Global and Stochastic Analysis
Vol. 5, No. 2, December (2018), 111-119

MUK PUBLICATIONS
Open Access Publisher

NEW FAMILY OF ONE-STEP PROCESSES ADMITTING SPECIAL INTERPOLATING MARTINGALE MEASURES

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Abstract. In this paper we present new sufficient conditions on parameters of one-step processes implementing noncoincidence barycenter condition (NBC) of martingale measures. The initial random variable of such processes is a constant a and the final one takes countable many different values b_k, $b_k \neq a$. It is proved that if b_k are rational and a is irrational, then there exist martingale measures satisfying NBC. If we mean that the process under consideration is discounted price of a stock, NBC provides the possibility to interpolate the corresponding arbitrage-free financial market to complete one.

1. Introduction

Denote by $Z = (Z_n, \mathcal{F}_n)_{n=0}^1$ a one-step process, where $\mathbf{F} = (\mathcal{F}_\mathbf{0}, \mathcal{F}_\mathbf{1})$ is a one-step filtration, $\mathcal{F}_0 = \{\Omega, \oslash\}$, $\mathcal{F}_1$ is generated by a decomposition of Ω into a countable number of atoms B_k^i ($k \in \mathbb{N} = \{1, 2, \dots\}$, $1 \le i < m_k + 1$, $1 \le m_k \le \infty$), $Z_0 = a, Z_1(B_k^i) = b_k$ (b_k are different real numbers, $b_k \neq a \ \forall k \in \mathbb{N}$). We say that m_k is order of the number b_k. We identify probability measures P on $(\Omega, \mathcal{F}_1)$ with the sequences $(p_k^i, k \in \mathbb{N}, 1 \le i < m_k + 1)$, where $p_k^i = P(B_k^i)$, and consider only measures $P = (p_k^i)$ with strictly positive components (non-degenerate measures). Denote by $\mathcal{P}(Z, \mathbf{F})$ the set of all non-degenerate **martingale** measures. It is obvious that $\mathcal{P}(Z, \mathbf{F}) \neq \oslash$ is equivalent to the inequality $\inf_k b_k < a < \sup_k b_k$. We suppose that this inequality is always satisfied. The set $\mathcal{P}(Z, \mathbf{F})$ coincides with the set of all solutions of the system:

$$\begin{cases} \sum\limits_{k=1}^{\infty} \sum\limits_{i=1}^{m_k} p_k^i = 1 \\ \sum\limits_{k=1}^{\infty} \sum\limits_{i=1}^{m_k} b_k p_k^i = a \\ \sum\limits_{k=1}^{\infty} \sum\limits_{i=1}^{m_k} |b_k| p_k^i < \infty \\ p_k^i > 0, \ k \in \mathbb{N}, \ 1 \le i < m_k + 1. \end{cases} \tag{1.1}$$

Date: Date of Submission May 31, 2017 ; Date of Acceptance August 30, 2017 , Communicated by Yuri E. Gliklikh .

2010 *Mathematics Subject Classification.* Primary 60G42; Secondary 60H30.

Key words and phrases. Martingale measure, special interpolating martingale measures, noncoincidence barycenter condition, arbitrage-free markets, complete markets.

* This research is supported by the RFBR (project 16-01-00184).

Denote $p_k := \sum\limits_{i=1}^{m_k} p_k^i$ and consider the system:

$$\begin{cases} \sum\limits_{k=1}^{\infty} p_k = 1 \\ \sum\limits_{k=1}^{\infty} b_k p_k = a \\ \sum\limits_{k=1}^{\infty} |b_k| p_k < \infty \\ p_k > 0, \ k \in \mathbb{N}. \end{cases} \tag{1.2}$$

Obtaining a solution $(p_k, \ \ k \in \mathbb{N})$ of the system (1.2) and writing down a decomposition $p_k := \sum\limits_{i=1}^{m_k} p_k^i$, we get so a solution of the system (1.1). It is clear that we can get in this manner all the solutions of the system (1.1).

Definition 1.1. We say that a martingale measure $P = (p_k^i)$ satisfies noncoincidence barycenter condition if $\forall l$, $l \in \mathbb{N}$, and for all subsets $J \subset \{(k,i), k \in \mathbb{N}, 1 \leq i < m_k + 1\}$ with finite complementation J^c the following inequalities hold:

$$b_l \neq \frac{\sum\limits_J b_k p_k^i}{\sum\limits_J p_k^i}. \tag{1.3}$$

The set of all martingale measures satisfying noncoincidence barycenter condition is denoted by $\mathcal{SP}(Z, \mathbf{F})$.

In this paper, we present new conditions on the parameters a and b_k under which the set $\mathcal{SP}(Z, \mathbf{F})$ is not empty. In [1] it was proved that if $\forall k \in \mathbb{N}$ $m_k = 1$, b_k increasises monotonically and exponentially, and $b_1 < a < b_2$, then $\mathcal{SP}(Z, \mathbf{F}) \neq \oslash$. The case where the set $\{b_k\}$ is finite but in this set there exist b_k of infinite order was studied in [2] and [3]. The most thoroughly studied case is when the space Ω is finite (c.f. [4]–[6]).

The aim of this paper is to prove the following theorem.

Theorem 1.2. *If number a is irrational and all numbers b_k $(k \in \mathbb{N})$ are rational, then $\mathcal{SP}(Z, \mathbf{F}) \neq \oslash$.*

2. The Main Lemma

We prove first the following technical lemma.

Lemma 2.1. *Let the series with positive terms $\sum_{k=1}^{\infty} \alpha_k = \alpha < \infty$ and $\sum_{k=1}^{\infty} \beta_k = \beta < \infty$ be satisfy the condition: $0 \leq \alpha_k < \beta_k$ $(\forall k \in \mathbb{N})$. If c is a real number, $\alpha < c < \beta$, then there exists a series $\sum_{k=1}^{\infty} c_k$ with rational terms such that $\alpha_k < c_k < \beta_k$ $(\forall k \in \mathbb{N})$ and $\sum_{k=1}^{\infty} c_k = c$.*

Proof. 1) Denote $q_k := \beta_k - \alpha_k$, $k \in \mathbb{N}$, and $r := c - \alpha > 0$. We have: $q := \sum_{k=1}^{\infty} q_k = \beta - \alpha > r$. Prove that there exists a series $\sum_{k=1}^{\infty} r_k$ with rational terms such that $0 < r_k < q_k$ $(\forall k \in \mathbb{N})$ and $\sum_{k=1}^{\infty} r_k = r$.

Let $r < q_1$. Choose a rational r_1, $0 < r_1 < r$, such that $r - r_1 < q_2$. Then find a rational r_2, $0 < r_2 < r - r_1$, in the manner that $r - r_1 - r_2 < q_3$. We have by induction: $0 < r - (r_1 + r_2 + \cdots + r_n) < q_{n+1}$. Hence $\sum_{k=1}^{\infty} r_k = r$, where $0 < r_k < q_k$, $k \in \mathbb{N}$.

Let now $r \geq q_1$. Choose a rational r_1, $0 < r_1 < q_1$, such that $r - r_1 < \sum_{k=2}^{\infty} q_k$. Two cases are possible. Case 1: $r - r_1 < q_2$. Then we put $\tilde{r} := r - r_1$ and (using the reasoning we made under supposition $r < q_1$) we find rational numbers $\tilde{r_2}, \tilde{r_3}, \ldots$ such that $0 < \tilde{r_k} < q_k$ and $\tilde{r} = \sum_{k=2}^{\infty} \tilde{r_k}$. It follows that $r = r_1 + \sum_{k=2}^{\infty} \tilde{r_k}$ as required. Case 2: $r - r_1 \geq q_2$. Choose a rational r_2, $0 < r_2 < q_2$, such that $r - r_1 - r_2 < \sum_{k=3}^{\infty} q_k$. We reason by induction. If there exists an integer n such that $0 < r - (r_1 + r_1 + \cdots + r_n) < q_{n+1}$, then doing the same as in the case 1, we obtain the required series. If for all integer n we have $0 < r - (r_1 + r_1 + \cdots + r_n) < \sum_{k=n+1}^{\infty} q_k$, then tending n to infinity we obtain $r = \sum_{k=1}^{\infty} r_k$.

2) At the first, let us suppose that all numbers α_k are rational. Applying the first part of this proof, we find such rational r_k that $0 < r_k < q_k$ $(\forall k \in \mathbb{N})$ and $\sum_{k=1}^{\infty} r_k = r$. Denote $c_k = \alpha_k + r_k$, $k \in \mathbb{N}$. It is obvious that the numbers c_k are rational. We have: $\alpha_k < c_k = \alpha_k + r_k < \alpha_k + q_k = \alpha_k + (\beta_k - \alpha_k) = \beta_k$, $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} r_k = \alpha + r = c$.

In general, let α_k be real numders (not necessary rational). Fix rational numbers $\tilde{\alpha}_k$ such that $\forall k \in \mathbb{N}$ $\alpha_k < \tilde{\alpha}_k < \beta_k$ and $\tilde{\alpha}_k - \alpha_k < (c - \alpha)2^{-k-1}$. Denote $\tilde{\alpha} := \sum_{k=1}^{\infty} \tilde{\alpha}_k$. We have:

$$0 < \sum_{k=1}^{\infty} (\tilde{\alpha}_k - \alpha_k) < \frac{c - \alpha}{2} \;\Leftrightarrow\; 0 < \tilde{\alpha} - \alpha < \frac{c - \alpha}{2} \;\Leftrightarrow$$

$$\alpha < \tilde{\alpha} < \frac{c - \alpha}{2} + \alpha = \frac{c + \alpha}{2} < c.$$

It remains to apply the arguments from the beginning of the part 2) of the proof for the series $\sum_{k=1}^{\infty} \tilde{\alpha}_k$, $\sum_{k=1}^{\infty} \beta_k$ and number c. Q.E.D.

$\square$

3. Proof of the Main Result

In this section we suppose that numbers b_k, $k \in \mathbb{N}$ are rational and a is a real number.

To prove theorem 1.2 we need some auxiliary results. We say that a solution $(p_1, p_2, \ldots)$ of the system (1.2) is rational if all the components of this solution are rational numbers. We shell use further the following trivial

Lemma 3.1. *Let $b_k > 0$, $k \in \mathbb{N}$. The system (1.2) has a rational solution if there exist an increasing subsequence of integers $(k_j \uparrow \infty, j \geq 0, k_0 = 0)$, a sequence $(r_j > 0, j \in \mathbb{N})$ of rational numbers such that $\sum_{j=1}^{\infty} r_j = 1$, and a sequence $(a_j > 0, j \in \mathbb{N})$ of rational numbers such that $\sum_{j=1}^{\infty} a_j = a$ possessing the following property: $\forall j \in \mathbb{N}$ the system*

$$\begin{cases} \sum\limits_{i=k_{j-1}+1}^{k_j} p_i = r_j \\ \sum\limits_{i=k_{j-1}+1}^{k_j} b_i p_i = a_j \\ p_i > 0 \;\; (k_{j-1} + 1 \leq i \leq k_j) \end{cases} \tag{3.1}$$

admits a rational solution.

Proof. The proof is omitted. □

Lemma 3.2. *The system* (3.1) *admits a rational solution iff* $\forall j \in \mathbb{N}$

$$\min_{k_{j-1}+1 \le i \le k_j} b_i < \frac{a_j}{r_j} < \max_{k_{j-1}+1 \le i \le k_j} b_i. \tag{3.2}$$

Proof. The proof is omitted. □

Proposition 3.3. *If* $\inf_k b_k > -\infty$ *and* $\sup_k b_k = +\infty$*, then the system* (1.2) *has a rational solution.*

Proof. Without loss of generality let us suppose that $\forall k \in \mathbb{N}$ $b_k > 0$ and $b_1 < a$. Define a rational number a_1 so that $b_1 < a_1 < a$. The remaining rational a_j $(j \ge 2)$ are chosen arbitrarily but in such a way that $\sum_{j=1}^{\infty} a_j = a$. Put $r'_j = \frac{1}{2^j}$. Since $\sup_k b_k = \infty$ we can find a number k_1 such that

$$\frac{a_1}{r'_1} < b_{k_1}. \tag{3.3}$$

Now consider the fraction $\frac{a_2}{r'_2}$. Replacing, if necessary, r'_2 by a rational r''_2 $(0 < r''_2 < r'_2)$, we achieve that $\frac{a_2}{r''_2} > b_{k_1+1}$. Let us find $k_2 > k_1 + 1$ such that $\frac{a_2}{r''_2} < b_{k_2}$. As a result we obtain the double inequality $b_{k_1+1} < \frac{a_2}{r''_2} < b_{k_2}$. Continuing in the same manner, we get for $j \ge 2$:

$$b_{k_{j-1}+1} < \frac{a_j}{r''_j} < b_{k_j}. \tag{3.4}$$

Consider two converging positive series $\sum_{j=2}^{\infty} \frac{a_j}{b_{k_j}}$ and $\sum_{j=2}^{\infty} r''_j$. It follows from (3.4) that $\forall j \ge 2$ $\frac{a_j}{b_{k_j}} < r''_j$. Using lemma 2.1, we find the series $\sum_{j=2}^{\infty} r_j$ such that $\forall j \ge 2$ $\frac{a_j}{b_{k_j}} < r_j < r''_j$ and $\sum_{j=2}^{\infty} r_j$ is a rational number. Hence, $\forall j \ge 2$ the inequality $\frac{a_j}{r_j} < b_{k_j}$ is fulfilled. Since $r_j < r''_j$, by (3.4) $\forall j \ge 2$ the inequality $b_{k_{j-1}+1} < \frac{a_j}{r_j}$ is true too. Thus $\forall j \ge 2$

$$b_{k_{j-1}+1} < \frac{a_j}{r_j} < b_{k_j}. \tag{3.5}$$

Denote by r_1 the rational number $1 - \sum_{j=2}^{\infty} r_j$. By construction, $0 < \sum_{j=2}^{\infty} r_j < \sum_{j=2}^{\infty} r'_j = 1 - r'_1$, so $r'_1 < 1 - \sum_{j=2}^{\infty} r_j = r_1$. Since $b_1 < a_1$ and $0 < r_1 < 1$, we have $b_1 < \frac{a_1}{r_1}$. On the other hand $b_{k_1} > \frac{a_1}{r_1}$ because of $r_1 > r'_1$ and (3.3). Therefore

$$b_1 < \frac{a_1}{r_1} < b_{k_1}. \tag{3.6}$$

From (3.6) and (3.5) it follows that the inequalities (3.2) are fulfilled. Applying lemmas 3.2 and 3.1, we obtain that the system (1.2) has a rational solution.

□

Proposition 3.4. *If* $\inf_k b_k = -\infty$ *and* $\sup_k b_k = +\infty$*, then the system* (1.2) *has a rational solution.*

Proof. Without loss of generality we can suppose that $\forall k \in \mathbb{N}\ b_{2k-1} \leq 0$, $b_{2k} > 0$, $a > 0$ and there exists an integer n such that $0 < b_{2n} < a$. Reduce the system (1.2) to the form:

$$\begin{cases} \sum\limits_{k=1}^{\infty} p_{2k-1} + \sum\limits_{k=1}^{\infty} p_{2k} = \varepsilon + (1-\varepsilon) \\ \sum\limits_{k=1}^{\infty} b_{2k-1}p_{2k-1} + \sum\limits_{k=1}^{\infty} b_{2k}p_{2k} = -1 + (a+1) \\ p_k > 0,\ k \in \mathbb{N}, \end{cases} \tag{3.7}$$

where ε is a small positive rational number and

$$\inf_k b_{2k} \leq b_{2n} < a < \sup_k b_{2k} = \infty. \tag{3.8}$$

Now consider two systems:

$$\begin{cases} \sum\limits_{k=1}^{\infty} p_{2k-1} = \varepsilon \\ \sum\limits_{k=1}^{\infty} (-b_{2k-1})p_{2k-1} = 1 \\ p_{2k-1} > 0,\ k \in \mathbb{N}, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \sum\limits_{k=1}^{\infty} p_{2k} = 1 - \varepsilon \\ \sum\limits_{k=1}^{\infty} b_{2k}p_{2k} = a + 1 \\ p_{2k} > 0,\ k \in \mathbb{N}. \end{cases} \tag{3.10}$$

The system (3.9) is equivalent to the system

$$\begin{cases} \sum\limits_{k=1}^{\infty} \frac{p_{2k-1}}{\varepsilon} = 1 \\ \sum\limits_{k=1}^{\infty} (-b_{2k-1})\frac{p_{2k-1}}{\varepsilon} = \frac{1}{\varepsilon} \\ p_{2k-1} > 0,\ k \in \mathbb{N}. \end{cases} \tag{3.11}$$

Solvability condition $\inf_k(-b_{2k-1}) < \frac{1}{\varepsilon} < \sup_k(-b_{2k-1}) = \infty$ is fulfilled for sufficiently small ε. Hence by proposition 3.3 the system (3.11) admits rational solutions.

The system (3.10) is equivalent to the system

$$\begin{cases} \sum\limits_{k=1}^{\infty} \frac{p_{2k}}{1-\varepsilon} = 1 \\ \sum\limits_{k=1}^{\infty} b_{2k}\frac{p_{2k}}{1-\varepsilon} = \frac{a+1}{1-\varepsilon} \\ p_{2k} > 0,\ k \in \mathbb{N}. \end{cases} \tag{3.12}$$

It follows from (3.8) that solvability condition $\inf_k b_{2k} < \frac{a+1}{1-\varepsilon} < \sup_k b_{2k} = \infty$ is also fulfilled and by proposition 3.3 the system (3.12) admits rational solutions.

Consequently, the system (1.2) has rational solutions. Q.E.D. $\square$

Proposition 3.5. *If $c := \sup_k |b_k| < \infty$ and the number a is rational, then the system* (1.2) *has a rational solution.*

Proof. Without loss of generality we suppose that $a = 0$, $b_1 < 0$ and the set $B := \{b_k : b_k > 0\}$ is infinite.

1) Suppose that the set B contains an increasing sequence. Fix such a sequence $(b_{k_j})_{j=1}^{\infty}$ with the following additional property: for all $j \in \mathbb{N}$ there exists a number $b_{i_{j+1}} \in B$ such that $b_{k_j} < b_{i_{j+1}} < b_{k_{j+1}}$. Let $\varepsilon > 0$ be a rational number. Consider a series $\sum\limits_{j=2}^{\infty} r_j$ with positive rational terms and with the sum that is equal to ε. Denote $r_1 := 1 - \varepsilon$. Then $\sum\limits_{j=2}^{\infty} r_j b_{k_j} < c\varepsilon$. Choose ε such that $c\varepsilon < (1-\varepsilon)|b_1|$.

Consider the series $\sum\limits_{j=2}^{\infty} r_j b_{i_j}$ and $\sum\limits_{j=2}^{\infty} r_j b_{k_j}$. Using lemma 2.1 find a series $\sum\limits_{j=2}^{\infty} a_j$ with rational terms, where $r_j b_{i_j} < a_j < r_j b_{k_j}$ and the sum $\sum\limits_{j=2}^{\infty} a_j$ is rational. Denote $a_1 := -\sum\limits_{j=2}^{\infty} a_j < 0$. We have:

$$\frac{-a_1}{r_1} = \frac{\sum\limits_{j=2}^{\infty} a_j}{1-\varepsilon} < \frac{\sum\limits_{j=2}^{\infty} r_j b_{k_j}}{1-\varepsilon} < \frac{(1-\varepsilon)|b_1|}{1-\varepsilon} = |b_1| = -b_1 \Leftrightarrow b_1 < \frac{a_1}{r_1}. \tag{3.13}$$

Consider now the systems (3.1). For $j = 1$ the solvability condition $b_1 < \frac{a_1}{r_1} < b_{k_1}$ follows from (3.13) and from positivity of b_{k_1}. For $j \geq 2$ we have:

$$\min_{k_{j-1}+1 \leq i \leq k_j} b_i < b_{i_j} < \frac{a_j}{r_j} < b_{k_j} < \max_{k_{j-1}+1 \leq i \leq k_j} b_i.$$

Applying lemmas 3.1 and 3.2, we obtain that the system (1.2) admits rational solutions.

2) Suppose that the set B contains a decreasing sequence. Here arguments are similar to those that were given in 1). Q.E.D. □

Lemma 3.6. *Let $b_k > 0$, $k \in \mathbb{N}$. Then the the system* (1.2) *has a rational solution if there exist a subsequence of integers $(k_j \uparrow \infty, j \geq 0, k_0 = 0)$, two sequences of rational numbers $(r_j > 0, j \in \mathbb{N})$ with $\sum_{j=1}^{\infty} r_j = 1$ and $(\lambda_j > 0, j \in \mathbb{N})$ with $0 < \lambda_j < 1$, and two sequences (b'_j), (b''_j), $b'_j < b''_j$, $b'_j, b''_j \in \{b_{k_{j-1}+1}, \dots, b_{k_j}\}$ possessing the property: $\sum_{j=1}^{\infty} r_j[\lambda_j b'_j + (1-\lambda_j)b''_j] = a$.*

Proof. The proof is omitted. □

Proposition 3.7. *If $\sup_k |b_k| < \infty$ and a is irrational, then the system* (1.2) *has a rational solution.*

Proof. Without loss of generality we can suppose that $\forall k \in \mathbb{N}\ b_k > 0$. Denote a limit point of the set $\{b_k\}_{k=1}^{\infty}$ by b. Consider several cases.

1) Let $a < b$ and there exists a subsequence $\{b'_i\}_{i=1}^{\infty} \subset \{b_k\}_{k=1}^{\infty}$ that converges to b on the left. We can assume that $b'_1 = b_1 < a < b'_2 < b'_3 < \cdots \to b$; $b'_j = b_{k_{j-1}+1}$, $j \in \mathbb{N}$; between b'_j and b'_{j+1} there exists a term $b''_j \in \{b_k\}_{k=1}^{\infty}$ such

that $b'_j < b''_j < b'_{j+1}$; $a < b''_1$. Denote $d_j = b''_j - b'_j > 0$. Let r ($b'_1 < r < a$) be a rational number. Consider the system:

$$\begin{cases} \sum\limits_{j=1}^{\infty} r_j = 1 \\ \sum\limits_{j=1}^{\infty} b'_j r_j = r \\ r_j > 0,\ j \in \mathbb{N}. \end{cases} \tag{3.14}$$

It follows from proposition 3.5 that the system (3.14) has a rational solution. Denote it by $(r_j)_{j=1}^{\infty}$. It is clear that $\sum\limits_{j=1}^{\infty} r_j d_j < \infty$. We have:

$$\sum_{j=1}^{\infty} r_j d_j = \sum_{j=1}^{\infty} r_j b''_j - \sum_{j=1}^{\infty} r_j b'_j > a - r.$$

By lemma 2.1 find the series $\sum\limits_{j=1}^{\infty} c_j$ with rational positive terms such that $c_j < r_j d_j$ and $\sum\limits_{j \to \infty} c_j = a - r$. Put $\lambda_j = 1 - \frac{c_j}{r_j d_j}$. It is clear that $0 < \lambda_j < 1$. Then

$$\sum_{j=1}^{\infty} r_j [\lambda_j b'_j + (1 - \lambda_j) b''_j] = \sum_{j=1}^{\infty} [r_j b'_j + c_j] = r + (a - r) = a.$$

Using lemma 3.6, we obtain the required result.

2) The case when $a \leq b$ and there exists a subsequence $\{\tilde{b}_i\}_{i=1}^{\infty} \subset \{b_k\}_{k=1}^{\infty}$ that converges to b on the right can be proved similarly.

3) The case $a \geq b$ is reduced to previous ones. Q.E.D. □

Now we can prove theorem 1.2.

Proof. It follows from propositions 3.3, 3.4 and 3.7 that system (1.2) has a rational solution. Fix such a solution $(p_1, p_2, \dots)$ and decompose p_k $\forall k \in \mathbb{N}$ in the form $p_k = \sum\limits_{i=1}^{m_k} p_k^i$, where p_k^i are rational numbers. We obtain a rational solution $P = \{p_k^i,\ k \in \mathbb{N}, 1 \leq i < m_k + 1\}$ of system (1.1). Represent condition (1.3) in the form

$$b_j \neq \frac{a - \sum\limits_T b_k p_k^i}{1 - \sum\limits_T p_k^i}, \tag{3.15}$$

$\forall j \in \mathbb{N}, \forall T \subset \{(k, i), k \in \mathbb{N}, 1 \leq i < r_k + 1\}$, T is finite. Since a is irrational and the numbers b_l, b_k, p_k^i are rational, the inequality (3.15) is fulfilled. Q.E.D. □

4. Application to mathematical finance

The importance of measures from $\mathcal{SP}(Z, \mathbf{F})$ consists in the following. Transform $P = \{p_k^i,\ k \in \mathbb{N}, 1 \leq i < m_k + 1\}$ to a sequence $(q_1, q_2, \dots)$. For any permutation $\{k_1, \dots, k_n, \dots\}$ of $\{1, 2, \dots, n, \dots\}$ introduce interpolating special Haar filtration (ISHF) of $\mathbf{F} = (\mathcal{F}_{\mathbf{0}}, \mathcal{F}_{\mathbf{1}})$ in the manner:

$\mathcal{H}_0 = \mathcal{F}_0$,

$\mathcal{H}_1 = \sigma\{B_{k_1}\}$,
. . . .
$\mathcal{H}_n = \sigma\{B_{k_1}, B_{k_2}, \ldots, B_{k_n}\}$,
. . . .
$\mathcal{H}_\infty = \mathcal{F}_1$.

Let $P \in \mathcal{P}(Z, \mathbf{F})$ and consider $Y_n := E^P[Z_1|\mathcal{H}_n]$. Then the process $Y = (Y_n, \mathcal{H}_n)_{n=0}^{\infty}$ is called a special martingale Haar interpolation of Z. We say that $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies uniqueness property if for $\mathbf{F}$ and *every* ISHF $\mathbf{H} = (\mathcal{H}_\mathbf{n})_{\mathbf{n=0}}^{\infty}$ $|\mathcal{P}(Y, \mathbf{H})| = \mathbf{1}$ (Y is a martingale *only* with respect to the initial measure P). It is easy to see that $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies uniqueness property if and only if $P \in \mathcal{SP}(Z, \mathbf{F})$. This fact can be used in the transformation of arbitrage-free financial (B,S)-markets to complete ones (c.f. [1], [4], [5], [6]).

References

1. Pavlov, I.V., Tsvetkova, I.V., Shamrayeva V.V.: Some results on martingale measures of static financial markets models relating noncoincidence barycenter condition, *Vestn. Rostov Gos. Univ. Putei Soobshcheniya* **45**, N3 (2012) 177–181.
2. Pavlov, I.V., Tsvetkova, I.V., Shamrayeva V.V.: On the existence of martingale measures satisfying weakened noncoincidence barycenter condition: constructivist approach, *Vestn. Rostov Gos. Univ. Putei Soobshcheniya* **47**, N4 (2014) 132–138.
3. Pavlov, I.V., Tsvetkova, I.V., Shamrayeva V.V.: On the existence of martingale measures satisfying the weakened condition of noncoincidence of barycenters in the case of countable probability space, *Theory Probab. Appl.* **61**, N1 (2017) 167-175.
4. Bogacheva, M.N., Pavlov, I.V.: Haar extensions of arbitrage-free financial markets to markets that are complete and arbitrage-free, *Russian Math. Surveys* **57**, N3 (2002) 581–583.
5. Bogacheva, M.N., Pavlov, I.V.: Haar extensions of arbitrage-free financial markets to markets that are complete and arbitrage-free, *Izvestiya VUZov, Severo-Kavkaz. Region, Estestvenn. Nauki*, N3 (2002) 16–24. I
6. Gorgorova, V.V., Pavlov, I.V.: On Haar uniqueness properties for vector-valued random processes, *Russian Math. Surveys* **62**, N6 (2007) 1202–1203.

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