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Solvability of the Nonlinear Competition Model with Variable Coefficients and Comparison of the Results with Modified Decomposition Method

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Abstract: The proof of new results concerning the conditions for solvability of the nonlinear competition model of two and three equations with variable coefficients are presented. The results obtained by the present method are compared with that obtained by modified decomposition method.

Keywords: Modified decomposition method, Nonlinear system of differential equations.

1. INTRODUCTION

The competition models constitute a system of nonlinear differential equations [3]

$$\begin{cases} \dfrac{dx_1}{dt} & = & x_1\left(a_1 - b_{11}x_1 - b_{12}x_2 - \ldots - b_{1n}x_n\right), \\ \dfrac{dx_2}{dt} & = & x_2\left(a_2 - b_{21}x_1 - b_{22}x_2 - \ldots - b_{2n}x_n\right), \\ \ldots & & \\ \dfrac{dx_n}{dt} & = & x_n\left(a_n - b_{n1}x_1 - b_{n2}x_2 - \ldots - b_{nn}x_n\right), \\ x_i(0) & = & \alpha_i, \imath = 1, \ldots, n, \end{cases}$$

where a_i and $b_{ij}, i, j = 1, \ldots n$ are constants. One of the main problems of mathematics appears when $a_i(t)$ and $b_{ij}(t), i, j = 1, \ldots n$ are analytic functions and added to the model. It may be worth while if the new model can be

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constructed in such a manner that it can be solved analytically or transformed into another system in which the equations are decoupled and solvable separately.

For the sake of clarity we begin with a system of two coupled differential equations, and we discuss a new result on the conditions of solvability of this system. It is shown how this result may be extended to the case of a system of three coupled equations. The modified decomposition method [1,2] is also employed to solve the discussed problems. These results obtained by the present method are compared with that obtained by modified decomposition method [1,2]. Two examples are given to illustrate the possible practical use of this comparison.

2. SOLVABILITY OF SYSTEM OF TWO COUPLED EQUATIONS

The new differential equations, incorporating the above functions are as follows:

$$\frac{dx}{dt} = x\left(a_1(t) - b_1(t)x - c_1(t)y\right), \tag{1}$$

$$\frac{dy}{dt} = y\left(a_2(t) - b_2(t)x - c_2(t)y\right), \tag{2}$$

$$x(0) = \alpha, y(0) = \beta. \tag{3}$$

The system (1)–(3) can be transformed into another system in which the equations are decoupled and solvable separately. This separability can be obtained if

$$x = \phi(t)y, \tag{4}$$

where $\phi(t)$ is unknown function with

$$\phi(0) = \frac{\alpha}{\beta}.$$

The substitution (4) into (1)–(2) gives a system which we write as:

$$\frac{dy}{dt} = \left(\frac{a_1(t)\phi(t) - \phi'(t)}{\phi(t)}\right)y - \left(b_1(t)\phi(t) + c_1(t)\right)y^2, \tag{5}$$

$$\frac{dy}{dt} = a_2(t)y - \left(b_2(t)\phi(t) + c_2(t)\right)y^2, \tag{6}$$

where $\phi' = \frac{d\phi}{dt}$. Equating coefficients of like terms of (5) and (6), it follows that

$$\frac{\phi'(t)}{\phi(t)} = a_1(t) - a_2(t), \tag{7}$$

and

$$\phi(t) = -\frac{c_1(t) - c_2(t)}{b_1(t) - b_2(t)}. \tag{8}$$

That is

$$\frac{d\phi}{dt} = \frac{(a_1 - a_2)(c_2 - c_1)}{b_1 - b_2}. \tag{9}$$

Then it may be shown that the coupled system (1)–(2) can be transformed into another system in which the equations are decoupled and solvable separately. Indeed, if we multiply both sides of (6) by y^{-2} and substitute $z = y^{-1}$ we obtain

$$z' + a_2(t)z = b_2(t)\phi(t) + c_2(t). \tag{10}$$

This first-order linear differential equation may be solved and gives us

$$z(t)\exp(\int a_2(t)dt) = \int \left(b_2(t)\phi(t) + c_2(t)\right)\exp(\int a_2(t)dt)dt + k. \tag{11}$$

Solving this equation for z leads to an explicit solution. After z has been found, the solution of (6) is given by $z = y^{-1}$, and by (4) we find $x(t)$. We have proved the following result.

Lemma 1. The system (1)-(3) may be transformed into another system in which the equations are decoupled and solvable separately by $x(t) = \phi(t)y(t)$ if and only if the following condition is satisfied

$$\frac{d\phi}{dt} = \frac{(a_1 - a_2)(c_2 - c_1)}{b_1 - b_2}.$$

3. SOLVABILITY OF SYSTEM OF THREE COUPLED EQUATIONS

We shall show how the previous result may be extended to the case of three coupled differential equations. In this case, the corresponding system of coupled equations will be

$$\frac{dx}{dt} = x\left(a_1(t) - b_1(t)x - c_1(t)y - d_1(t)z\right), \tag{12}$$

$$\frac{dy}{dt} = y\left(a_2(t) - b_2(t)x - c_2(t)y - d_2(t)z\right), \tag{13}$$

$$\frac{dz}{dt} = z\left(a_3(t) - b_3(t)x - c_3(t)y - d_3(t)z\right), \tag{14}$$

$$x(0) = \alpha, y(0) = \beta, z(0) = \gamma. \tag{15}$$

Let

$$x = \phi_1(t)z \text{ and } y = \phi_2(t)z \tag{16}$$

where $\phi_i(t), i = 1, 2$ are unknown functions with

$$\phi_1(0)=\frac{\alpha}{\gamma} \text{ and } \phi_2(0)=\frac{\beta}{\gamma}.$$

Replacing (16) in (12)–(14), the following system of equations are obtained:

$$\frac{dz}{dt}+\left(\frac{\phi_1^{'}(t)-a_1(t)\phi_1(t)}{\phi_1(t)}\right)z+\left(b_1(t)\phi_1(t)+c_1(t)\phi_2+d_1(t)\right)z^2=0, \tag{17}$$

$$\frac{dz}{dt}+\left(\frac{\phi_2^{'}(t)-a_2(t)\phi_2(t)}{\phi_2(t)}\right)z+\left(b_2(t)\phi_1(t)+c_2(t)\phi_2+d_2(t)\right)z^2=0, \tag{18}$$

$$\frac{dz}{dt}-a_3(t)z+\left(b_3(t)\phi_1(t)+c_3(t)\phi_2+d_3(t)\right)z^2=0. \tag{19}$$

It appears now clearly that, from what was stated in the case of two nonlinear equations, system (17)–(19) may be decoupled if and only if the following conditions are satisfied simultaneously:

$$\phi_i^{'}(t)=\left(a_i(t)-a_3(t)\right)\phi_i(t), i=1,2, \tag{20}$$

and

$$\left(b_i(t)-b_3(t)\right)\phi_1(t)+\left(c_i(t)-c_3(t)\right)\phi_2(t)=d_3(t)-d_i(t), i=1,2. \tag{21}$$

Solving Eq. (20), we obtain

$$\phi_i(t)=N_i\exp\left[\int(a_i-a_3)dx\right], i=1,2. \tag{22}$$

Returning to (17)–(19), we may conclude that these equations are similarly, and after solving one equation of them, we then obtain $x(t)$ and $y(t)$.

The result on the separation of the equations of (12)–(15) states the following:

Lemma 2: The system (12)–(15) may always be completely separated by $x=\phi_1(t)z$, and $y=\phi_2(t)z$ if and only if the following conditions are satisfied

$$\phi_i^{'}(t)=\left(a_i(t)-a_3(t)\right)\phi_i(t), i=1,2,$$

and

$$\left(b_i(t)-b_3(t)\right)\phi_1(t)+\left(c_i(t)-c_3(t)\right)\phi_2(t)=d_3(t)-d_i(t), i=1,2.$$

In the following section, we will try to present the modified decomposition method to solve (1)–(3) and (12)–(15).

4. MODIFIED DECOMPOSITION METHOD

We present here a variation of the decomposition method [1,2], we define the operator L in (1)–(2) by $L = \frac{d}{dt}$ and we expect the decomposition of the solutions $x(t)$, $y(t)$ as a sum of components to be defined by.

$$x(t) = \sum_{n=0}^{\infty} x_n t^n,$$

$$y(t) = \sum_{n=0}^{\infty} y_n t^n,$$

we let

$$a_i(t) = \sum_{n=0}^{\infty} a_{i,n} t^n, i = 1,2,$$

$$b_i(t) = \sum_{n=0}^{\infty} b_{i,n} t^n, i = 1,2,$$

$$c_i(t) = \sum_{n=0}^{\infty} c_{i,n} t^n, i = 1,2.$$

The substitution yields

$$\sum_{n=0}^{\infty} x_n t^n = \alpha + \int_0^t \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \left(a_{1,k} x_{n-k} - b_{1,k} A_{1,n-k} - c_{1,k} B_{n-k} \right) dt, \tag{23}$$

$$\sum_{n=0}^{\infty} y_n t^n = \beta + \int_0^t \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \left(a_{2,k} y_{n-k} - b_{2,k} B_{n-k} - c_{2,k} A_{2,n-k} \right) dt, \tag{24}$$

where the polynomials $A_{i,k}, i = 1,2$ and B_k are called Adomian polynomials [1,2]:

$$A_{1,0} = x_0^2, A_{2,0} = y_0^2,$$

$$A_{1,1} = 2x_0 x_1, A_{2,1} = 2y_0 y_1,$$

$$A_{1,2} = x_1^2 + 2x_0 x_2, A_{2,2} = y_1^2 + 2y_0 y_2,$$

$$A_{1,3} = 2x_1 x_2 + 2x_0 x_3, A_{2,3} = 2y_1 y_2 + 2y_0 y_3,$$

$$\ldots$$

and

$$B_0 = x_0 y_0,$$

$$B_1 = x_1 y_0 + x_0 y_1,$$

$$B_2 = x_2 y_0 + x_1 y_1 + x_0 y_2,$$

$$\ldots$$

We now carry out the above integrations to write

$$\sum_{n=0}^{\infty} x_n t^n = \alpha + \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \sum_{k=0}^{n} \left(a_{1,k} x_{n-k} - b_{1,k} A_{1,n-k} - c_{1,k} B_{n-k} \right), \tag{25}$$

$$\sum_{n=0}^{\infty} y_n t^n = \beta + \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \sum_{k=0}^{n} \left(a_{2,k} y_{n-k} - b_{2,k} B_{n-k} - c_{2,k} A_{2,n-k} \right). \tag{26}$$

Replacing n by n–1 in the summations on the right, to write

$$\sum_{n=0}^{\infty} x_n t^n = \alpha + \sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{k=0}^{n-1} \left(a_{1,k} x_{n-1-k} - b_{1,k} A_{1,n-1-k} - c_{1,k} B_{n-1-k} \right), \tag{27}$$

$$\sum_{n=0}^{\infty} y_n t^n = \beta + \sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{k=0}^{n-1} \left(a_{2,k} y_{n-1-k} - b_{2,k} B_{n-1-k} - c_{2,k} A_{2,n-1-k} \right). \tag{28}$$

Finally, we can equate coefficients of like powers of t on the left side and on the right side to obtain the recurrence relations for the coefficients. Thus

$$\begin{cases} x_0 &= & \alpha, \\ y_0 &= & \beta, \\ x_n &= & \dfrac{\sum_{k=0}^{n-1} \left(a_{1,k} x_{n-1-k} - b_{1,k} A_{1,n-1-k} - c_{1,k} B_{n-1-k} \right)}{n}, n \geqslant 1 \\ y_n &= & \dfrac{\sum_{k=0}^{n-1} \left(a_{2,k} y_{n-1-k} - b_{2,k} B_{n-1-k} - c_{2,k} A_{2,n-1-k} \right)}{n}, n \geqslant 1. \end{cases}$$

The final solutions are now given by $x(t) = \sum_{n=0}^{\infty} x_n t^n$ and $y(t) = \sum_{n=0}^{\infty} y_n t^n$.

Similarly, the solution of (12)–(15) can be obtained by modified decomposition method and solving the following recurrence relations:

$$\begin{cases} x_0 &= & \alpha, \\ y_0 &= & \beta, \\ z_0 &= & \gamma, \\ x_n &= & \dfrac{\sum_{k=0}^{n-1} \left(a_{1,k} x_{n-1-k} - b_{1,k} A_{1,n-1-k} - c_{1,k} B_{n-1-k} - d_{1,k} C_{1,n-1-k} \right)}{n}, n \geqslant 1 \\ y_n &= & \dfrac{\sum_{k=0}^{n-1} \left(a_{2,k} y_{n-1-k} - b_{2,k} B_{n-1-k} - c_{2,k} A_{2,n-1-k} - d_{2,k} C_{2,n-1-k} \right)}{n}, n \geqslant 1, \\ z_n &= & \dfrac{\sum_{k=0}^{n-1} \left(a_{3,k} z_{n-1-k} - b_{3,k} B_{n-1-k} - c_{3,k} C_{2,n-1-k} - d_{3,k} A_{3,n-1-k} \right)}{n}, n \geqslant 1, \end{cases}$$

where the polynomials

$$A_{1,n} = \sum_{k=0}^{n} x_k x_{n-k}, A_{2,n} = \sum_{k=0}^{n} y_k y_{n-k}, A_{3,n} = \sum_{k=0}^{n} z_k z_{n-k},$$

$$B_n = \sum_{k=0}^{n} x_k y_{n-k}, C_{1,n} = \sum_{k=0}^{n} x_k z_{n-k}, \text{ and } C_{2,n} = \sum_{k=0}^{n} y_k z_{n-k}$$

are Adomian polynomials [1,2]. The final solutions are now given by $x(t) = \sum_{n=0}^{\infty} x_n t^n$, $y(t) = \sum_{n=0}^{\infty} y_n t^n$, and $z(t) = \sum_{n=0}^{\infty} z_n t^n$.

5. APPLICATIONS

Two simple examples are discussed in order to prove that the results obtained by modified decomposition are just the same as those given in our method.

Example 1. Consider system (1)-(3) with

$$a_1(t) = \frac{1}{1-t}, b_1(t) = c_1(t) = e^t,$$

$$a_2(t) = \frac{1}{1-t}, b_2(t) = c_2(t) = e^{-t},$$

$$\alpha = 1, \beta = -1.$$

We see that the conditions of Lemma 1 are fulfilled and straightforward computation yields

$$\varphi(t) = -1, x(t) = \frac{1}{1-t}, y(t) = -\frac{1}{1-t}.$$

Now we solve this system by modified decomposition method. According to the coefficients introduced in this example, we have

$$a_{1,k} = 1, b_{1,k} = c_{1,k} = \frac{1}{k!}, k = 0,1,...$$

$$a_{2,k} = 1, b_{2,k} = c_{2,k} = \frac{(-1)^k}{k!}, k = 0,1,...$$

Direct calculation produces

$$x_n = 1, y_n = -1, n = 0,1,...$$

Then the terms of $x(t)$ and $y(t)$ can be written as

$$x(t) = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

$$y(t) = -\left(1 + t + t^2 + t^3 + \ldots + t^n + \ldots\right)$$

which are the partial sum of the Taylor series of the solutions $x(t)$ and $y(t)$ respectively.

Example 2. Consider system (12)–(15) with

$$a_1(t) = t+1, b_1(t) = t, c_1(t) = -t, d_1(t) = t,$$

$$a_2(t) = t+1, b_2(t) = -2t, c_2(t) = t, d_2(t) = t + te^t$$

$$a_3(t) = t, b_3(t) = t, c_3(t) = t, d_3(t) = t - 2e^t$$

$$\alpha = 1, \beta = 1, \gamma = 1.$$

We see that the conditions of Lemma 2 are fulfilled and it can be shown that

$$\varphi_1(t) = e^t, \varphi_2(t) = e^t, x(t) = e^t, y(t) = e^t, \text{ and } z(t) = 1.$$

Now we solve this system by modified decomposition method. According to the coefficients introduced in this example, we have

$$a_{1,0} = 1, a_{1,1} = 1, a_{1,k} = 0, k = 2, 4, \ldots$$

$$b_{1,0} = 0, b_{1,1} = 1, b_{1,k} = 0k = 2, 3, \ldots$$

$$c_{1,0} = 0, c_{1,1} = -1, c_{1,k} = 0, k = 2, 3, \ldots$$

$$d_{1,0} = 0, d_{1,1} = 1, d_{1,k} = 0, k = 2, 3, \ldots$$

$$a_{2,0} = 1, a_{2,1} = 1, a_{2,k} = 0, k = 3, 4, \ldots$$

$$b_{2,0} = 0, b_{2,1} = -2, b_{2,k} = 0, k = 2, 3, \ldots$$

$$c_{2,0} = 0, c_{2,1} = 1, c_{2,k} = 0, k = 2, 3, \ldots$$

$$d_{2,0} = 0, d_{2,1} = 2, d_{2,k} = \frac{1}{(k-1)!}, k = 2, 3, \ldots$$

$$a_{3,0} = 1, a_{3,1} = 1, a_{3,k} = 0, k = 2, 3, \ldots$$

$$b_{3,0} = 0, b_{3,1} = 1, b_{3,k} = 0, k = 2, 3, \ldots$$

$$c_{3,0} = 0, c_{3,1} = 1, c_{3,k} = 0, k = 2, 3, \ldots$$

$$d_{3,0} = 0, d_{3,1} = -1, d_{3,k} = \frac{-2}{(k-1)!}, k = 2, 3, \ldots$$

Direct calculation produces

$$x_n = \frac{1}{n!}, y_n = \frac{1}{n!}, n = 0,1,\ldots$$

and

$$z_0 = 1, \text{ and } z_n = 0, n = 1, 2, \ldots$$

Then the terms of $x(t)$, $y(t)$, and $z(t)$ can be written as

$$x(t) = 1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} + \ldots$$

$$y(t) = 1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} + \ldots$$

which are the partial sum of the Taylor series of the solutions $x(t)$, $y(t)$ and

$$z(t) = 1.$$

6. CONCLUSION

In the above discussion it was shown that, the nonlinear competition model with variable coefficients can be solved by Modified decomposition method, and show that, under some conditions, separation of the system of two and three equations is also possible and the results obtained are just the same as those given from applying the above method.

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