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# A New Quasi-Filled Function Method for Constrained Global Optimization

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**Abstract**: In this paper, a class of new auxiliary function with one parameter on constrained problem for escaping the current local minimizer of global optimization problem is proposed. Firstly, we give a definition of the quasi-filled function for constrained problem and under some mild assumptions, we prove the proposed auxiliary function is a quasi-filled function. Then, a new algorithm is presented according to the theoretical analysis. We also report the results of a preliminary numerical experience.

**Keywords.** Local minimizer, Global optimization, Quasi-filled function, Nonlinear programming.

## 1. INTRODUCTION

In many real world optimization problems it is essential or desirable to determine the global minimum of the objective function. In the past thirty years methods that aim to find the global minimum of a function have attracted a lot of interest. Numerous algorithms have been studied and can be found in the surveys by Horst and Pardalos [14] and Levy Montalvo [10], Ge [5] as well as in the introductory textbook by Horst et al.[7]. These global optimization methods can be classified into two groups: stochastic(see.e.g.,[3], [9], [11], [12], [16]) and deterministic methods (see, e.g., [1], [5], [7], [8], [15]). Stochastic methods employ random factors, for example points are chosen randomly. Thus every run of such an algorithm could yield a different result. However, results obtained by a deterministic method are always reproducible. In this paper, we focus on filled function method, which was proposed by Ge in [5].

In [5], the following global optimization problem is considered

$$\min_{x \in X} f(x) \tag{1}$$

where $X \subset R^n$ is a closed bounded box containing all global minimizers of $f(x)$ in its interior. It is assumed in this paper that $f(x)$ has only a finite number of local minimizers on $X$.

The basin of $f(x)$ at an isolated local minimizer of $f(x)$, $x^*$, is defined in [4]. The concept of the filled functions was introduced in [5]. Suppose $x^*$ is a current local minimizer of $f(x)$ . $P(x, x^*)$ is said to be a filled function of $f(x)$ at $x^*$, if it satisfies the following properties:

(P1) $x^*$ is a strictly maximizer of $P(x, x^*)$ and the whole basin $B^*$ of $f(x)$ at $x^*$ becomes a part of a hill of $P(x, x^*)$:

(P2) $P(x, x^*)$ has no minimizers or saddle points in any basin of $f(x)$ higher than $B^*$;

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(P3) If $f(x)$ has a lower basin than $B^*$, then there is a point $x'$ in such a basin that minimizer $P(x, x^*)$ on the line through $x$ and $x_1^*$.

Ge specifically proposed the following two-parameter filled function in [5]:

$$P(x, x^*, r, \rho) = \frac{1}{r + f(x)} \exp(-\frac{\| x - x^* \|^2}{\rho^2}), \tag{2}$$

where the parameters $r$ and ρ need to be chosen appropriately.

In recent years, rapid progress has been made both in theory and in practical applications for filled function methods (see, e.g., [2], [6], [13], [17], [18], [19]). What'more, all the open literatures devote to unconstrained global optimization need the following assumption: $f(x): R^n \to R$ is coercive, i.e., $f(x) \to +\infty$ as $\| x \| \to +\infty$. And up to now, to our knowledge, there are fewer literatures for finding a global minimizer on constrained global optimization problem. This consideration motivates the study reported in this paper.

In this paper, we consider the following global optimization:

$$(P) \min_{x \in S} f(x) \tag{3}$$

where $S = \{x \in X \mid g_i(x) \le 0, i = 1, \cdots, m\}$, $X$ is a bounded box set, and the functions $f, g_1, g_2, ..., g_m$ are continuously differentiable in $R^n$. Let $g(x) = (g_1(x) \cdots g_m(x))^T$. Assume that $x^*$ is a local minimizer of $f(x)$, the number of the different value of minimizer of $(P)$ is finite. Now, we give a definition of the quasi-filled function as the follows:

Suppose $x^*$ is a current local minimizer of (P). $P(x, x^*)$ is said to be a quasi-filled function of $f(x)$ at $x^*$, if it satisfies the following properties:

(P1) $\nabla P(x, x^*) \ne 0$, when $x \in S_1$ or $x \in X \setminus S$ and $x \ne x^*$, where $S_1 = \{x \in S \mid f(x) \ge f(x^*)\}$;

(P2) If $x^*$ is a local minimizer of (P) and $S_2 = \{x \mid f(x) < f(x^*), x \in S\}$ is not empty, then there exists a point $x_1^* \in S_2$ such that $x_1^*$ is a local minimizer of $P(x, x^*)$.

Adopting the concept of the quasi-filled functions, a global optimization problem can be solved via a two-phase cycle. In Phase 1, we start from a given point and use any local minimization method to find a local minimizer $x^*$ of $f(x)$. In Phase 2, we construct a quasi-filled function at $x^*$ and minimize it in order to identify a point $x' \in S$ with $f(x') < f(x^*)$. If such an $x'$ is found, $x'$ is certainly in a lower basin than $B^*$. We can then use $x'$ as the initial point in Phase 1 again, and hence we can find a better local minimizer $x^{**}$ of $f(x)$ with $f(x^{**}) < f(x^*)$. This process repeats until the time when minimizing a quasi-filled function does not yield a better solution. The current local minimum will be then taken as a global minimizer of $f(x)$.

This paper is organized as follows. Following this introduction, a class of new quasi-filled function is proposed and the properties of it are investigated. In Section 3, a solution algorithm is suggested and the result of a preliminary numerical experience is given to show the practicability of the proposed approach. Finally, some conclusions are drawn in Section 4.

## 2. A QUASI-FILLED FUNCTION AND ITS PROPERTIES

Consider the one parameter quasi-filled function for problem (3)

$$T(x,x^*,\rho)=\eta(\|x-x_0\|^2)-\varphi(\rho[e^{(min(0,max(f(x)-f(x^*),g_1(x),g_2(x),\dots,g_m(x))))^2}-1])$$

where $x^*$ is a current local minimizer of $f(x)$, $\|\cdot\|$ indicates the Euclidean vector norm.

In order to guarantee the theoretical properties of our quasi-filled function, $\eta(t)$ and $\varphi(t)$ need to satisfy the following conditions:

(1): $\eta(0)=0$, and $\eta\ '(t)>0$, for all $t>0$.

(2): $\varphi(0)=0$, and $\varphi\ '(t)>0$, for all $t>0$.

From (2), we know that function $\varphi(t)$ must have its inverse function $\varphi^{-1}(t)$.

Some examples of $\eta(t)$ and $\varphi(t)$ are $t$, $1-\exp(-t)$.

The following theorems show that $T(x,x^*,\rho)$ satisfies the conditions to be qualified as a quasi-filled function under some conditions on parameter ρ.

**Theorem 2.1.** Given that $x^*$ is a local minimizer of problem $(P)$, then $T(x,x^*,\rho)$ has no stationary points in the union of $\{x\in S: f(x)\geq f(x^*)\}$ and $X\setminus S$.

**Proof.** For any $x\in\{x\in S: f(x)\geq f(x^*)\}$ or $x\in X\setminus S$,

$$T(x,x^*,\rho)=\eta(\|x-x_0\|^2).$$

So

$$\nabla T(x,x^*,\rho)=2(x-x_0)\eta\ '(\|x-x_0\|^2)\neq 0.$$

This completes the theorem.

**Theorem 2.2:** Given that $x^*$ is a local minimizer but not a global minimizer of problem $(P)$. Suppose that $cl(intS)=clS$. If $\rho>0$ is sufficient large, then there exists a point $x^{'}\in S$ such that $x^{'}$ is a local minimizer of $T(x,x^*,\rho)$ and $f(x^{'})<f(x^*)$.

**Proof.** By conditions, there exists another local minimizer of $(P)$, $x_1^*$, such that

$$f(x_1^*)<f(x^*)\quad and\quad x_1^*\in S$$

Due to $cl(intS) = clS$, we conclude that there exists a point $\overline{x_1^*}$ and a neighborhood $O(x_1^*, \sigma)$ of $x_1^*$ with $\sigma > 0$ such that

$$\overline{x_1^*} \in intS \cap O(x_1^*, \sigma),\ f(\overline{x_1^*}) < f(x^*), g_i(\overline{x_1^*}) < 0, i = 1, \cdots, m.$$

Let

$$\rho > \frac{\varphi^{-1}(\eta(\| \overline{x_1^*} - x_0 \|^2)}{e^{(max(f(\overline{x_1^*}) - f(x^*), g_i(\overline{x_1^*})))^2 - 1}},$$

then

$$T(\overline{x_1^*}, x^*, \rho) = \eta(\| \overline{x_1^*} - x_0 \|^2) - \varphi(\rho[e^{(max(f(\overline{x_1^*}) - f(x^*), g_i(\overline{x_1^*})))^2} - 1]) < 0.$$

Let

$$x' = argmin_{x \in S} T(x, x^*, \rho),$$

then

$$T(x', x^*, \rho) \leq T(\overline{x_1^*}, x^*, \rho) < 0.$$

Since when $x \in X \setminus S$ or $x \in \partial S$ or $x \in \{x \in S : f(x) \geq f(x^*)\}$, where $x \in \partial S$ denotes the boundary of set *S*

$$T(x, x^*, \rho) = \eta(\| x - x_0 \|^2) > 0.$$

Therefore, we have $f(x') < f(x^*)$ and $x' \in intS$.

This completes the proof of the theorem.

## 3. SOLUTION ALGORITHM AND NUMERICAL IMPLEMENTATION

The theoretical properties of the proposed quasi-filled function $T(x, x^*, \rho)$ were discussed in the last section. The development in this section is to further study the properties of the proposed quasi-filled function in numerical implementation and to suggest an efficient algorithm. Based on the theoretical results on the previous section, a solution algorithm is proposed as follows:

**Quasi-Filled Function Algorithm**

1. Initialization :

   Given a prefixed point $x_0 \in R^n \setminus X$ in which $\| x - x_0 \| \geq 1$ holds for any $x \in X$, choose a tolerance $\rho_U > 0$ and an initial point $x^0 \in X$, set $k = 1$.

2. Start from an initial point $x^0$, solve the problem of minimizing $f(x)$ over $S$ by using penalty function method and obtain a minimizer $x_k^*$.

3. choose a set of initial points $\{x_{k+1}^{(0)i} : i = 1, 2, ..., r\}$ such that $x_{k+1}^{(0)i} \in X \setminus N(x_k^*, \epsilon_k)$ for some $\epsilon_k > 0$.

4. set $\rho = 1$.
5. set $i = 1$.

6. If $i \le r$, then set $x = x_{k+1}^{(0)i}$ and goto 7; otherwise, goto 9.

7. If $f(x) < f(x^*)$ and $x \in S$, then use $x$ as an initial point to find $x_{k+1}^*$ such that $f(x_{k+1}^*) < f(x_k^*)$ and $x_{k+1}^* \in S$. Then set $k =: k+1$ and goto 3; otherwise, goto 8.

8. Find a new $x$ in the direction $D = -\nabla T(x, x_k^*, \rho)$ such that $T(x, x_k^*, \rho)$ can reduce to a certain extent. If $x$ attains the boundary of $X$ during minimization, then set $i := i+1$ and goto 6; otherwise, goto 7.

9. Set $\rho = 10\rho$. If $\rho \le \rho_U$, then goto 5; otherwise, the algorithm is incapable of finding a better minimizer starting from the initial points, $\{x_{k+1}^{(0)i} : i = 1, 2, \ldots, r\}$. The algorithm stops and $x_k^*$ is taken as a global minimizer.

The idea and mechanism of algorithm are explained as follows:

There are two phrases in the algorithm. One is to minimize the original function $f(x)$ over $S$, the other is to minimize the quasi-filled function $T(x, x^*, \rho)$ over $X$. We let $\rho > 0$ in the initialization, afterwards, it is gradually increased via the two-phase cycle until it is more than sufficiently large positive scale. If the parameters ρ is sufficiently large, and we can't find the point $x \in S$ such that $f(x) < f(x^*)$ yet, then we believe that there doesn't exist a better minimizer of $f(x)$ over $S$, then the algorithm is terminated.

Although the focus of this paper is more theoretical than computational, we still test our algorithm on several global minimization problems to have an initial feeling of the potential practical value of the quasi-filled function algorithm. All the numerical experiments are implemented in Fortran 95, under Windows XP and Pentium (R) 4 CPU 2.80GMHZ. The Fortran 95 function 'nconf' in module 'IMSL' is used in the algorithm to find local minimizers of (P).

The form of $T(x, x^*, \rho)$ is chosen as follows:

$$T(x, x^*, \rho) = \| x - x_0 \|^2 - \rho[e^{(min(0, max(f(x) - f(x^*), g_i(x))))^2} - 1].$$

**Problem 1**

$$f(x) = x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3$$

s.t

$$g_1(x) = (x_1 - 2)^2 + x_2^2 - 1.6^2 \le 0.$$

$$g_2(x) = x_1^2 + (x_2 - 3)^2 - 2.7^2 \le 0.$$

The global minimum solutions: $x^* = (0.7250289, 0.3991602)$, and $f^* = 1.837504$.

**Problem 2**

$$f(x) = (4 - 2.1x_1^2 + \frac{x_1^4}{3})x_1^2 + x_1 x_2 + (-4 + 4x_2^2)x_2^2$$

s.t

$$g(x) = -\sin(4\pi x_1) + 2\sin^2(2\pi x_2) \le 0$$

$$-1 \le x_1, x_2 \le 1$$

The global minimum solutions: $x^* = (0.1090018, -0.62233971)$, and $f^* = -0.9711032$.

**Problem 3**

$$f(x) = -25(x_1 - 2)^2 - (x_2 - 2)^2 - (x_3 - 1)^2 - (x_4 - 4)^2 - (x_5 - 1)^2 - (x_6 - 4)^2$$

s.t

$$g_1(x) = (x_3 - 3)^2 + x_4 \ge 4$$

$$g_2(x) = (x_5 - 3)^2 + x_6 \ge 4$$

$$g_3(x) = x_1 - 3x_2 \le 2$$

$$g_4(x) = -x_1 + x_2 \le 2$$

$$g_5(x) = x_1 + x_2 - 6 \le 0$$

$$g_6(x) = x_1 + x_2 \ge 2$$

$$0 \le x_1, x_4 \le 6$$

$$0 \le x_2 \le 8$$

$$1 \le x_3, x_5 \le 5$$

$$0 \le x_6 \le 10$$

The global minimum solutions: $x^*$ = (5.000000, 0.9999999, 5.000000, 0.0000000, 5.000000, 10.00000), and $f^* = -310.0000$.

The main iterative results are summarized in tables for each function. The symbols used are shown as follows:

$x_k^0$ : The $k$-th initial point

$k$ : The iteration number in finding the $k$-th local minimizer

$x_k^*$ : The $k$-th local minimizer

$f(x_k^*)$ : The function value of the $k$-th local minimizer

**Table 1**
**Problem 1**

| $k$ | $x_k^0$ | $x_k^*$ | $f(x_k^*)$ |
|---|---|---|---|
| 1 | (2,1.5) | (1.835084,1.100764) | 5.612519 |
| 2 | (1.516314,1.182714) | (0.4396141,0.3538329) | 1.982739 |
| 3 | (0.7104309,0.3993877) | (0.7250289,0.3991602) | 1.837504 |

**Table 2**
**Problem 2**

| $k$ | $x_k^0$ | $x_k^*$ | $f(x_k^*)$ |
|---|---|---|---|
| 1 | (0.5,-0.9) | (0.5734327,-0.8912419) | -0.064501 |
| 2 | (0.1714329,-0.8912776) | (0.1248393,-0.8749353) | -0.7654098 |
| 3 | (0.1248175,-0.6229354) | (0.1090018,-0.62233971) | -0.9711032 |

**Table 3**
**Problem 3**

| k | $x_k^0$ | $x_k^*$ | $f(x_k^*)$ |
|---|---|---|---|
| 1 | (3.000000, 3.000000, 3.000000, 3.000000, 3.000000, 3.000000) | (5.000000, 1.000000, 5.000000, 3.997948, 5.000000, 3.997948) | -258.0000 |
| 2 | (5.000000, 1.000000, 5.105109, 3.997957, 5.000009, 3.997957) | (5.000000, 1.000000, 5.000000, 0.000005, 5.000000, 3.997987) | -274.0000 |
| 3 | (5.000000, 1.000000, 5.000000, 0.1104090, 5.000000, 10.11041) | (5.000001, 1.000000, 5.000000, 0.0000000, 5.000000, 10.00000) | -310.0002 |

## 4. CONCLUSIONS

A definition of the quasi-filled functions for constrained nonlinear programming is given in this paper, it is an extension of the idea of the filled functions for unconstrained or box constrained global optimization problems. Based on this definition, a quasi-filled function satisfying this definition is given, an algorithm for constrained global optimization is also developed. The results tested on several examples show the practicability of this quasi-filled function method.

Recently, global optimality conditions have been derived in [20] for quadratic binary integer programming problems. However, A global optimality conditions for continuous variables are still an open problem, in general. The criterion of a global minimizer will provide solid stopping conditions for a continuous quasi-filled function method.

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