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The Cauchy Problem for Nonlinear Hyperbolic Equations of Higher order using Modified Decomposition Method

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Abstract: In this paper modified decomposition method is applied to the solvability of nonlinear hyperbolic equations of higher order with initial conditions and illustrated with a few simple examples. The results obtained indicate that Modified Decomposition Method is very effective and simple.

Keywords: Adomian decomposition method, nonlinear hyperbolic equation of higher order.

1. INTRODUCTION

We shall consider the Cauchy problem for the nonlinear hyperbolic equation of higher order [1,2,3] in $[0,T]\times\Re$

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)^n u = F(u), n\geq 1, \tag{1}$$

where the nonlinear term is represented by $F(u)$.

To equation (1) we attach the initial conditions

$$\frac{\partial^i u}{\partial t^i}(0,x)=\varphi_i(x), i=0,...,2n-1. \tag{2}$$

The solution of (1)-(2) for $F(u)= f(t,x)\in L_2([0,T]\times\Re)$ is always unique follows from a general theorem due to Holmgren.

Eq.(1) can be written as follows:

$$\frac{\partial^{2n}u}{\partial t^{2n}}+\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\frac{\partial^{2n}u}{\partial t^{2k}\partial x^{2n-2k}}=F(u), \tag{3}$$

where $\binom{n}{k}=\frac{n!}{(n-k)!k!}$.

The Adomian decomposition method (in short ADM) is a non numerical method for solving a wide variety of functional equations and usually gets the solution in a series form which converges rapidly to accurate solutions. This method was successfully employed to a large amount of applications in applied sciences. Furthermore, many mathematicians devoted many papers for ADM [4, 6, 13, 14, 15, 16, 10, 11, 12, 13, 14, 15,

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16, 17, 18] to some more general types of parabolic and hyperbolic equations, and various problems.

In this paper we shall apply Modified Decomposition Method [4,6] to find solutions of (1)-(2). The decomposition scheme will be illustrated by studying suitable examples for nonlinear hyperbolic equations of fourth and sixth - orders. The solutions are obtained in the form of rapidly convergent power series with elegantly computable terms showing that the new technique is reliable, powerful and promising.

2. MODIFIED DECOMPOSITION METHOD

The modified decomposition method [4,6] may be used to solve the nonlinear problem given by (1) subject to the initial conditions (2). Defining the differential operator $L_{tt\ldots t}^{-1}$ of $L_{tt\ldots t}=\frac{\partial^{2n}}{\partial t^{2n}}$ as the $2n$ fold integration from 0 to t, we rewrite Eq. (3) in the operator form

$$L_{tt\ldots t}u=\sum_{k=0}^{n-1}(-1)^{n-k+1}\binom{n}{k}\frac{\partial^{2n}u}{\partial t^{2k}\partial x^{2n-2k}}+F(u) \tag{4}$$

Applying the inverse operator $L_{tt\ldots t}^{-1}$

$$L_{tt\ldots t}^{-1}(.)=\int_0^t\int_0^t\ldots\int_0^t(.)dtdt\ldots dt,$$

to Eq.(4) and using the initial conditions (2) we obtain

$$u=\sum_{i=0}^{2n-1}\varphi_i(x)\frac{t^i}{i!}+L_{tt\ldots t}^{-1}\left(\sum_{k=0}^{n-1}(-1)^{n-k+1}\binom{n}{k}\frac{\partial^{2n}u}{\partial t^{2k}\partial x^{2n-2k}}\right) \tag{5}$$

$$+L_{tt\ldots t}^{-1}F(u),$$

or

$$u=\sum_{i=0}^{2n-1}\varphi_i(x)\frac{t^i}{i!}+L_{tt\ldots t}^{-1}\left[(-1)^{n+1}\binom{n}{0}\frac{\partial^{2n}u}{\partial x^{2n}}+(-1)^{n}\binom{n}{1}\frac{\partial^{2n}u}{\partial t^2\partial x^{2n-2}}+\ldots+(-1)^{n}\binom{n}{n-1}\frac{\partial^{2n}u}{\partial t^{2n-2}\partial x^{2}}\right]$$

$$+L_{tt\ldots t}^{-1}F(u).$$

The Adomian decomposition method defines the unknown solution *u* by series of the form

$$u=\sum_{k=0}^{\infty}a_k(x)t^k,$$

and write

$$F(u)=\sum_{k=0}^{\infty}A_k(x)t^k=\sum_{k=0}^{\infty}A_k(a_0,a_1,\ldots,a_k,\ldots)t^k,$$

where A_n are called Adomian polynomials and can be generated for all types of nonlinearities according to algorithm set by Adomian [4,6],

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} F\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]_{\lambda=0}, n = 0,1,2,...$$

The substitution yields

$$\sum_{k=0}^{\infty} a_k(x)t^k = \sum_{i=0}^{2n-1}\varphi_i(x)\frac{t^i}{i!} + L_{tt...t}^{-1} S_1 + L_{tt...t}^{-1}\sum_{k=0}^{\infty} A_k(a_0, a_1, ..., a_k, ...)t^k, \tag{6}$$

where

$$S_1 = (-1)^{n+1}\binom{n}{0}\sum_{k=0}^{\infty}\frac{\partial^{2n}}{\partial x^{2n}} a_k(x)t^k + (-1)^n\binom{n}{1}\sum_{k=2}^{\infty} k(k-1)\frac{\partial^{2n-2}}{\partial x^{2n-2}} a_k(x)t^{k-2} \tag{7}$$

$$+ ... + \binom{n}{n-1}\sum_{k=2n-2}^{\infty} k(k-1)...(k-2n+3)\frac{\partial^2}{\partial x^2} a_k(x)t^{k-2n+2},$$

In the summations we now replace $k-2n+2, n=0,1,...$ by k on the right to get

$$S_1 = (-1)^{n+1}\binom{n}{0}\sum_{k=0}^{\infty}\frac{\partial^{2n}}{\partial x^{2n}} a_k(x)t^k + (-1)^n\binom{n}{1}\sum_{k=0}^{\infty}(k+2)(k+1)\frac{\partial^{2n-2}}{\partial x^{2n-2}} a_{k+2}(x)t^k \tag{8}$$

$$+ ... + \binom{n}{n-1}\sum_{k=0}^{\infty}(k+2n-2)(k+2n-3)...(k+1)\frac{\partial^2}{\partial x^2} a_{k+2n-2}(x)t^k.$$

We now carry out the above integrations in (6) to write

$$\sum_{k=0}^{\infty} a_k(x)t^k = \sum_{i=0}^{2n-1}\varphi_i(x)\frac{t^i}{i!} + \sum_{k=0}^{\infty}\frac{t^{k+2n}}{(k+1)(k+2)...(k+2n)} S_2 \tag{9}$$

$$+\sum_{k=0}^{\infty}\frac{A_k(a_0, a_1, ..., a_k, ...)t^{k+2n}}{(k+1)(k+2)...(k+2n)},$$

where

$$S_2 = (-1)^{n+1}\binom{n}{0}\frac{\partial^{2n}}{\partial x^{2n}} a_k(x) + ... + \binom{n}{n-1}(k+2n-2)(k+2n-3)...(k+1)\frac{\partial^2}{\partial x^2} a_{k+2n-2}(x). \tag{10}$$

In the summation on the right, k can be replaced by $k-2n$,, to write

$$\sum_{k=0}^{\infty} a_k(x)t^k = \sum_{i=0}^{2n-1}\varphi_i(x)\frac{t^i}{i!} + \sum_{k=2n}^{\infty}\frac{t^k}{(k-2n+1)(k-2n+2)...k}\left(S_3 + A_{k-2n}\right), \tag{11}$$

where

$$S_3 = (-1)^{n+1}\binom{n}{0}\frac{\partial^{2n}}{\partial x^{2n}}a_{k-2n}(x) + \ldots + \binom{n}{n-1}(k-2)(k-3)\ldots(k-2n+1)\frac{\partial^2}{\partial x^2}a_{k-2}(x). \qquad (12)$$

Finally, we can equate coefficients of like powers of t on the left side and on the right side to obtain the recurrence relations for the coefficients. Thus

$$\begin{cases} a_0 & = & \varphi_0(x), \\ a_1 & = & \varphi_1(x), \\ a_2 & = & \dfrac{\varphi_2(x)}{2!}, \\ a_3 & = & \dfrac{\varphi_3(x)}{3!}, \\ \ldots & & \\ a_{2n-1} & = & \dfrac{\varphi_{2n-1}(x)}{(2n-1)!}, \\ a_k & = & \dfrac{(-1)^{n+1}\binom{n}{0}\frac{\partial^{2n}a_{k-2n}(x)}{\partial x^{2n}} + \ldots + N\frac{\partial^2 a_{k-2}(x)}{\partial x^2} + A_{k-2n}}{(k-2n+1)(k-2n+2)\ldots k}, k \geq 2n, \end{cases}$$

where $N = \binom{n}{n-1}(k-2)(k-3)\ldots(k-2n+1)$.

The final solution is now given by $u(t,x) = \sum_{k=0}^{\infty} a_k(x)t^k$.

This method is illustrated in the following cases.

1. Hyperbolic equation of fourth-order

For $n = 2$ problem (1)-(2) becomes

$$\frac{\partial^4 u}{\partial t^4} - 2\frac{\partial^4 u}{\partial t^2 \partial x^2} + \frac{\partial^4 u}{\partial x^4} = F(u), \qquad (13)$$

$$\frac{\partial^i u}{\partial t^i}(0,x) = \varphi_i(x), i = 0,\ldots,3, \qquad (14)$$

which is the Cauchy problem for hyperbolic of fourth order [19,20,21]. The recurrence relations for the coefficients can be put in the form.

$$\begin{cases} a_0 &= \varphi_0(x), \\ a_1 &= \varphi_1(x), \\ a_2 &= \dfrac{\varphi_2(x)}{2!}, \\ a_3 &= \dfrac{\varphi_3(x)}{3!}, \\ a_k &= \dfrac{-(\frac{\partial^4}{\partial x^4})a_{k-4} + 2(k-2)(k-3)(\frac{\partial^2}{\partial x^2})a_{k-2} + A_{k-4}}{(k-3)(k-2)(k-1)k}, k \geq 4. \end{cases}$$

Example 1. Consider problem (13)-(14) with

$$\varphi_i(x) = e^x, i = 0, \ldots, 3,$$

and

$$F(u) = u^2 - (\frac{\partial u}{\partial t})^2.$$

Now we solve this problem by modified decomposition method. Direct computation of some of the Adomian's polynomials for the nonlinear term $F(u) = u^2 - (\frac{\partial u}{\partial t})^2$ are given as follows:

$$A_0 = a_0^2 - a_1^2,$$

$$A_1 = 2a_0a_1 - 4a_1a_2,$$

$$A_2 = a_1^2 + 2a_0a_2 - 6a_1a_3 - 4a_2^2,$$

$$A_3 = 2a_1a_2 + 2a_0a_3 - 8a_1a_4 - 12a_2a_3,$$

$$A_4 = a_2^2 + 2a_1a_3 + 2a_0a_4 - 10a_1a_5 - 14a_2a_4 - 9a_3^2,$$

$$\ldots$$

The above recurrence relations for the coefficients gives

$$a_k = \frac{e^x}{k!}, k \geq 0,$$

and

$$u(t,x) = \sum_{k=0}^{\infty} a_k(x)t^k = \sum_{k=0}^{\infty} \frac{e^x}{k!} t^k,$$

which is the partial sum of the Taylor series of the exact solution $\exp(x+t)$.

Example 2. Consider problem (13)-(14) with

$$\varphi_0(x) = -x^4, \varphi_i(x) = 0, i = 1, 2, 3,$$

and

$$F(u) = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 144u.$$

The Adomian's polynomials for $F(u) = \left(\frac{\partial^2 u}{\partial t^2}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 144u$ are

$$A_0 = 4a_2^2 - a_0^{''2} - 144a_0,$$

$$A_1 = 24a_2a_3 - 2a_0^{''}a_1^{''} - 144a_1,$$

$$A_2 = 48a_2a_4 + 36a_3^2 - 2a_0^{''}a_2^{''} - a_1^{''2} - 144a_2,$$

$$\dots$$

The above recursive algorithm and the computed polynomials A_n yield

$$a_0 = -x^4, a_1 = a_2 = a_3 = 0,$$

and

$$a_4 = 1, a_k = 0, k = 5, 6, \dots$$

Thus, the solution for $u(t,x)$ in series form

$$u(t,x) = \sum_{k=0}^{\infty} a_k(x)t^k = -x^4 + t^4,$$

with only two terms, which can be verified through substitution to be the exact solution of this problem.

2. Hyperbolic Equation of Sixth-Order

For $n = 3$, problem (1)-(2) becomes

$$\frac{\partial^6 u}{\partial t^6} - 3\frac{\partial^6 u}{\partial t^4 \partial x^2} + 3\frac{\partial^6 u}{\partial t^2 \partial x^4} - \frac{\partial^6 u}{\partial x^6} = F(u), \tag{15}$$

$$\frac{\partial^i u}{\partial t^i}(0,x) = \varphi_i(x), i = 0, \dots, 5, \tag{16}$$

and the recurrence relations for the coefficients for problem (15)-(16) can be written as follows.

$$\begin{cases} a_0 & = & \varphi_0(x), \\ a_1 & = & \varphi_1(x), \\ a_2 & = & \dfrac{\varphi_2(x)}{2!}, \\ a_3 & = & \dfrac{\varphi_3(x)}{3!}, \\ a_4 & = & \dfrac{\varphi_4(x)}{4!}, \\ a_5 & = & \dfrac{\varphi_5(x)}{5!}, \\ a_k & = & \dfrac{\frac{\partial^6 a_{k-6}}{\partial x^6} - 3(k-5)(k-4)\frac{\partial^4 a_{k-4}}{\partial x^4} + 3(k-2)(k-3)(k-4)(k-5)\frac{\partial^2 a_{k-2}}{\partial x^2} + A_{k-6}}{(k-5)(k-4)(k-3)(k-2)(k-1)k}, k \geq 6. \end{cases}$$

Example 3. Consider problem (15)-(16) with

$$\varphi_0(x) = \varphi_2(x) = \varphi_4(x) = 0,$$

$$\varphi_1(x) = \cos x, \varphi_3(x) = -\cos x, \varphi_5(x) = \cos x,$$

and

$$F(u) = u\frac{\partial^2 u}{\partial t^2} - u\frac{\partial^2 u}{\partial x^2}.$$

By modified decomposition method, the components $a_k, k = 0,1...$ are determined by the recursive algorithm

$$\begin{cases} a_{2k} & = & 0, k0, \\ a_{2k+1} & = & \cos x \dfrac{(-1)^k}{(2k+1)!}, k \geq 0, \end{cases}$$

where the Adomian polynomials can be determined for $F(u) = u\frac{\partial^2 u}{\partial t^2} - u\frac{\partial^2 u}{\partial x^2}$ as

$$A_0 = 2a_0a_2 - a_0a_0'',$$

$$A_1 = 6a_0a_3 + 2a_1a_2 - a_0a_1'' - a_1a_0'',$$

$$A_2 = 12a_0a_4 + 6a_1a_3 + 2a_2^2 - a_0a_2'' - a_1a_1'' - a_2a_0'',$$

$$\ldots$$

Thus the solution

$$u(t,x)=\sum_{k=0}^{\infty} a_k(x)t^k=\cos x\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!}=\cos x\sin t,$$

which can be verified through substitution to be the exact solution.

3. CONCLUSION

The modified decomposition method has been proved to be reliable in handing the initial value problems for nonlinear hyperbolic equations of higher order. Some examples with closed form solutions are studied, and the results obtained are just the same as those given from applying the modified decomposition method.

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