.

MR(2000) Subject Classification: 39A10

Keywords: Rational difference equation; Trajectory structure rule; Global asymptotic stability; Semi-cycle.

1. INTRODUCTION AND PRELIMINARIES

Motivated by the work [1,2,3], we consider in this paper the following fifth-order rational difference equation

$$x_{n+1} = \frac{F(x_n, x_{n-2}, x_{n-3}, x_{n-4})}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \cdots, \tag{1}$$

where the functions

$$F(x, y, z, w) = xy + xz + xw + yz + yw + zw + xyzw + 1 + a$$

and

$$G(x, y, z, w) = x + y + z + w + xyz + xyw + xzw + yzw + a$$

the parameters $a \in [0, +\infty)$, and the initial values $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, +\infty)$.

* Institute of Agricultural Engineering, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R.China

** School of Economics and Management, Nanhua University, Hengyang, Hunan 421001, P.R.China

***Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R. China, *E-mail: lds1010@sina.com*

It is easy to see that the unique positive equilibrium $\bar{x}$ of Eq. (1) is $\bar{x} = 1$.

Here, for readers' convenience, we give some corresponding definitions.

Definition 1.1: A solution $\{x_n\}_{n=-4}^{\infty}$ of Eq. (1) is said to be eventually trivial if x_n is eventually equal to $\bar{x} = 1$; otherwise, the solution is said to be nontrivial.

Definition 1.2: A solution $\{x_n\}_{n=-4}^{\infty}$ of Eq. (1) is said to be eventually positive if x_n is eventually greater than $\bar{x} = 1$.

Definition 1.3: A positive semi-cycle of a solution $\{x_n\}_{n=-4}^{\infty}$ of Eq. (1) consists of a string of terms $\{x_l, x_{l+1}, \ldots x_m\}$, all greater than or equal to the equilibrium $\bar{x}$ with $l \geq -4$ and $m \leq \infty$ such that either $l = -4$ or $l > -4$ and $x_{l-1} < \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}$.

Definition 1.4: A negative semi-circle of a solution $\{x_n\}_{n=-4}^{\infty}$ of Eq. (1) consists of astring of terms $\{x_l, x_{l+1}, \ldots x_m\}$, all less than the equilibrium $\bar{x}$, with $l \geq -4$ and $m \leq \infty$ such that either $l = -4$ or $l > -4$ and $x_{l-1} \geq \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \bar{x}$. The length of a semi-cycle is the number of the total terms contained in it.

For the other concepts in this paper, see Refs. [4, 5].

2. NONTRIVIAL SOLUTION

Theorem 2.1: A positive solution $\{x_n\}_{n=-4}^{\infty}$ of Eq. (1) is eventually trivial if and only if

$$(x_{-4}-1)(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1) = 0. \tag{2}$$

Proof: Sufficiency. Assume that Eq. (2) holds. Then according to Eq. (1), we know that the following conclusions are true:

(i) If $x_{-4} = 1$, then $x_n = 1$ for $n \geq 1$.

(ii) If $x_{-3} = 1$, then $x_n = 1$ for $n \geq 1$.

(iii) If $x_{-2} = 1$, then $x_n = 1$ for $n \geq 1$.

(iv) If $x_{-1} = 1$, then $x_n = 1$ for $n \geq 1$.

(v) If $x_0 = 1$, then $x_n = 1$ for $n \geq 1$.

Necessity. Conversely, assume that

$$(x_{-4}-1)(x_{-3}-1)(x_{-2}-1)(x_{-1}-1)(x_0-1) \neq 0. \tag{3}$$

Then we can show $x_n \neq 0$ for any $n \geq 1$. For the sake of contradiction, assume that for some $N \geq 1$,

$$x_N = 1 \text{ and that } x_n \neq 1 \text{ for any } -4 \leq n \leq N-1. \tag{4}$$

Clearly,

$$1 = x_N = \frac{F(x_{N-1}, x_{N-3}, x_{N-4}, x_{N-5})}{G(x_{N-1}, x_{N-3}, x_{N-4}, x_{N-5})}.$$

From this we can know that

$$0 = x_N - 1 = \frac{(x_{N-1}-1)(x_{N-3}-1)(x_{N-4}-1)(x_{N-5}-1)}{G(x_{N-1}, x_{N-3}, x_{N-4}, x_{N-5})},$$

which implies $x_{N-1} = 1$, or $x_{N-3} = 1$, or $x_{N-4} = 1$, or $x_{N-5} = 1$. This contradicts with Eq. (4).

Remark 2.2: Theorem (2.1) actually demonstrates that a positive solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(1) is eventually nontrivial if $(x_{-4} - 1)(x_{-3} - 1)(x_{-2} - 1)(x_{-1} - 1)(x_0 - 1) \neq 0$. So, if a solution is a nontrivial one, then $x_n \neq 1$ for any $n \leq -4$.

3. OSCILLATION AND NON-OSCILLATION

Before stating the oscillation and non-oscillation of solutions, we need the following key lemmas.

Lemma 3.1: Let $\{x_n\}_{n=-4}^{\infty}$ be a positive solution of Eq.(1) which is not eventually equal to 1, then the following conclusions are valid:

(a) $(x_{n+1} - 1)\ (x_n - 1)\ (x_{n-2} - 1)\ (x_{n-3} - 1)\ (x_{n-4} - 1) > 0$, for $n \geq 0$;

(b) $(x_{n+1} - x_n)(x_n - 1) < 0$, for $n \geq 0$;

(c) $(x_{n+1} - x_{n-1})(x_{n-1} - 1) < 0$, for $n \geq 0$;

(d) $(x_{n+1} - x_{n-2})(x_{n-2} - 1) < 0$, for $n \geq 0$;

(e) $(x_{n+1} - x_{n-3})(x_{n-3} - 1) < 0$, for $n \geq 0$;

(f) $(x_{n+1} - x_{n-4})(x_{n-4} - 1) < 0$, for $n \geq 0$.

(g) $(x_{n+1} - x_{n-5})(x_{n-5} - 1) < 0$, for $n \geq 0$

Proof: First, we investigate (a). According to Eq.(1), we have that

$$x_{n+1} - 1 = \frac{(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1)(x_{n-4} - 1)}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})}, n = 0, 1, \cdots.$$

So

$$(x_{n+1} - 1)(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1)(x_{n-4} - 1) > 0.$$

Second, it comes (b). From Eq.(1) we obtain

$$x_{n+1} - x_n = \frac{(1 - x_n)(a + (1 + x_n)(1 + x_{n-2}x_{n-3} + x_{n-2}x_{n-4} + x_{n-3}x_{n-4}))}{G(x_n, x_{n-2}, x_{n-3}, x_{n-4})},$$

This teaches us that $(x_{n+1} - x_n)(1 - x_n) > 0$, $n = 0, 1, \ldots$. That's to say, $(x_{n+1} - x_n)(x_n - 1) < 0.n = 0, 1, \ldots$. So, the proof of (b) is complete. The proofs for (c), (d), (e), (f), (g) are similar to that of (b). The proof for Lemma (3.1) is complete.

Theorem 3.2: There exist non-oscillatory solutions of Eq.(1) with $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (1, +\infty)$, which must be eventually positive. There don't exist eventually negative non-oscillatory solutions of Eq. (1).

Proof: Consider a solution of Eq.(1) with

$$x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \in (1,+\infty).$$

We then know from Lemma (3.1)(a) that $x_n > 1$ for $x_n \in [-4,+\infty)$. So, this solution is just a non-oscillatory solution and furthermore eventually positive.

Suppose that there exist eventually negative non-oscillatory of Eq.(1). Then, there exists a positive integer N such that $x_n < 1$ for $n \geq N$. Thereout, for $n \geq N + 4$,

$$(x_{n+1} - 1)(x_n - 1)(x_{n-2} - 1)(x_{n-3} - 1)(x_{n-4} - 1) \leqslant 0.$$

This contradicts Lemma (3.1)(a). So, there don't exist eventually negative non-oscillatory of Eq.(1), as desired.

4. RULE OF CYCLE LENGTH

Theorem 4.1: Let $\{x_n\}_{-4}^{\infty}$ be a strictly oscillatory of Eq.(1), then the rule for the lengths of positive and negative semi-cycles of this solution to occur successively is … , $4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-$, … .

Proof: By Lemma (3.1) (a), one can see that the length of a negative semi-cycle is at most 5, and a positive semi-cycle is at most 4. On the basis of the strictly oscillatory character of the solution, we see that, for some integer $p \geq 0$, one of the following 2 cases must occur:

case 1: $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} > 1$, and $x_{p+4} > 1$;

case 2: $x_p > 1$, $x_{p+1} > 1$, $x_{p+2} > 1$, $x_{p+3} > 1$, and $x_{p+4} < 1$;

Case 1 can't occur. Otherwise, the solution is a non-oscillatory solution of Eq.(1).

If Case 2 occurs, it follows from Lemma (3.1)(a) that $x_{p+5} < 1$, $x_{p+6} < 1$, $x_{p+7} > 1$, $x_{p+8} > 1$, $x_{p+9} < 1$, $x_{p+10} < 1$, $x_{p+11} > 1$, $x_{p+12} < 1$, $x_{p+13} < 1$, $x_{p+14} < 1$, $x_{p+15} < 1$, $x_{p+16} < 1$, $x_{p+17} > 1$, $x_{p+18} < 1$, $x_{p+19} > 1$, $x_{p+20} > 1$, $x_{p+21} > 1$, $x_{p+22} < 1$, $x_{p+23} > 1$, $x_{p+24} > 1$, $x_{p+25} < 1$, $x_{p+26} > 1$, $x_{p+27} < 1$, $x_{p+28} > 1$, $x_{p+29} < 1$, $x_{p+30} < 1$, $x_{p+31} > 1$, $x_{p+32} > 1$, $x_{p+33} > 1$, $x_{p+34} > 1$, $x_{p+35} < 1$, $x_{p+36} < 1$, $x_{p+37} < 1$, $x_{p+38} > 1$, $x_{p+39} > 1$, $x_{p+40} < 1$, $x_{p+41} < 1$, $x_{p+42} > 1$, $x_{p+43} < 1$, $x_{p+44} < 1$, $x_{p+45} < 1$, $x_{p+46} < 1$, $x_{p+47} < 1$, $x_{p+48} > 1$, $x_{p+49} < 1$, $x_{p+50} > 1$, $x_{p+51} > 1$, $x_{p+52} > 1$, $x_{p+53} < 1$, $x_{p+54} > 1$, $x_{p+55} > 1$, $x_{p+56} < 1$, $x_{p+57} > 1$, $x_{p+58} < 1$, $x_{p+59} > 1$, $x_{p+60} < 1$, $x_{p+61} < 1$, … .

This means that rule for the lengths of positive and negative semi-cycles of the solution of Eq.(1) to occur successively is … , $4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-$, … . So, the proof for this theorem is complete.

5. GLOBAL ASYMPTOTIC STABILITY

First, we consider the local asymptotic stability for unique positive equilibrium point $\overline{x}$ of Eq.(1). We have the following results.

Theorem 5.1: The positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof: The linearized equation of Eq.(1) about the positive equilibrium point $\overline{x}$ is

$$y_{n+1} = 0 \cdot y_n + 0 \cdot y_{n-2} + 0 \cdot y_{n-3} + 0 \cdot y_{n-4},\ n = 0, 1, \dots,$$

and so it is clear from the paper [5, Remark 1.3.7] that the positive equilibrium point $\overline{x}$ of Eq.(1) is locally asymptotically stable. The proof is complete.

We are now in a position to study the global asymptotically stability of positive equilibrium point $\overline{x}$.

Theorem 5.2: The positive equilibrium point of Eq. (1) is globally asymptotically stable.

Proof: We must prove that the positive equilibrium point $\overline{x}$ of Eq.(1) is both locally asymptotically stable and globally attractive. Theorem (5.1) has shown the local asymptotic stability of $\overline{x}$. Hence it remains to verify that every positive solution $\{x_n\}_{n=-4}^{\infty}$ of Eq.(1) converges to $\overline{x}$ as $n \longrightarrow \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \overline{x} = 1. \tag{5}$$

We can divide the solutions into two kinds of types.

i) Trivial solutions;

ii) Nontrivial solutions.

If the the solutions is a trivial solutions, then it is obvious for (5) to hold because $x_n = 1$ is eventually.

If the the solutions is a nontrivial solutions, then we can further divide the solution into two cases.

a) Non-oscillatory solution;

b) Oscillatory solution.

Consider now $\{x_n\}$ to be non-oscillatory about the positive equilibrium point $\overline{x}$ of Eq.(1). By virtue of Lemma (3.1)(b), it follows that the solution is monotonic and bounded. So $\lim_{n\to\infty} x_n$ exists and is finite. Taking limits on both sides of Eq.(1), one can easily see that (5) holds.

Now let $\{x_n\}$ be strictly oscillatory about the positive equilibrium point of Eq.(1). By virtue of Theorem (4.1), one understands that the rule for the lengths of positive and negative semi-cycles occurring successively is Case (C)$\cdots$, 4^+, 3^-, 2^+, 2^-, 1^+, 5^-, 1^+, 1^-, 3^+, 1^-, 2^+, 1^-, 1^+, 1^-, 1^+, 2^-, $\ldots$. For simplicity, for some nonnegative integer p, we denote by $\{x_p, x_{p+1}, x_{p+2}, x_{p+3}\}^+$ the terms of a positive semicycle of length four, followed by $\{x_{p+4}, x_{p+5}, x_{p+6}\}^-$, a negative semicycle with semicycle length three, then a positive semicycle of length two and a negative semicycle of length two, and so on. Namely, the rule for the lengths of positive and negative semicycles to occur successively can be periodically expressed as follows:

$$\begin{aligned}
&\{x_{p+31n}, x_{p+31n+1}, x_{p+31n+2}, x_{p+31n+3}\}^+, \{x_{p+31n+4}, x_{p+31n+5}, x_{p+31n+6}\}^-,\\
&\{x_{p+31n+7}, x_{p+31n+8}\}^+, \{x_{p+31n+9}, x_{p+31n+10}\}^-, \{x_{p+31n+11}\}^+,\\
&\{x_{p+31n+12}, x_{p+31n+13}, x_{p+31n+14}, x_{p+31n+15}, x_{p+31n+16}\}^-, \{x_{p+31n+17}\}^+,\\
&\{x_{p+31n+18}\}^-, \{x_{p+31n+19}, x_{p+31n+20}, x_{p+31n+21}\}^+, \{x_{p+31n+22}\}^-,\\
&\{x_{p+31n+23}, x_{p+31n+24}\}^+, \{x_{p+31n+25}\}^-, \{x_{p+31n+26}\}^+, \{x_{p+31n+27}\}^-,\\
&\{x_{p+31n+28}\}^+, \{x_{p+31n+29}, x_{p+31n+30}\}^-, n = 0, 1, 2, \ldots .
\end{aligned}$$

Lemma (3.1)(b), (c), (d), (e), (f), (g) teaches us that the following results are true:

(A)

$$\begin{aligned}
x_{p+31n} &> x_{p+31n+1} > x_{p+31n+2} > x_{p+31n+3} > x_{p+31n+7}\\
&> x_{p+31n+8} > x_{p+31n+11} > x_{p+31n+17} > x_{p+31n+19}\\
&> x_{p+31n+20} > x_{p+31n+21} > x_{p+31n+23} > x_{p+31n+24}\\
&> x_{p+31n+26} > x_{p+31n+28} > x_{p+31(n+1),} n = 0, 1, 2, \cdots .
\end{aligned}$$

(B)

$$
\begin{aligned}
& x_{p+31n+4} < x_{p+31n+5} < x_{p+31n+6} < x_{p+31n+9} < x_{p+31n+10} \\
& < x_{p+31n+12} < x_{p+31n+13} < x_{p+31n+14} < x_{p+31n+15} \\
& < x_{p+31n+16} < x_{p+31n+18} < x_{p+31n+22} < x_{p+31n+25} \\
& < x_{p+31n+27} < x_{p+31n+29} < x_{p+31n+30} < x_{p+31(n+1)+4,} \\
& n = 0, 1, 2, \cdots .
\end{aligned}
$$

So, from (A) one can see that $\{x_{p+31n}\}_{n=0}^{\infty}$ is decreasing with lower bound 1. So, the limit $S = \lim_{n\to\infty} x_{p+31n}$ exist and is finite.

Furthermore, From (A) one can further obtain

$$
\begin{aligned}
S &= \lim_{n\to\infty} x_{p+31n+1} = \lim_{n\to\infty} x_{p+31n+2} = \lim_{n\to\infty} x_{p+31n+3} \\
&= \lim_{n\to\infty} x_{p+31n+7} = \lim_{n\to\infty} x_{p+31n+8} = \lim_{n\to\infty} x_{p+31n+11} \\
&= \lim_{n\to\infty} x_{p+31n+17} = \lim_{n\to\infty} x_{p+31n+19} = \lim_{n\to\infty} x_{p+31n+20} \\
&= \lim_{n\to\infty} x_{p+31n+21} = \lim_{n\to\infty} x_{p+31n+23} = \lim_{n\to\infty} x_{p+31n+24} \\
&= \lim_{n\to\infty} x_{p+31n+26} = \lim_{n\to\infty} x_{p+31n+28}
\end{aligned}
$$

Similarly, by (B) one can see that $\{x_{p+31n+4}\}_{n=0}^{\infty}$ is increasing with upper bound 1. So, the limit $T = \lim_{n\to\infty} x_{p+31n+4}$ exist and is finite.

Furthermore, from (B) one can further obtain

$$
\begin{aligned}
T &= \lim_{n\to\infty} x_{p+31n+4} = \lim_{n\to\infty} x_{p+31n+5} = \lim_{n\to\infty} x_{p+31n+6} \\
&= \lim_{n\to\infty} x_{p+31n+7} = \lim_{n\to\infty} x_{p+31n+9} = \lim_{n\to\infty} x_{p+31n+10} \\
&= \lim_{n\to\infty} x_{p+31n+12} = \lim_{n\to\infty} x_{p+31n+13} = \lim_{n\to\infty} x_{p+31n+14} \\
&= \lim_{n\to\infty} x_{p+31n+15} = \lim_{n\to\infty} x_{p+31n+16} = \lim_{n\to\infty} x_{p+31n+18} \\
&= \lim_{n\to\infty} x_{p+31n+22} = \lim_{n\to\infty} x_{p+31n+25} = \lim_{n\to\infty} x_{p+31n+27} \\
&= \lim_{n\to\infty} x_{p+31n+29} = \lim_{n\to\infty} x_{p+31n+30}.
\end{aligned}
$$

Easily, we can prove $S = T = 1$. The proof for Theorem (5.2) is complete.

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