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COMMON FIXED POINT THEOREMS FOR ADMISSIBLE MAPPINGS WITH WEAKER CONTROL FUNCTIONS

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Abstract. We propose to derive common fixed point theorems for a pair of mappings satisfying a new generalized (α, φ, ϕ)-weakly contractive condition under weaker control functions with $\mathcal{S}$-α-admissible condition in the framework of metric space. Our results generalize several well-known comparable results in the literature. As an application of our main result, we further establish some common fixed point theorems in metric spaces endowed with a partial order. We supply some illustrative examples to highlight the realized improvements in our results over the corresponding relevant results in the existing literature.

1. Introduction and Preliminaries

The celebrated Banach Contraction Principle is one of the cornerstones in the development of Nonlinear Analysis. In fact, the fixed point theorems have applications not only in the various branches of mathematics but also in economics, chemistry, biology, computer science, engineering, etc. In particular, such theorems are used to demonstrate the existence and uniqueness of solutions of differential equations, integral equations, functional equations and partial differential equations. Therefore, generalizations of the Banach Contraction Principle have been explored heavily by many authors. This famous theorem can be stated as follows.

Theorem 1.1. [4]. *Let $(\mathcal{X}, d)$ be a complete metric space and $\mathcal{T}$ be a mapping of $\mathcal{X}$ into itself satisfying:*

$$d(\mathcal{T}x, \mathcal{T}y) \leq kd(x, y), \ \forall\, x, y \in \mathcal{X}, \tag{1.1}$$

where k is some constant in $(0, 1)$. Then, $\mathcal{T}$ has a unique fixed point $x^* \in \mathcal{X}$.

In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. There are control functions which alter the distance between two points in a metric space. In this direction, Khan et al. [12] addressed a new category of fixed point problems for a single self-map with the help of a control function which they called an altering distance function.

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Definition 1.2. (altering distance function [12]). A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(a) φ is continuous and non-decreasing,
(b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

In [3], Alber and Guerre-Delabriere introduced the concept of weak contraction in Hilbert spaces. Rhoades [17] showed that the result which Alber et al. had proved in [3] is also valid in complete metric spaces.

Definition 1.3. (weakly contractive mapping). Let $\mathcal{X}$ be a metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called weakly contractive if

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x, y) - \varphi(d(x, y)), \ \forall\, x, y \in \mathcal{X}, \tag{1.2}$$

where φ is an altering distance function.

Theorem 1.4. [17, Theorem 2] . *Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a weakly contractive mapping, then $\mathcal{T}$ has a unique fixed point.*

Note that Alber et al. [3] assumed an additional condition on φ which is $\lim\limits_{t \to +\infty} \varphi(t) = +\infty$. But Rhoades [17] obtained the result noted in Theorem 1.4 without using this particular assumption. If one takes $\varphi(t) = (1 - k)t$, where $0 < k < 1$, then (1.2) reduces to (1.1).

Dutta and Choudhoury [6] generalized Theorem 1.4 as follows.

Theorem 1.5. *Let $(\mathcal{X}, d)$ be a complete metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping satisfying the inequality*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in \mathcal{X}$, where $\varphi, \phi : [0, +\infty) \to [0, +\infty)$ are both continuous and nondecreasing functions with $\varphi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then $\mathcal{T}$ has exactly one fixed point.

Đorić [7] gave the following generalized version of Theorem 1.5 and Theorem 1.4.

Theorem 1.6. *Let $(\mathcal{X}, d)$ be a nonempty complete metric space an let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self-mapping such that for each $x, y \in \mathcal{X}$,*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Phi(x, y)) - \phi(\Phi(x, y)),$$

where

(i) $\Phi(x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)]\}$.
(ii) $\varphi : [0, +\infty) \to [0, +\infty)$ *is a continuous, nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$,*
(iii) $\phi : [0, +\infty) \to [0, +\infty)$ *is a lower semi-continuous function with $\phi(0) = 0$ if and only if $t = 0$.*

Then there exists a point $z \in \mathcal{X}$ such that $z = \mathcal{T}z$.

Abbas and Khan [1] and Abbas and Đorić [2] extended Theorem 1.5 to obtain common fixed points for a pair of mappings. Popescu [15, Theorem 4] proved Theorem 1.6 under some weaker conditions for control functions which was extended for a pair of maps in [13].

Samet et al. introduced in [19] the notion of α-admissible mappings and proved some fixed point theorems using this notion. After that, several other authors used α-admissible mappings to obtain various (common) fixed point results (see, e.g., [11, 18] and the references cited therein).

In this paper, we propose to introduce the concept of generalized (α, φ, ϕ)-weakly contractive mapping, and we study the existence and uniqueness of fixed points for $\mathcal{S}$-α-admissible mappings. Also, our results improve [1, Theorem 2.1] and [15, Theorem 4] by considering weaker conditions for control functions φ and ϕ. As an application of our main result, we further establish common fixed point theorems for metric spaces endowed with a partial order. We furnish some illustrative examples to highlight the realized improvements in our results over the corresponding relevant results in the existing literature.

2. Main Results

In this section, we propose new contraction conditions under which a pair of mappings has a common fixed point. To achieve our goal, we recall some important definition.

Definition 2.1. [18] Let $\mathcal{X}$ be a non-empty set, let $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$. The mapping $\mathcal{T}$ is called $\mathcal{S}$-α-admissible if, for all $x, y \in \mathcal{X}$, $\alpha(\mathcal{S}x, \mathcal{S}y) \geq 1$ implies $\alpha(\mathcal{T}x, \mathcal{T}y) \geq 1$. If $\mathcal{S}$ is the identity mapping, then $\mathcal{T}$ is called α-admissible [19].

Definition 2.2. [18] Let $(\mathcal{X}, d)$ be a metric space and $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$. $\mathcal{X}$ is called α-regular if, for every sequence $\{x_n\} \subset \mathcal{X}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

Our main result is the following

Theorem 2.3. *Let $(\mathcal{X}, d)$ be a metric space. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are mappings such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, that $\mathcal{S}(\mathcal{X})$ is complete and that they satisfy generalized (α, φ, ϕ)-weakly contractive condition, that is,*

$$\varphi(\alpha(\mathcal{S}x, \mathcal{S}y) d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \tag{2.1}$$

for all $x, y \in \mathcal{X}$, where $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ and

(a)

$$\Theta(x, y) = \max\{d(\mathcal{S}x, \mathcal{S}y), d(\mathcal{S}x, \mathcal{T}x), d(\mathcal{S}y, \mathcal{T}y), \frac{1}{2}[d(\mathcal{S}y, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{T}y)]\}. \tag{2.2}$$

(b) $\varphi : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing function with* $\varphi(t) = 0$ *if and only if* $t = 0$.

(c) $\phi : [0, +\infty) \to [0, +\infty)$ *is a function with* $\phi(t) = 0$ *if and only if* $t = 0$, *and* $\liminf_{n\to\infty} \phi(t_n) > 0$ *if* $\lim_{n\to\infty} t_n = t > 0$,

(d) $\phi(t) > \varphi(t) - \varphi(t-)$ *for any* $t > 0$, *where* $\varphi(t-)$ *is the left limit of* φ *at* t.

Assume also that the following conditions hold:

(i) $\mathcal{T}$ *is* $\mathcal{S}$-α-*admissible;*
(ii) *there exists* $x_0 \in \mathcal{X}$ *such that* $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) \geq 1$*;*
(iii) $\mathcal{X}$ *is* α-*regular;*
(iv) *either* $\alpha(\mathcal{S}u, \mathcal{S}v) \geq 1$ *or* $\alpha(\mathcal{S}v, \mathcal{S}u) \geq 1$ *whenever* $\mathcal{S}u = \mathcal{T}u$ *and* $\mathcal{S}v = \mathcal{T}v$.

Then $\mathcal{T}$ *and* $\mathcal{S}$ *have a unique point of coincidence in* $\mathcal{X}$*. Moreover, if* $\mathcal{T}$ *and* $\mathcal{S}$ *commute at their coincidence points then* $\mathcal{T}$ *and* $\mathcal{S}$ *have exactly one common fixed point.*

Proof. It should be noted that there exists the left limit of φ at each $t > 0$ by the monotonicity of φ.

If $\mathcal{T}x^* = \mathcal{S}x^*$, then we have a coincidence point. Suppose $\mathcal{T}x \neq \mathcal{S}x$ for all $x \in \mathcal{X}$. Let $x_0 \in \mathcal{X}$ be an arbitrary point such that $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) > 1$. Now since $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, we can choose $x_1 \in \mathcal{X}$ so that $\mathcal{S}x_1 = \mathcal{T}x_0$. Again, from $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, we can find $x_2 \in \mathcal{X}$ so that $\mathcal{S}x_2 = \mathcal{T}x_1$. Continuing this process we find a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{S}x_{n+1} = \mathcal{T}x_n \text{ for all } n \geq 0.$$

If there exists $n_0 \in \{1, 2, \cdots\}$ such that $\Theta(x_{n_0}, x_{n_0-1}) = 0$ then it is clear that $\mathcal{S}x_{n_0-1} = \mathcal{T}x_{n_0} = \mathcal{T}x_{n_0-1}$, contrary to the assumption. Hence, we can suppose

$$\Theta(x_n, x_{n-1}) > 0 \tag{2.3}$$

for all $n \geq 1$.

Step 1. We claim that

$$\alpha(\mathcal{T}x_n, \mathcal{T}x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Using condition (ii), we have $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) = \alpha(\mathcal{T}x_0, \mathcal{T}x_1) \geq 1$. Since, by hypothesis, $\mathcal{T}$ is $\mathcal{S}$-α-admissible, we obtain

$$\alpha(\mathcal{T}x_0, \mathcal{T}x_1) = \alpha(\mathcal{S}x_1, \mathcal{S}x_2) \geq 1, \;\; \alpha(\mathcal{T}x_1, \mathcal{T}x_2) = \alpha(\mathcal{S}x_2, \mathcal{S}x_3) \geq 1.$$

By induction, we get

$$\alpha(\mathcal{T}x_n, \mathcal{T}x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Step 2. We claim that

$$\lim_{n\to\infty} d(\mathcal{T}x_{n+1}, \mathcal{T}x_n) = 0.$$

First of all, by (2.2), we have for $n \geq 1$

$$\begin{aligned}\Theta(x_n, x_{n-1}) &= \max\{d(\mathcal{S}x_n, \mathcal{S}x_{n-1}), d(\mathcal{S}x_n, \mathcal{T}x_n), d(\mathcal{S}x_{n-1}, \mathcal{T}x_{n-1}),\\ &\qquad \frac{1}{2}[d(\mathcal{S}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}x_{n-1})]\}\\ &= \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n), \frac{1}{2}d(\mathcal{T}x_{n-2}, \mathcal{T}x_n)\}\\ &\leq \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n)\}.\end{aligned}$$

We will prove that

$$d(\mathcal{T}x_{n+1}, \mathcal{T}x_n) \leq d(\mathcal{T}x_n, \mathcal{T}x_{n-1}) \tag{2.4}$$

for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that $d(\mathcal{T}x_{n_0+1}, \mathcal{T}x_{n_0}) > d(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0-1})$. Substituting $x = x_{x_{n_0+1}}$ and $y = x_{n_0}$ in the inequality (2.1), we have

$$\begin{aligned}
\varphi(d(\mathcal{T}x_{n_0+1}, \mathcal{T}x_{n_0})) &\leq \varphi(\alpha(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0-1})d(\mathcal{T}x_{n_0+1}, \mathcal{T}x_{n_0}))\\
&\leq \varphi(\Theta(x_{n_0+1}, x_{n_0})) - \phi(\Theta(x_{n_0+1}, x_{n_0}))\\
&\leq \varphi(\max\{d(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0-1}), d(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0+1})\}) - \phi(\Theta(x_{n_0+1}, x_{n_0}))\\
&= \varphi(d(\mathcal{T}x_{n_0}, \mathcal{T}x_{n_0+1})) - \phi(\Theta(x_{n_0+1}, x_{n_0})).
\end{aligned}$$

This implies $\phi(\Theta(x_{n_0+1}, x_{n_0})) = 0$. By the properties of ϕ, we have $\Theta(x_{n_0+1}, x_{n_0}) = 0$, which contradicts (2.3).

Therefore, (2.4) is true and so the sequence $\{d(\mathcal{T}x_{n+1}, \mathcal{T}x_n)\}$ is nonincreasing and bounded. Thus there exists $\rho \geq 0$ such that $\lim_{n\to\infty} d(\mathcal{T}x_{n+1}, \mathcal{T}x_n) = \rho$. Therefore by (2.2),

$$\begin{aligned}
\lim_{n\to\infty} d(\mathcal{T}x_n, \mathcal{T}x_{n-1}) &\leq \lim_{n\to\infty} \Theta(x_n, x_{n-1})\\
&= \lim_{n\to\infty} \max\{d(\mathcal{S}x_n, \mathcal{S}x_{n-1}), d(\mathcal{S}x_n, \mathcal{T}x_n), d(\mathcal{S}x_{n-1}, \mathcal{T}x_{n-1}),\\
&\qquad \frac{1}{2}[d(\mathcal{S}x_{n-1}, \mathcal{T}x_n) + d(\mathcal{S}x_n, \mathcal{T}x_{n-1})]\}\\
&= \lim_{n\to\infty} \max\{d(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-2}), d(\mathcal{T}x_{n-1}, \mathcal{T}x_n), \frac{1}{2}d(\mathcal{T}x_{n-2}, \mathcal{T}x_n)\}.
\end{aligned}$$

This implies $\rho \leq \lim_{n\to\infty} \Theta(x_n, x_{n-1}) \leq \rho$ and so $\lim_{n\to\infty} \Theta(x_n, x_{n-1}) = \rho$.

Now we claim that $\rho = 0$. By (2.1), we have

$$\varphi(d(\mathcal{T}x_n, \mathcal{T}x_{n-1})) \leq \varphi(\Theta(x_n, x_{n-1})) - \phi(\Theta(x_n, x_{n-1}))$$

and taking limit as $n \to \infty$, we have

$$\varphi(\rho+) \leq \varphi(\rho+) - \liminf_{n\to\infty} \phi(\Theta(x_n, x_{n+1}))$$

which is contradictory, unless $\rho = 0$. Hence

$$\rho = 0 = \lim_{n\to\infty} d(\mathcal{T}x_{n+1}, \mathcal{T}x_n). \tag{2.5}$$

Step 3. We show that $\{\mathcal{T}x_n\}$ is a Cauchy sequence.

Suppose this is not true. Then there is an $\varepsilon > 0$ such that for an integer k there exist integers $m(k) > n(k) > k$ such that

$$d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)}) > \varepsilon. \tag{2.6}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.6) and such that

$$d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) < \varepsilon. \tag{2.7}$$

Then

$$\begin{aligned}
\varepsilon &\leq d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})\\
&\leq d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) + d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)}).
\end{aligned}$$

Then by (2.6) and (2.7) it follows that

$$\lim_{k\to\infty} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)}) = \varepsilon. \tag{2.8}$$

Also, by the triangle inequality, we have

$$\left|d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) - d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})\right| < d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)}).$$

By using (2.8) we get

$$\lim_{k\to\infty} d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) = \varepsilon. \tag{2.9}$$

Now by (2.2) we get

$$\begin{aligned}
d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) &\le \Theta(x_{n(k)}, x_{m(k)-1}) \\
&= \max\{d(\mathcal{S}x_{n(k)}, \mathcal{S}x_{m(k)-1}), d(\mathcal{S}x_{n(k)}, \mathcal{T}x_{n(k)}), d(\mathcal{S}x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), \\
&\quad \frac{1}{2}[d(\mathcal{S}x_{m(k)-1}, \mathcal{T}x_{n(k)}) + d(\mathcal{S}x_{n(k)}, \mathcal{T}x_{m(k)-1})]\} \\
&\le \max\{d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2}), d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}), d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{m(k)-1}), \\
&\quad \frac{1}{2}[d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{n(k)}) + d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-1})]\} \\
&\le \max\{d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2}), d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}), \\
&\quad d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{m(k)-1}), \frac{1}{2}[d(\mathcal{T}x_{m(k)-2}, \mathcal{T}x_{n(k)-1}) \\
&\quad + d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}) + d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-1})]\}
\end{aligned}$$

and letting $k \to \infty$ and using (2.8) and (2.9), we have

$$\varepsilon \le \lim_{k\to\infty} \Theta(x_{n(k)}, x_{m(k)-1}) \le \varepsilon$$

and so

$$\lim_{k\to\infty} \Theta(x_{n(k)}, x_{m(k)-1}) = \varepsilon.$$

If there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\varepsilon < d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p))})$ for any p, then by (2.2) we get

$$\begin{aligned}
\varphi(\varepsilon+) &= \limsup_{k\to\infty} \varphi(d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})) \\
&\le \limsup_{n} \phi(\alpha(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2})d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)})) \\
&\le \limsup_{n} \phi(\alpha(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2})[d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1}) + d(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{m(k)})]) \\
&= \limsup_{n} \phi(\alpha(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{m(k)-2})d(\mathcal{T}x_{n(k)}, \mathcal{T}x_{m(k)-1})) \\
&\le \limsup_{k\to\infty}[\varphi(\Theta(x_{n(k)}, x_{m(k)-1})) - \phi(\Theta(x_{n(k)}, x_{m(k)-1}))] \\
&= \varphi(\varepsilon+) - \liminf_{k\to\infty} \phi(\Theta(x_{n(k)}, x_{m(k)-1})),
\end{aligned}$$

which is a contradiction. We repeat the procedure if there exists a subsequence $\{k(p)\}$ of $\{k\}$ such that $\varepsilon < d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p)+1)})$ for any p or $\varepsilon < d(\mathcal{T}x_{n(k(p)+1)}, \mathcal{T}x_{m(k(p))})$ for any p. Therefore, we can suppose that $d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p))}) = \varepsilon$, $d(\mathcal{T}x_{n(k(p)+1)}, \mathcal{T}x_{m(k(p))}) \le \varepsilon$ and $d(\mathcal{T}x_{n(k(p))}, \mathcal{T}x_{m(k(p)+1)}) \le \varepsilon$ for any $k \ge k_1$. Then $\Theta(x_{n(k)}, x_{m(k)}) = \varepsilon$ for $k \ge k_3 = \max\{k_1, k_2\}$, where k_2 is such that $d(\mathcal{T}x_k, \mathcal{T}x_{k+1}) < \varepsilon$ for all $k \ge k_2$. Substituting $x = x_{n(k)}$, $y = x_{m(k)}$ in (2.1), we have

$$\varphi(d(\mathcal{T}x_{n(k)+1}, \mathcal{T}x_{m(k)+1})) \le \varphi(\varepsilon) - \phi(\varepsilon)$$

for any $k \geq k_2$. Obviously $d(\mathcal{T}x_{n(k)+1}, \mathcal{T}x_{m(k)+1}) < \varepsilon$, otherwise we have $\phi(\varepsilon) = 0$. Letting $k \to \infty$ we obtain

$$\varphi(\varepsilon-) \leq \varphi(\varepsilon) - \phi(\varepsilon),$$

which contradicts hypothesis (c). Thus $\{\mathcal{T}x_n\}$ is a Cauchy sequence.

Step 4. Existence of a coincidence point of $\mathcal{T}$ and $\mathcal{S}$.

From the completeness of $\mathcal{S}(\mathcal{X})$, it follows that there exists $z \in \mathcal{S}(\mathcal{X})$ such that $\mathcal{S}x_n \to z$ as $n \to \infty$. Let $u \in \mathcal{X}$ be such that $\mathcal{S}u = z$. We claim that $\mathcal{T}u = z$.

Since $\mathcal{X}$ is α-regular there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_n, u) > 1$. For each n, applying (2.1) with $x = x_{n(k)}$ and $y = u$, since

$$\begin{aligned}\Theta(u, x_{n(k)}) &= \max\{d(\mathcal{S}u, \mathcal{S}x_{n(k)}), d(\mathcal{S}u, \mathcal{T}u), d(\mathcal{S}x_{n(k)}, \mathcal{T}x_{n(k)}), \frac{1}{2}[d(\mathcal{S}x_{n(k)}, \mathcal{T}u) + d(\mathcal{S}u, \mathcal{T}x_{n(k)})]\}\\ &= \max\{d(z, \mathcal{T}x_{n(k)-1}), d(z, \mathcal{T}u), d(\mathcal{T}x_{n(k)-1}, \mathcal{T}x_{n(k)}), \frac{1}{2}[d(\mathcal{T}x_{n(k)-1}, \mathcal{T}u) + d(z, \mathcal{T}x_{n(k)})]\},\end{aligned}$$

we have that $\lim_{k\to\infty} \Theta(u, x_{n(k)}) = d(z, \mathcal{T}u)$. Therefore, we have

$$\begin{aligned}\varphi(d(\mathcal{T}u, z)-) &\leq \limsup_{k\to\infty} \varphi(\alpha(\mathcal{S}u, \mathcal{S}x_{n(k)+1})d(\mathcal{T}u, \mathcal{S}x_{n(k)+1}))\\ &= \limsup_{k\to\infty}(\varphi(d(\mathcal{T}u, \mathcal{T}x_{n(k)})))\\ &\leq \limsup_{k\to\infty}[\varphi(\Theta(u, x_{n(k)})) - \phi(\Theta(u, x_{n(k)}))]\\ &\leq \varphi(d(\mathcal{T}u, z)) - \phi(d(\mathcal{T}u, z)).\end{aligned}$$

which contradicts hypothesis (c). Hence $\mathcal{T}u = z$. Therefore, $\mathcal{T}u = \mathcal{S}u = z$. Thus we have proved that $\mathcal{T}$ and $\mathcal{S}$ have a coincidence point.

The uniqueness of the point of coincidence is a consequence of the conditions (2.1) and (iv), we omit the details.

If $\mathcal{S}$ and $\mathcal{T}$ commute at their coincidence points, then by a well-known result of Jungck [10], they have a unique common fixed point. Thus, the proof is complete. □

An immediate consequence of Theorem 2.3 is as follows.

Corollary 2.4. *Let $(\mathcal{X}, d)$ be a metric space. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are mappings such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, that $\mathcal{S}(\mathcal{X})$ is complete and that the following condition holds:*

$$\varphi(\alpha(\mathcal{S}x, \mathcal{S}y)d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(\mathcal{S}x, \mathcal{S}y)) - \phi(d(\mathcal{S}x, \mathcal{S}y))$$

for all $x, y \in \mathcal{X}$, where $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ and

(a) *$\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$,*
(b) *$\phi : [0, +\infty) \to [0, +\infty)$ is a function with $\phi(t) = 0$ if and only if $t = 0$, and $\liminf_{n\to\infty} \phi(t_n) > 0$ if $\lim_{n\to\infty} t_n = t > 0$,*
(c) *$\phi(t) > \varphi(t) - \varphi(t-)$ for any $t > 0$.*

Assume also that the following conditions hold:

(i) *$\mathcal{T}$ is $\mathcal{S}$-α-admissible;*
(ii) *there exists $x_0 \in \mathcal{X}$ such that $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) \geq 1$;*

(iii) *$\mathcal{X}$ is α-regular and, for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1$, we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$;*

(iv) *either $\alpha(\mathcal{S}u, \mathcal{S}v) \geq 1$ or $\alpha(\mathcal{S}v, \mathcal{S}u) \geq 1$ whenever $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$.*

Then $\mathcal{T}$ and $\mathcal{S}$ have a unique point of coincidence. Moreover, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point.

If $\mathcal{S} = I$, an identity mapping, in Theorem 2.3, then we have [15, Theorem 4] as corollary:

Corollary 2.5. *Let $(\mathcal{X}, d)$ be a complete metric space. Suppose $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a mapping and $\alpha : X \times X \to [0, +\infty)$, such that the following condition holds:*

$$\varphi(\alpha(x, y)d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y))$$

for all $x, y \in \mathcal{X}$, where

(a)

$$\Theta(x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{T}y), \frac{1}{2}[d(y, \mathcal{T}x) + d(x, \mathcal{T}y)]\},$$

(b) *$\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$,*

(c) *$\phi : [0, +\infty) \to [0, +\infty)$ is a function with $\phi(t) = 0$ if and only if $t = 0$, and $\liminf_{n\to\infty} \phi(t_n) > 0$ if $\lim_{n\to\infty} t_n = t > 0$,*

(d) *$\phi(t) > \varphi(t) - \varphi(t-)$ for any $t > 0$.*

Assume also that the following conditions hold:

(i) *$\mathcal{T}$ is α-admissible;*

(ii) *there exists $x_0 \in \mathcal{X}$ such that $\alpha(x_0, \mathcal{T}x_0) \geq 1$;*

(iii) *$\mathcal{X}$ is α-regular;*

(iv) *either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $u = \mathcal{T}u$ and $v = \mathcal{T}v$.*

Then $\mathcal{T}$ has a unique fixed point.

Now, we give an example involving mappings $\mathcal{T}$ and $\mathcal{S}$ that are not continuous and show that Theorem 2.3 can be used in the situations when [1, Theorem 2.1] and [15, Theorem 4] cannot. The example is inspired by [13, Example 2.4].

Example 2.6. Let $\mathcal{X} = [0, 1]$ be equipped with the standard metric and consider the following mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ and functions $\varphi, \phi : [0, +\infty) \to [0, +\infty)$:

$$\mathcal{S}x = \begin{cases} 1, & 0 \leq x < 1/2 \\ 1/2, & x = 1/2, \\ 1/10, & 1/2 < x \leq 2/3, \\ 0, & 2/3 < x \leq 1, \end{cases} \qquad \mathcal{T}x = \begin{cases} 1/2, & 0 \leq x \leq 1/2, \\ 1, & 1/2 < x \leq 1, \end{cases}$$

$$\varphi(t) = \begin{cases} (7/5)t, & 0 \leq t < 1/2, \\ (2-\sqrt{2})t + (\sqrt{2}-1), & 1/2 \leq t < +\infty, \end{cases} \qquad \phi(t) = (1/10)t^2.$$

Consider $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty[$ given by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [0, \frac{1}{2}], \\ 0, & \text{otherwise} \end{cases}$$

We will prove that:

(A) $\phi(t) > \varphi(t) - \varphi(t-)$ for any $t > 0$, where $\varphi(t-)$ is the left limit of φ at t.
(B) $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ satisfy generalized (α, φ, ϕ)-weakly contractive condition.
(C) $\mathcal{T}$ is $\mathcal{S}$-α-admissible;
(D) there exists $x_0 \in \mathcal{X}$ such that $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) \geq 1$;
(E) $\mathcal{X}$ is α-regular;
(F) either $\alpha(\mathcal{S}u, \mathcal{S}v) \geq 1$ or $\alpha(\mathcal{S}v, \mathcal{S}u) \geq 1$ whenever $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$.

Proof. (A) The only point of discontinuity of φ is $1/2$ and it is $\phi(1/2) = 0.025 > \sqrt{2}/2 - 0.7 = \varphi(1/2) - \varphi(1/2-)$, hence condition (A) is satisfied.

(B) Since $\phi(t) \leq \varphi(t)$ for all $t \in [0, 1]$, the only nontrivial cases when the contractive condition (2.1) has to be checked are when $x \in [0, 1/2)$, $y \in (1/2, 2/3]$ and $x \in [0, 1/2)$, $y \in (2/3, 1]$ (or vice versa).

In the first case (2.1) becomes $\varphi(1/2) \leq \varphi(9/10) - \phi(9/10)$ and in the second $\varphi(1/2) \leq \varphi(1) - \phi(1)$, and both of these inequalities are easily verified, hence $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ is a generalized (α, φ, ϕ)-weakly contraction.

(C) Let $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $\alpha(\mathcal{T}x, \mathcal{T}y) < 1$. From the definition of α, it follows that either $\mathcal{T}x, \mathcal{T}y \in [0, 1/2)$ or $\mathcal{T}x, \mathcal{T}y \in (1/2, 1]$. By definition of $\mathcal{T}$, the first case is not possible. In the second case it follows that $\mathcal{S}x, \mathcal{S}y < 1/2$ and, hence, $\alpha(\mathcal{S}x, \mathcal{S}y) = 0 < 1$.

Thus $\mathcal{T}$ is $\mathcal{S}$-α-admissible.

(D) Taking $x_0 = \frac{1}{2}$, we have $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) = \alpha(\frac{1}{2}, \frac{1}{2}) = 1$.

(E) Let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$ for some $x \in \mathcal{X}$. From the definition of α, for all n, we have

$$(x_n, x_{n+1}) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [0, \frac{1}{2}].$$

Since $[0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ is a closed set with respect to the Euclidean metric, we get that

$$(x, x) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \cup [\frac{1}{2}, 1] \times [0, \frac{1}{2}].$$

Then, the only possibility is that $x = \frac{1}{2}$. Thus X is α-regular for all n.

(F) It is easy to show that only for $u = v = \frac{1}{2}$ we have $\mathcal{S}u = \mathcal{T}u (= \frac{1}{2})$ and $\mathcal{S}v = \mathcal{T}v (= \frac{1}{2})$. Then, $\alpha(\mathcal{S}u, \mathcal{S}v) = \alpha(\frac{1}{2}, \frac{1}{2}) = 1$. So, condition (F) is satisfied.

Now, all the hypotheses of Theorem 2.3 are satisfied; thus $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{X}$ (which is 1/2). □

It is easy to show that several existing fixed point results in the literature can be deduced from our Theorem 2.3.

Taking in Theorem 2.3, $\alpha(\mathcal{S}x, \mathcal{S}y) = 1$ for all $x, y \in \mathcal{X}$, we obtain immediately the following fixed point theorem.

Corollary 2.7. [13] *Let $(\mathcal{X}, d)$ be a metric space. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are mappings such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is complete and that the following condition holds:*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y))$$

for all $x, y \in \mathcal{X}$*, where*

(a)

$$\Theta(x,y) = \max\{d(\mathcal{S}x,\mathcal{S}y), d(\mathcal{S}x,\mathcal{T}x), d(\mathcal{S}y,\mathcal{T}y), \frac{1}{2}[d(\mathcal{S}y,\mathcal{T}x) + d(\mathcal{S}x,\mathcal{T}y)]\}$$

(b) $\varphi : [0,+\infty) \to [0,+\infty)$ *is a nondecreasing function with* $\varphi(t) = 0$ *if and only if* $t = 0$,

(c) $\phi : [0,+\infty) \to [0,+\infty)$ *is a function with* $\phi(t) = 0$ *if and only if* $t = 0$, *and* $\liminf_{n\to\infty} \phi(t_n) > 0$ *if* $\lim_{n\to\infty} t_n = t > 0$,

(d) $\phi(t) > \varphi(t) - \varphi(t-)$ *for any* $t > 0$.

Then $\mathcal{T}$ *and* $\mathcal{S}$ *have a unique point of coincidence. Further, if* $\mathcal{T}$ *and* $\mathcal{S}$ *commute at their coincidence points then* $\mathcal{T}$ *and* $\mathcal{S}$ *have exactly one common fixed point.*

Corollary 2.8. [15] *Let* $(\mathcal{X}, d)$ *be a complete metric space. Suppose that* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is a mapping satisfying the following condition:*

$$\varphi(d(\mathcal{T}x,\mathcal{T}y)) \leq \varphi(\Theta(x,y)) - \phi(\Theta(x,y))$$

for all $x, y \in \mathcal{X}$*, where*

(a)

$$\Theta(x,y) = \max\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \frac{1}{2}[d(y,\mathcal{T}x) + d(x,\mathcal{T}y)]\},$$

(b) $\varphi : [0,+\infty) \to [0,+\infty)$ *is a nondecreasing function with* $\varphi(t) = 0$ *if and only if* $t = 0$,

(c) $\phi : [0,+\infty) \to [0,+\infty)$ *is a function with* $\phi(t) = 0$ *if and only if* $t = 0$, *and* $\liminf_{n\to\infty} \phi(t_n) > 0$ *if* $\lim_{n\to\infty} t_n = t > 0$,

(d) $\phi(t) > \varphi(t) - \varphi(t-)$ *for any* $t > 0$.

Then $\mathcal{T}$ *has a unique fixed point.*

3. Fixed Point Theorems on Metric Spaces Endowed with a Partial Order

The technique of contraction mappings and the abstract monotone iterative technique are well known and are applicable to a variety of situations. It is well-known that there is a possibility to combine these two techniques. In the context of ordered metric spaces, the usual contraction conditions are weakened but at the expense that the operator is supposed to be monotone.

The first result in this direction was given by Ran and Reurings [16, Theorem 2.1] who presented an analogue of Banach's fixed point theorem in partially ordered sets. It was applied to the resolution of matrix equations. Further, Harjani and Sadarangani [8, 9] used the above discussed concept and proved some fixed point theorems for weakly contractive operators in ordered metric spaces. Thereafter many work has been done in this direction.

We will show that the results of Section 2 can be used to obtain new (common) fixed point results in ordered metric spaces.

Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

If $(\mathcal{X}, \preceq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Let $(\mathcal{X}, \preceq)$ is a partially ordered set and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, then $\mathcal{T}$ is said to be non-decreasing, if for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$.

Definition 3.1. [9]. Suppose $(\mathcal{X}, \preceq)$ is a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are self-mappings on $\mathcal{X}$. One says that $\mathcal{T}$ is $\mathcal{S}$-non-decreasing if for $x, y \in \mathcal{X}$,

$$\mathcal{S}x \preceq \mathcal{S}y \text{ implies } \mathcal{T}x \preceq \mathcal{T}y.$$

Theorem 3.2. *Let $(\mathcal{X}, d, \preceq)$ be an ordered metric space. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are mappings such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is complete and generalized ordered (φ, ϕ)-weakly contractive condition is satisfied, that is, for every pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $\mathcal{S}x$ and $\mathcal{S}y$ are comparable,*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y)) \tag{3.1}$$

where conditions (a)–(d) of Theorem 2.3 are satisfied.

Assume also that the following conditions hold:

(i) *$\mathcal{T}$ is $\mathcal{S}$-nondecreasing;*
(ii) *there exists $x_0 \in \mathcal{X}$ such that $\mathcal{S}x_0 \preceq \mathcal{T}x_0$;*
(iii) *if $\{x_n\} \subset \mathcal{X}$ such that $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*
(iv) *for all $u, v \in \mathcal{X}$, if $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$, then $\mathcal{S}u$ and $\mathcal{S}v$ are comparable.*

Then $\mathcal{T}$ and $\mathcal{S}$ have a unique point of coincidence in $\mathcal{X}$. Moreover, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have exactly one common fixed point.

Proof. Define the mapping $\alpha : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in \mathcal{S}\mathcal{X} \text{ and } x \preceq y \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $(\mathcal{T}, \mathcal{S})$ is a generalized (α, φ, ϕ)-weakly contractive mapping, since $\alpha(\mathcal{S}x, \mathcal{S}y) = 1$ for all $x, y \in \mathcal{X}$ such that $\mathcal{S}x \preceq \mathcal{S}y$. Otherwise $\varphi(\alpha(\mathcal{S}x, \mathcal{S}y) d(\mathcal{T}x, \mathcal{T}y)) = 0$ and so condition (2.1) holds.

For all $x, y \in \mathcal{S}\mathcal{X}$, from the $\mathcal{S}$-nondecreasing property of $\mathcal{T}$, we have

$$\alpha(x, y) \geq 1 \Rightarrow x \preceq y \Rightarrow \mathcal{T}x \preceq \mathcal{T}y \Rightarrow \alpha(\mathcal{T}x, \mathcal{T}y) \geq 1.$$

Thus $\mathcal{T}$ is an $\mathcal{S}$-α-admissible mapping. Hence, (i) of Theorem 2.3 holds.

From condition (ii), for $x_0 \in \mathcal{X}$ we have $\alpha(\mathcal{S}x_0, \mathcal{T}x_0) \geq 1$. Hence, (ii) of Theorem 2.3 holds.

Now, let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in \mathcal{X}$ as $n \to \infty$. By the definition of α, we have

$$x_n, x_{n+1} \in \mathcal{S}\mathcal{X} \text{ and } x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N}.$$

Since $\mathcal{S}\mathcal{X}$ is complete, we deduce that $x \in \mathcal{S}\mathcal{X}$. By (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and so $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$ and so $\mathcal{X}$ is α-regular. Moreover, $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$. Hence, (iii) of Theorem 2.3 holds.

From condition (iv) and definition of α, $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$ implies that $\alpha(\mathcal{S}u, \mathcal{S}v) \geq 1$ or $\alpha(\mathcal{S}v, \mathcal{S}u) \geq 1$. Hence, (iv) of Theorem 2.3 holds.

Thus the hypotheses (i)–(iv) of Theorem 2.3 are satisfied and by Theorem 2.3, $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point. □

Remark 3.3. 1. Theorem 3.2 is ordered version generalization of [2, Theorem 2.1] (for two maps) with weaker control function.

2. Theorem 3.2 is generalization of [8, Theorem 2.1] in the sense of using generalized weakly contraction condition with weaker control function.

The following results are immediate consequences of Theorem 3.2.

Corollary 3.4. *Let $(\mathcal{X}, d, \preceq)$ be an ordered metric space. Suppose that $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are mappings such that $\mathcal{T}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$, $\mathcal{S}(\mathcal{X})$ is complete and for every pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $\mathcal{S}x$ and $\mathcal{S}y$ are comparable,*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(d(\mathcal{S}x, \mathcal{S}y)) - \phi(d(\mathcal{S}x, \mathcal{S}y))$$

where conditions (b)–(d) of Theorem 2.3 are satisfied.

Assume also that the following conditions hold:

(i) *$\mathcal{T}$ is $\mathcal{S}$-nondecreasing;*
(ii) *there exists $x_0 \in \mathcal{X}$ such that $\mathcal{S}x_0 \preceq \mathcal{T}x_0$;*
(iii) *if $\{x_n\} \subset \mathcal{X}$ such that $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*
(iv) *for all $u, v \in \mathcal{X}$ such that $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$, then $\mathcal{S}u$ and $\mathcal{S}v$ are comparable.*

Then $\mathcal{T}$ and $\mathcal{S}$ have a unique point of coincidence in $\mathcal{X}$. Moreover, if $\mathcal{T}$ and $\mathcal{S}$ commute at their coincidence points then $\mathcal{T}$ and $\mathcal{S}$ have exactly one common fixed point.

Corollary 3.5. *Let $(\mathcal{X}, d, \preceq)$ be an ordered complete metric space. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying for every pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ such that $\mathcal{S}x$ and $\mathcal{S}y$ are comparable,*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(\Theta(x, y)) - \phi(\Theta(x, y))$$

where conditions (a)–(d) of Theorem 2.3 are satisfied.

Assume also that the following conditions hold:

(i) *$\mathcal{T}$ is nondecreasing;*
(ii) *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq \mathcal{T}x_0$;*
(iii) *if $\{x_n\} \subset \mathcal{X}$ such that $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;*
(iv) *for all $u, v \in \mathcal{X}$ such that $u = \mathcal{T}u$ and $v = \mathcal{T}v$, then u and v are comparable.*

Then $\mathcal{T}$ has exactly one fixed point.

The above Corollary 3.5 is ordered version generalization of [15, Theorem 4] with weaker control function.

The following example is inspired by [14, Example 2].

Example 3.6. Let $\mathcal{X} = [0,1]$ be endowed with the usual order $\leq$ in $\mathbb{R}$ and the standard metric $d(x,y) = |x-y|$ for all $x, y \in X$. Consider the following discontinuous mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ and functions $\varphi, \phi : [0,+\infty) \to [0,+\infty)$:

$$\mathcal{S}x = \begin{cases} 0, & 0 \leq x < 1/2 \\ 1/2, & x = 1/2, \\ 2/3, & 1/2 < x \leq 2/3, \\ 4/5, & 2/3 < x \leq 1, \end{cases} \qquad \mathcal{T}x = \begin{cases} 1/2, & 0 \leq x \leq 1/2, \\ 1, & 1/2 < x \leq 1, \end{cases}$$

$$\varphi(t) = \begin{cases} (7/5)t, & 0 \leq t < 1/2, \\ \sqrt{2}/2, & t = 1/2, \\ (2t+3)/5, & 1/2 < t < +\infty, \end{cases} \qquad \phi(t) = (1/10)t^2.$$

We will prove that:

(A) $\phi(t) > \varphi(t) - \varphi(t-)$ for any $t > 0$, where $\varphi(t-)$ is the left limit of φ at t.
(B) $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ satisfy generalized (φ, ϕ)-weakly contractive condition.
(C) $\mathcal{T}$ is $\mathcal{S}$-nondecreasing;
(D) there exists $x_0 \in \mathcal{X}$ such that $\mathcal{S}x_0 \preceq \mathcal{T}x_0$;
(E) if $\{x_n\} \subset \mathcal{X}$ is such that $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;
(F) for all $u, v \in \mathcal{X}$, if $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$, then $\mathcal{S}u$ and $\mathcal{S}v$ are comparable.

Proof. (A) It is clear.

(B) Since $\phi(t) \leq \varphi(t)$ for all $t \in [0,1]$, the only nontrivial cases when the contractive condition (2.1) has to be checked are when $x \in [0,1/2)$, $y \in (1/2, 2/3]$ and $x \in [0,1/2)$, $y \in (2/3, 1]$ (or vice versa).

In both these cases, (3.1) becomes $\varphi(1/2) \leq \varphi(1) - \phi(1)$, and this inequality is easily verified. Hence $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ satisfies generalized (φ, ϕ)-weakly contractive condition.

(C) It is clear from the definition of $\mathcal{S}$ and $\mathcal{T}$ that for all $x, y \in X$, $\mathcal{S}x \preceq \mathcal{S}y$ implies that $\mathcal{T}x \preceq \mathcal{T}y$.

(D) Taking $x_0 = \frac{1}{2}$, we have $\mathcal{S}x_0 = \frac{1}{2} = \mathcal{T}x_0$.

(E) Let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x$ as $n \to \infty$ for some $x \in \mathcal{X}$. Since $x_n \in [0,1]$ for n and $[0,1]$ is a closed set with respect to the Euclidean metric, we get that $x \in [0,1]$. Then, $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$.

(F) It is easy to show that, only for $u = v = \frac{1}{2}$, we have $\mathcal{S}u = \mathcal{T}u$ and $\mathcal{S}v = \mathcal{T}v$. So $\mathcal{S}u \preceq \mathcal{S}v$. So, condition (F) is satisfied.

Now, all the hypotheses of Theorem 2.3 are satisfied; thus $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point in $\mathcal{X}$ (which is 1/2).

Note that this example is not covered when $\mathcal{S} = I$ by [8, Theorem 2.1], since the function φ is not right-continuous at the point 1/2. □

Remark 3.7. Similar to section 3, we can give application to cyclic mapping as in [5].

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