Global and Stochastic Analysis
Special Issue: *25th International Conference of Forum for Interdisciplinary Mathematics*

MUK PUBLICATIONS
Open Access Publisher

A STUDY ON UNIFORMITY TESTS UNDER VARIOUS ALTERNATIVES

TRIPAKSHI BORTHAKUR AND BIPIN GOGOI

Abstract: The Uniform distribution appears due to natural random events or to the application of methods for transforming samples from any other distribution to samples with uniformly distributed values in the interval [0, 1]. Thus, in order to test whether a sample comes from a given distribution, one can test whether its transformed sample is distributed according to the Uniform distribution or not. Several test procedures are developed to test the goodness-of-fit for uniformity. In this paper, we want to study the performance of five different tests for uniformity by considering different sample sizes as well as different alternatives. The results so obtained are displayed in various tables and graphs. Finally, conclusions are made on the basis of the results.
2010 Mathematics Subject Classification: Primary 62G35; Secondary 62G30; 65C05.

1. Introduction

The Uniform distribution appears due to natural random events or to the application of methods for transforming samples from any other distribution to samples with uniformly distributed values in the interval $[0, 1]$. Uniform distribution is often used to describe the measurement errors of some devices or systems, which is not least due to the lack of information. Naturally, its unjustified use can cause problems Goodness-of-fit tests are frequently used to decide if an observed sample X_i, $i = 1, 2, ..., n$ can be considered as a set of independent realizations of a given cumulative distribution function (cdf) $F_0(x)$. More precisely, they are used to test the hypothesis H_0: $F = F_0$, where F is the true cdf of the observations. Let us suppose that F_0 is a Uniform distribution in the interval [0, 1]. For testing uniformity, a number of authors proposed different statistical tests. Some of them are Anderson-Darling test, Kolmogorov-Smirnov test, Cramer-von-Mises test etc. Several tests are available for testing uniformity. Generally, a simple testable hypothesis of uniformity of the sample $X_1, X_2, ..., X_n$ of independent observations of a random value of X has the form: H_0: $F(x) = x$, $x \in [0, 1]$. Most of the tests for the hypothesis of uniformity on the interval $[0, 1]$ are based on the ordered samples $X_{(1)} < X_{(2)} < \dots < X_{(n)}$.

In this paper, a simulation study is carried out to know the power of five different tests by taking various sample sizes against four different alternative distributions. In the proposed study, section 2 and 3 represents a general description of the uniformity tests and the types of alternatives considered here, section 4 presents the simulation approach considered in the study and also the power results.

Keyword. Goodness-of-fit; Uniform distribution; Monte Carlo technique; level and power.

2. Test Procedure

Let F be a continuous cumulative distribution function. Let $X_1, X_2, \ldots, X_n$ be a random sample from F. We are interested in testing H_0: $F \sim U[0, 1]$, where U [0, 1] denotes the Uniform distribution in the interval [0, 1]. Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the ordered samples and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean.

Generally, the use of nonparametric goodness of fit tests for composite hypotheses with regard to different parametric models of probability distribution laws is seriously complicated due to the dependence of test statistic distribution on a number of factors. But, in case of nonparametric tests used for uniformity, such type of problem does not arise. Therefore, in many situations, a sample that is belonged to some parametric law comes down to test the hypothesis of uniformity on the interval [0, 1].

In our proposed study, the five well-known test statistics K-S(D_n), AD(A_n^2), CvM(W_n^2), Watson(U_n^2) and ZhangA(Z_A), defined in terms of a Uniform distribution function, have been considered. These statistics are described below:

2.1. Tests based on Empirical Distribution Function (EDF). The test statistics based on empirical distribution function (EDF) measure the discrepancy between the EDF, $F_n(x)$, and the distribution function, $F(x)$, and are based on the vertical differences between these two functions. In our proposed study, the test statistics based on EDF which are considered herein may be subclassified into two groups:

(a) **Supremum test statistics:** The most well known EDF statistic is D_n, introduced by Kolmogorov (1933) is given by

$$D_n = \sup_{-\infty < x < \infty} \left| F_n(x) - F(x) \right| = \max(D_n^+, D_n^-)$$

where, $D_n^+ = \max\limits_{1 \le i \le n}\left\{\frac{i}{n} - U_i\right\}$ and $D_n^- = \max\limits_{1 \le i \le n}\left\{U_i - \frac{i-1}{n}\right\}$.

D_n^+ and D_n^- are calculated as the largest vertical difference between $F_n(x)$ and $F(x)$ when $F_n(x)$ is greater or smaller than $F(x)$, respectively.

(b) **Quadratic test statistics:** This wide family of discrepancy measures is given by the Cramer-von Mises family $Q_n = n\int_0^1 \{F_n(x) - F(x)\}^2 \ \psi(x)dF(x)$, where $\psi(x)$ is a suitable function giving weights to $\{F_n(x) - F(x)\}^2$. When $\psi(x) = 1$, this statistic is the Cramer-von Mises statistic W_n^2 and when $\psi(x) = [\{F(x)\}\{1 - F(x)\}]^{-1}$, this statistic is the Anderson-Darling (1954) statistic A_n^2. The test statistic is defined as follows:

$$W_n^2 = \sum_{i=1}^{n}\left\{U_{(i)} - \frac{2i-1}{2n}\right\}^2 + \frac{1}{12n}$$

$$A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\,[\ln U_{(i)} + \ln\{1 - U_{(n+1-i)}\}]$$

Lewis (1961) demonstrated that for $n > 3$, the asymptotic distribution of A_n^2 is a good approximation to its distribution and he proportionated a table containing the q-values of this asymptotic distribution for the lower tail, where $q = P(A_n^2 < t)$.

Watson (1961) proposed the statistic U_n^2, which is a modification of W_n^2, which is used with observations on a circle. This statistic is defined as

$$U_n^2 = W_n^2 - n(\bar{U} - .5)^2$$

2.2. Other Approach. Zhang Test: Zhang (2002) proposed a method to construct new goodness of fit tests, derived from classical ones, and is defined as

$$Z_A = -\sum_{i=1}^{n}\left[\frac{\ln\{U_{(i)}\}}{[n - i + (1/2)]} + \frac{\ln\{1 - U_{(i)}\}}{[i - (1/2)]}\right]$$

The statistic Z_A is distribution symmetric. It appears similar to the tests Kolmogorov-Smirnov, Anderson-Darling and Cramer-von-Mises. The critical values for this statistic are found in Zhang (2002).

3. Types of Alternatives

If $F(x)$ is completely specified, the Z, should be uniformly distributed as U(0, 1). Power studies hence therefore been confined to a test of the hypothesis concerning Z, where Z_i's are drawn from alternative distributions. If the variance of the hypothesized $F(x)$ is correct, but the mean is wrong, the point Z_i will tend towards 0 and 1. Again if mean is correct, but the variance is wrong, then the point Z_i will move to each end, or will move towards 0.5.

The following alternatives Type A, Type B and Type C proposed by Stephens (1974) are given as follows:

$$A_k : F(z) = 1 - (1 - z)^k,\ 0 \le z \le 1$$

$$B_k : F(z) = \begin{Bmatrix} 2^{k-1}z^k, \text{ for } 0 \le z \le 0.5 \\ 1 - 2^{k-1}(1 - z)^k, \text{ for } 0.5 \le z \le 1 \end{Bmatrix}$$

$$C_k : F(z) = \begin{Bmatrix} 0.5 - 2^{k-1}(0.5 - z)^k, \text{ for } 0 \le z \le 0.5 \\ 0.5 + 2^{k-1}(z - 0.5)^k, \text{ for } 0.5 \le z \le 1 \end{Bmatrix}$$

where, $k > 0$. For $k > 1$, the family A_k gives points closer to zero than expected under the hypothesis of uniformity, B_k gives points near to 0.5 and C_k gives two clusters close to 0 and 1. For $k < 1$, the behavior is opposite, that is, the family A_k gives points closer to 1, B_k gives high probability to intervals near to 0 and 1 and C_k gives more probability to intervals around 0.5 than expected under the uniform distribution.

Also the p.d.f. of Beta distribution is given by

$$f(z : \beta, \gamma) = \frac{z^{\beta - 1}(1 - z)^{\gamma - 1}}{\int_0^1 u^{\beta - 1}(1 - u)^{\gamma - 1}\, du}$$

$$= \frac{z^{\beta - 1}(1 - z)^{\gamma - 1}}{B(\beta, \gamma)}; \beta, \gamma > 0, 0 \le z \le 1$$

4. Simulation Study

To study the empirical level and power of the five test statistics we have generated samples from different distributions. The study was carried out for six different sample sizes (n = 10, 20, 25, 30, 50, 100) and considering significance levels 0.10, 0.05 and 0.01 (for 10 percent and 1 percent levels are not shown in the table due to space) and by considering four different alternative distributions viz., Type A, Type B, Type C and Beta. Here, the uniform variates are generated by RAND function using BASIC and for the other distributions, method of inverse integral transformation is used. The ratio of the value of the test statistic greater than the critical value divided by the total number of repetition gives the empirical level under null case and power of the test statistics under the alternative hypothesis.

5. Results

Table 1. Empirical Power of the Tests under Type A Distribution (α = 0.05)

Sample size (n)	k_1	*Test Statistic*				
		Z_A	W_n^2	A_n^2	D_n	U_n^2
10	1.5	.1715	.1743	.1542	.1507	.1067
	2.0	.4334	.4462	.4133	.3898	.2308
	3.0	.8689	.8752	.8448	.8129	.5637
20	1.5	.3377	.3224	.3094	.2778	.1664
	2.0	.7811	.7693	.7553	.6899	.4457
	3.0	.9974	.9959	.9948	.9888	.9027
25	1.5	.3830	.3920	.3786	.3446	.1964
	2.0	.8596	.8616	.8525	.8037	.5458
	3.0	.9995	.9994	.9995	.9975	.9635
30	1.5	.4804	.4603	.4520	.3974	.2385
	2.0	.9311	.9226	.9194	.8753	.6519
	3.0	1.000	1.000	1.000	.9992	.9876
50	1.5	.7102	.6869	.6970	.6036	.3798
	2.0	.9954	.9931	.9941	.9847	.8861
	3.0	1.000	1.000	1.000	1.000	.9998
100	1.5	.9560	.9433	.9536	.9055	.7149
	2.0	1.000	1.000	1.000	1.000	.9984
	3.0	1.000	1.000	1.000	1.000	1.000

Table 2. Empirical Power of the Tests under Type B Distribution (α = 0.05)

Sample size (n)	k_1	*Test Statistic*				
		U_n^2	Z_A	D_n	W_n^2	A_n^2
10	1.5	.1394	.0642	.0380	.0266	.0143
	2.0	.3431	.1418	.0485	.0265	.0092
	3.0	.7778	.4411	.0940	.0629	.0197
20	1.5	.2450	.1899	.0576	.0441	.0331
	2.0	.6519	.5471	.1251	.0997	.0972
	3.0	.9847	.9620	.4012	.5077	.5359
25	1.5	.2949	.2378	.0739	.0538	.0476
	2.0	.7635	.6624	.1857	.1710	.1840
	3.0	.9962	.9907	.5968	.7366	.7918
30	1.5	.3685	.3344	.0826	.0625	.0605
	2.0	.8561	.8129	.2425	.2499	.3001
	3.0	.9997	.9989	.7507	.8883	.9334
50	1.5	.5809	.5566	.1377	.1189	.1685
	2.0	.9855	.9758	.4987	.6123	.7598
	3.0	1.000	1.000	.9851	.9993	.9998
100	1.5	.9052	.8954	.3420	.3795	.5655
	2.0	1.000	1.000	.9350	.9872	.9983
	3.0	1.000	1.000	1.000	1.000	1.000

Table 3. Empirical Power of the Tests under Type C Distribution (α = 0.05)

Sample size (n)	k_1	*Test Statistic*				
		U_n^2	A_n^2	D_n	W_n^2	Z_A
10	1.5	.1389	.1168	.1105	.0973	.0677
	2.0	.3429	.2217	.1956	.1554	.1175
	3.0	.7745	.4914	.3701	.2997	.2750
20	1.5	.2495	.1525	.1480	.1165	.0705
	2.0	.6567	.3731	.3106	.2442	.1768
	3.0	.9863	.8437	.6521	.6688	.5626
25	1.5	.2969	.1714	.1721	.1288	.0709
	2.0	.7639	.4519	.3862	.3131	.1873
	3.0	.9963	.9271	.7858	.8307	.6576
30	1.5	.3648	.1887	.1862	.1379	.0852
	2.0	.8533	.5335	.4408	.3846	.2657
	3.0	.9994	.9722	.8702	.9251	.8063

Sample size (*n*)	k_1	*Test Statistic*				
		U_n^2	A_n^2	D_n	W_n^2	Z_A
50	1.5	.5833	.2899	.2592	.2049	.1187
	2.0	.9799	.8102	.6497	.6866	.4785
	3.0	1.000	.9996	.9910	.9987	.9831
100	1.5	.9033	.5766	.4817	.4518	.2817
	2.0	1.000	.9952	.9609	.9899	.9084
	3.0	1.000	1.000	1.000	1.000	1.000

Table 4. Empirical Power of the Tests under Beta Distribution ($\alpha = 0.05$)

Sample size (*n*)	β, γ	*Test Statistic*				
		W_n^2	Z_A	A_n^2	D_n	U_n^2
10	1.1	.0552	.0517	.0526	.0542	.0554
	3.5	.9973	.9979	.9975	.9893	.8872
	4.1	.9861	.9827	.9790	.9650	.8206
20	1.1	.0513	.0510	.0500	.0525	.0506
	3.5	1.000	1.000	1.000	1.000	.9978
	4.1	.9999	.9999	.9999	.9998	.9946
25	1.1	.0483	.0396	.0477	.0504	.0455
	3.5	1.000	1.000	1.000	1.000	.9996
	4.1	1.000	1.000	1.000	1.000	.9990
30	1.1	.0486	.0503	.0469	.0503	.0497
	3.5	1.000	1.000	1.000	1.000	.9999
	4.1	1.000	1.000	1.000	1.000	.9999
50	1.1	.0454	.0485	.0487	.0446	.0493
	3.5	1.000	1.000	1.000	1.000	1.000
	4.1	1.000	1.000	1.000	1.000	1.000
100	1.1	.0464	.0538	.0513	.0490	.0540
	3.5	1.000	1.000	1.000	1.000	1.000
	4.1	1.000	1.000	1.000	1.000	1.000

6. Discussions

Table 1 shows the empirical power of five tests under the alternative of Type A distribution for three different values of the parameter (k_1). It is seen that power of all the tests increases as the sample size increases. However, the power of Z_A and CvM(W_n^2) seems to be higher than the other tests in most of the situation followed by AD(A_n^2) and K-S(D_n) and the power of Watson test is found to be the lowest among all the tests considered here.

Table 2 shows the empirical power of five tests under the alternative of Type B distribution for three different values of the parameter (k_1). It is seen that power of all the tests increases as the sample size increases. Here, it is found that the power of Watson test is higher among all the tests considered here. The Z_A test gives greater power as the sample size increases. Also for large sample sizes the power of K-S(D_n), CvM(W_n^2) and AD(A_n^2) tests also increases and finally it becomes exactly one.

Table 3 shows the empirical power of five tests under the alternative of Type C distribution for three different values of the parameter (k_1). It is seen that power of all the tests increases as the sample size increases and also the power becomes exactly one for large sample sizes. The power of Watson(W_n^2), AD(A_n^2) and K-S(D_n) tests are found to be higher than the other tests and the power of Z_A is the lowest among all the tests.

Table 4 shows the empirical power of five tests under the alternative of Beta distribution for three different set of the parameters (β, γ). It is found that the tests viz., CvM(W_n^2), Z_A and AD (A_n^2) gives comparatively higher power than the other tests in all the situations. Also the powers of all the tests become exactly one for large sample sizes and for large values of the parameter.

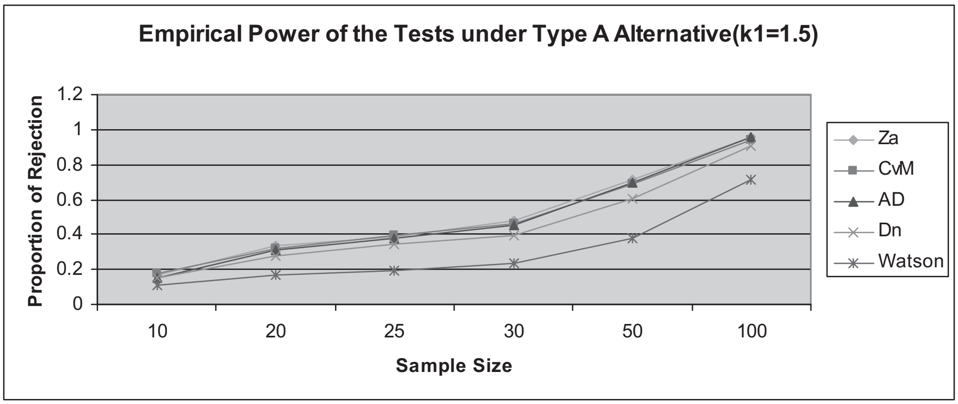

Figure 1

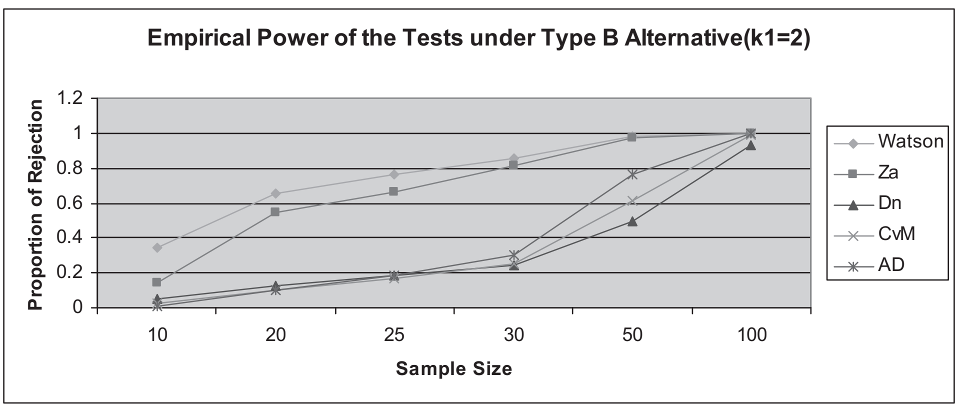

Figure 2

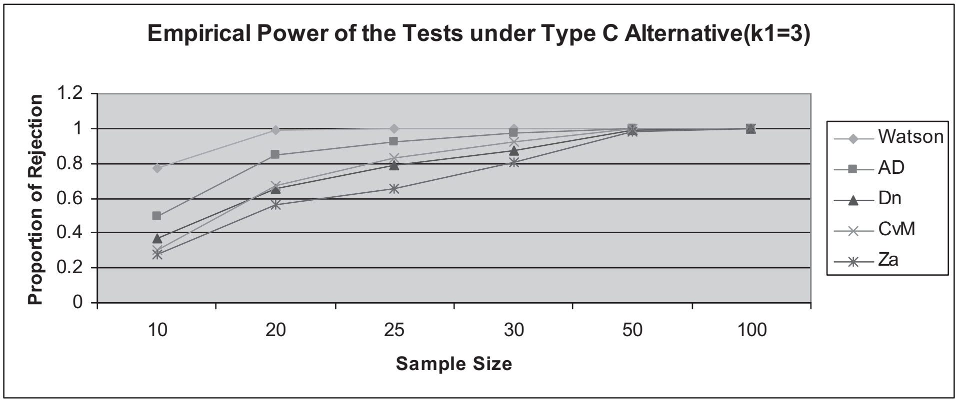

Figure 3

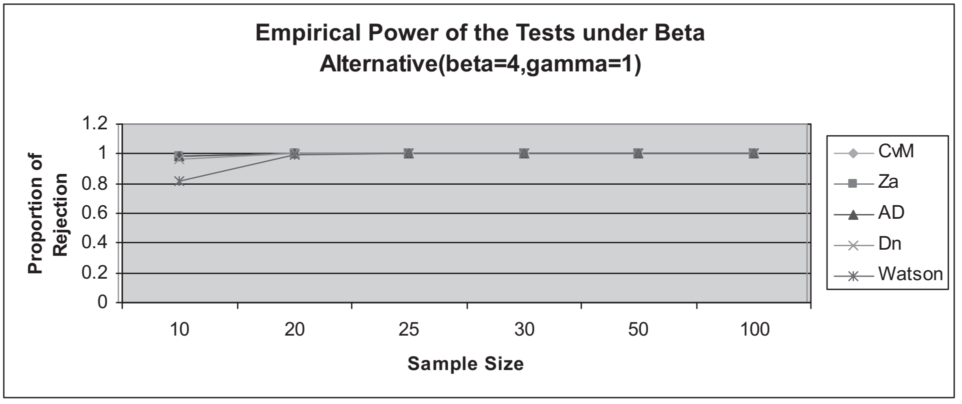

Figure 4

7. Conclusion

Power of Z_A test is found to be higher than the other tests in most of the alternatives except for the alternative of Type C. The test statistics CvM(W_n^2), AD(A_n^2) and K-S(D_n) give almost same power in most of the situations. The power of Watson(U_n^2) test is also good for some alternatives. Finally, we arrive at the conclusion that Z_A test may be recommended in most of the situations except for the alternative of Type C. We may give second preference to the tests CvM, AD and K-S.

References

1. Anderson, T.W. and Darling, D.A: A test of goodness-of-fit, Jour. Amer Stat. Assoc., 49(1954), 765-769.
2. Kolmogorov, A.N.: Sulla Determinazione Empirica Di una Legge di Distribuziane, Giornale Dell' Instituta Degli Attuari, 4 (1933), 83-91.
3. Lemeshko, Yu.B., Blinov, Yu.P. and Lemeshko, S.B.: Goodness of Fit Tests for Uniformity of Probability Distribution Law, Optoelectronics, Instrumentation and Data Processing, 52 (2) (2016), 128-140.

4. Lewis, P.A.W.: Distribution of the Anderson-Darling statistic, Ann. Math. Stat., 32(1961), 1118-1124.
5. Stephens, M.A.: EDF Statistic for Goodness of Fit and Some Comparisons, Jour. Amer. Stat. Assoc., 69(1974), 730-737.
6. Watson, G.S.: Goodness-of-fit on a circle, Biometrika, 48(1961), 109-114.
7. Zhang, J. : Powerful goodness-of-fit tests based on the likelihood ratio, JRSR, Series B, 64(2002), 281-294.

Tripakshi Borthakur: Research Scholar, Department Of Statistics, Dibrugarh University, Dibrugarh (Assam)

E-mail address: **borthakur.tripakshi06@gmail.com**

E-mail address:

Bipin Gogoi: Professor, Department Of Statistics, Dibrugarh University, Dibrugarh (Assam)

E-mail address: **bipin.gogoi@rediffmail.com**