Global and Stochastic Analysis
Vol. 5, No. 2, December (2018), 101-109

MUK PUBLICATIONS
Open Access Publisher

ON EXISTENCE OF OPTIMAL SOLUTIONS TO STOCHASTIC DIFFERENTIAL INCLUSIONS OF GEOMETRIC BROWNIAN MOTION TYPE WITH CURRENT VELOCITIES

YURI E. GLIKLIKH & OLGA O. ZHELTIKOVA

Abstract. We investigate the so-called stochastic inclusions of geometric Brownian motion type that are natural generalizations of the equations describing processes of geometric Brownian motion. The latter processes are frequently used in mathematical models of economy. The new point here is that the inclusions under consideration are given in terms of Nelson's mean derivatives, namely the current velocities – symmetric mean derivatives that are natural analogues of ordinary velocities of deterministic processes. We prove the existence of solutions for those inclusions that minimize a certain cost criterion. If the inclusion is obtained from the equation with feedback control, this result yields the existence of control that realises for the equation the minimizing solution of the corresponding inclusion.

1. Introduction

The notion of mean derivative (forward, backward, symmetric and antisymmetric) is introduced by E. Nelson (see [1, 2, 3]). In particular, forward derivatives give information about the drift of an Itô diffusion type process. Later, in [4], on the basis of some Nelson's idea, an additional mean derivative (called quadratic) giving information about the diffusion term, was introduced. After that it became in principle possible to recover a stochastic process from its mean derivatives. The differential equations and inclusions with mean derivatives arise in many subjects, especially in mathematical physics (the first example was the so-called Newton-Nelson equation, describing the motion of a quantum particle, see [1, 2, 3]). Many examples can be found, e.g., in [5]. The most important (but also the most complicated for investigation) case is the equations with symmetric derivatives (called current velocities) since they are the direct analogs of ordinary physical velocities of deterministic processes.

Inclusions of geometric Brownian motion type in terms of forward mean derivatives are introduced and investigated in [6] as a natural generalization of the well-known equation that describes the so-called geometric Brownian motion, the process that is frequently used in mathematical models of economy. Transition to the

Date: Date of Submission June 14, 2017 ; Date of Acceptance September 29, 2017 , Communicated by Yana I. Belopolskaya .

2010 *Mathematics Subject Classification.* Primary 60H10; Secondary 60H30; 60G99.

Key words and phrases. Mean derivatives; current velocity; inclusions of geometric Brownian motion type; optimal solutions.

* The work is supported in part by RFBR Grant 15-01-00620.

corresponding inclusions allows one to apply the methods of set-valued analysis to investigate optimal control problems for controlled equations with feedback and many others.

In this paper there are several new points that make the problem much more complicated and more convenient for applications. First of all, here we deal with inclusions of geometric Brownian motion type given in terms of current velocities (symmetric mean derivatives). It is important since current velocities are natural analogues of ordinary velocity of deterministic processes. That is why it is not surprising that there are a lot of economical models given in terms of current velocities (see e.g. [7]). But equations and inclusions with current velocities are difficult for investigation. In particular here we have to consider the inclusions on flat n-dimensional torus, not in a linear space – otherwise we cannot prove the existence of solutions for equations with the single-valued approximations and so for the inclusions. The second new point is that here the right-hand sides of equations and inclusions are not autonomous that requires serious modification of proofs.

Some remarks on notation. Vectors in $\mathbb{R}^n$ are considered as coordinate columns. If X is such a vector, the transposed row vector is denoted by X^*. Linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ are represented as $n\times n$ matrices, the symbol $*$ means transposition of a matrix (pass to the matrix of conjugate operator). The space of $n\times n$ matrices is denoted by $L(\mathbb{R}^n,\mathbb{R}^n)$. By $\mathrm{S}(n)$ we denote the linear space of symmetric $n\times n$ matrices that is a subspace in $L(\mathbb{R}^n,\mathbb{R}^n)$. The symbol $\mathrm{S}_+(n)$ denotes the set of positive definite symmetric $n\times n$ matrices that is a convex open set in $\mathrm{S}(n)$. Its closure, i.e., the set of positive semi-definite symmetric $n\times n$ matrices, is denoted by $\bar{\mathrm{S}}_+(n)$.

2. Preliminaries on the mean derivatives

Let $\xi(t)$ be a stochastic process in $\mathbb{R}^n$, $t\in[0,T]$, that is given on a certain probability space $(\Omega,\mathcal{F},\mathsf{P})$ so that $\xi(t)$ is L_1-random variable for all t.

Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t\in[0,T]$, determines three families of σ-subalgebras of σ-algebra $\mathcal{F}$:

(i) the "past" $\mathcal{P}^\xi_t$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s):\Omega\to\mathbb{R}^n$ for $0\le s\le t$;

(ii) the "future" $\mathcal{F}^\xi_t$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s):\Omega\to\mathbb{R}^n$ for $t\le s\le T$;

(iii) the "present" ("now") $\mathcal{N}^\xi_t$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.

All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}^\xi_t$ by $E^\xi_t(\cdot)$.

Ordinary ("unconditional") expectation is denoted by E.

Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all t. Thus its "classical" derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [1, 2, 3]) we give

Definition 2.1. (i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time t is an L_1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^{\xi}\left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t}\right) \tag{2.1}$$

where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0 and $\Delta t > 0$.

(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at t is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^{\xi}\left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t}\right) \tag{2.2}$$

where the conditions and the notation are the same as in (i).

Note that mainly $D\xi(t) \neq D_*\xi(t)$, but if, say, $\xi(t)$ a.s. has smooth sample paths, these derivatives evidently coinside.

From the properties of conditional expectation (see [8]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$\begin{aligned} Y^0(t,x) &= \lim_{\Delta t \to +0} E\left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} \Big| \xi(t) = x\right) \\ Y^0_*(t,x) &= \lim_{\Delta t \to +0} E\left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} \Big| \xi(t) = x\right) \end{aligned} \tag{2.3}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y^0_*(t, \xi(t))$.

Definition 2.2 ([1, 5])**.** The derivative $D_S = \frac{1}{2}(D + D_*)$ is called symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called anti-symmetric mean derivative .

Consider the vector fields

$$v^{\xi}(t,x) = \frac{1}{2}(Y^0(t,x) + Y^0_*(t,x))$$

and

$$u^{\xi}(t,x) = \frac{1}{2}(Y^0(t,x) - Y^0_*(t,x)).$$

Definition 2.3. $v^{\xi}(t) = v^{\xi}(t, \xi(t)) = D_S\xi(t)$ is called current velocity of $\xi(t)$;
$u^{\xi}(t) = u^{\xi}(t, \xi(t)) = D_A\xi(t)$ is called osmotic velocity of $\xi(t)$.

For stochastic processes the current velocity is a direct analogue of ordinary physical velocity of deterministic processes (see, e.g., [1, 2, 3, 5]). The osmotic velocity measures how fast the "randomness" grows up.

Following [4, 5] we introduce the differential operator D_2 that differentiates an L_1 random process $\xi(t)$, $t \in [0, T]$ according to the rule

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\left(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\right), \tag{2.4}$$

where $(\xi(t+\triangle t) - \xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t) - \xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$. We emphasize that this matrix product of a column on the left and a row on the right is a matrix of rank 1 but after passing to the limit, $D_2\xi(t)$ becomes a symmetric positive semi-definite matrix function on $[0,T]\times\mathbb{R}^n$. We call D_2 the quadratic mean derivative. For the diffusion process $\xi(t)$ it is proven that the regression of $D_2\xi(t)$ coincides with the diffusion coefficient.

3. Inclusions of geometric Brownian motion type with current velocities

A set-valued mapping F from a set X into a set Y is a correspondence that assigns a non-empty subset $F(x)\subset Y$ to every point $x\in X$; $F(x)$ is called the image of x. If in the right-hand side of a differential equation the single-valued function is replaced by a set-valued one, the equation is transformed into an iclusion.

Some preliminaries from set-valued analysis in the amount, enough for our purpose, can be found, e.g, in [5]. For convenience we recall the following two notions.

Definition 3.1. A set-valued mapping F is called upper semicontinuous at the point $x\in X$ if for each $\varepsilon>0$ there exists a neighbourhood $U(x)$ of x such that from $x'\in U(x)$ it follows that $F(x')$ belongs to the ε-neighbourhood of the set $F(x)$. F is called upper semicontinuous on X if it is upper semicontinuous at every point of X

Definition 3.2. Let X and Y be metric spaces. For a given $\varepsilon>0$ a continuous single-valued mapping $f_\varepsilon : X\to Y$ is called an ε-approximation of the set-valued mapping $F : X\to Y$, if the graph of f, as a set in $X\times Y$, lies in ε-neighbourhood of the graph of F.

Let $\mathbf{v}(t,m)$ be a set-valued vector field and $\mathbf{B}(t,m)$ be a set-valued symmetric positive semi-definite $(2,0)$-tensor field on $\mathcal{T}^n$. The system of the form

$$\left\{\begin{array}{l} D_S\xi(t)+\frac{1}{2}\mathrm{diag}D_2\xi(t)\in\mathbf{v}(t,\xi(t)),\\ D_2\xi(t)\in\mathbf{B}(t,\xi(t)).\end{array}\right. \tag{3.1}$$

is called a first order differential inclusion of geometric Brownian motion type with current velocities.

By some technical reasons we consider inclusions of (3.1) type on the flat n-dimensional torus $\mathcal{T}^n$. This means that the torus is obtained as a quotient space of $\mathbb{R}^n$ relative to the integral lattice and that the Riemannian metric on $\mathcal{T}^n$ is inherited from the Euclidean metric in $\mathbb{R}^n$. Everywhere below we use the operations of addition and subtraction of points and integration in $\mathcal{T}^n$ as in $\mathbb{R}^n$ modulo factorization relative to the integral lattice. Note that for such processes the notions of "past", "present" and "future" σ algebras are the same as for $\mathbb{R}^n$. The construction and notation of stochastic integrals and stochastic differential equations on $\mathcal{T}^n$ are the same as in $\mathbb{R}^n$ because of the use of Euclidean metric.

Definition 3.3. We say that (3.1) on $\mathcal{T}^n$ has a solution on $[0,T]$ with initial condition $\xi(0)=\xi_0$ if there exists a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a process $\xi(t)$ given on $(\Omega,\mathcal{F},\mathsf{P})$ and taking values in $\mathcal{T}^n$ such that $\xi(0)=\xi_0$ and for almost all $t\in[0,T]$ equation (3.1) is satisfied P-a.s. by $\xi(t)$.

Definition 3.4 ([9])**.** The perfect solution of (3.1) is a stochastic process with continuous sample paths such that it is a solution in the sense of Definition 3.3 and the measure corresponding to it on the space of continuous curves, is a weak limit of measures generated by solutions of a sequence of equations of the form analogous to (3.1), with continuous coefficients.

Theorem 3.5. *Let the set-valued vector field* $\mathbf{v}(t,m)$ *and the set-valued* $(2,0)$*-tensor field* $\mathbf{B}(t,m)$*, taking values in symmentric positive definite matrices, on* $\mathcal{T}^n$ *be uniformly bounded, have closed convex images and be upper-semicontinuous. Then for any initial condition* $\xi(0)=\xi_0$ *with smooth density that nowhere equals zero, inclusion (3.1) has a perfect solution well-posed on the entire interval* $t\in[0,T]$.

Proof. First of all, by [5, Theorem 4.11], under the hypothesis of the Theorem, for every sequence of positive numbers $\varepsilon_q\to 0$ there exists a sequence of single-valued continuous ε_q-approximations $v_q(t,m)$ of $\mathbf{v}(t,m)$ ($B_q(t,m)$ of $\mathbf{B}(t,m)$) that point-wise converges to a Borel measurable selector $v(t,m)$ of $\mathbf{v}(t,m)$ ($B(t,m)$ of $\mathbf{B}(t,m)$, respectively). Then $v_q(t,m)-\frac{1}{2}\mathrm{diag}B_q(t,m)$ point-wise converges to a Borel measurable selector $v(t,m)-\frac{1}{2}\mathrm{diag}B(t,m)$. Without loss of generality we can suppose those ε_q-approximations to be smooth.

Consider the sequence of equations

$$\left\{\begin{array}{rcl} D_S\xi(t)+\frac{1}{2}\mathrm{diag}B_q & = & v_q(t,\xi(t)) \\ D_2\xi(t) & = & B_q(t,\xi(t)) \end{array}\right. . \tag{3.2}$$

We consider the same initial condition ξ_0 for all those equations. Note that all v_q and B_q are uniformly bounded by the same constant since they are ε-approximations of uniformly bounded set-valued mappings. Since all those ε_q-approximations are at least C^1-smooth and given on the compact torus, their first partial derivatives are uniformly bounded for every k (by a constant depending on k). Thus all equations (3.2) satisfy the hypothesis of [10, Theorem 3], i.e. for every equation there exists a solution. We denote by $\xi_q(t)$ the solution of the q-th equation.

Introduce on $(C^0([0,T],\mathcal{T}^n)$ the σ-algebra $\mathcal{C}$ generated by cylinder sets. By $\mathcal{P}_t$ we denote the σ-algebra generated by cylinder sets with bases in $[0,t]$. Its restriction to the time instant t is denoted by $\mathcal{N}_t$. By [11, Lemma 3] the set $\{\mu_q\}$ of measures on $(C^0([0,T],\mathcal{T}^n),\mathcal{C})$, corresponding to ξ_q, is weakly compact. Hence, one can choose a subsequence that weakly converges so a certain measure μ. Without loss of generality we can suppose that the sequence μ_q weakly converges to μ. Consider the coordinate process $\xi(t)$ on the probability space $(C^0([0,T],\mathcal{T}^n),\mathcal{C},\mu)$, i.e. for every elementary event $x(\cdot)\in C^0([0,T],\mathcal{T}^n)$ by definition $\xi(t,x(\cdot))=x(t)$. Recall that $\mathcal{P}_t$ is the "past" for $\xi(t)$, while $\mathcal{N}_t$ is the "present" for this coordinate process.

If for any specified t we introduce $B_q(t,m(\cdot))=B_q(t,m(t))$ and $B(t,m(\cdot))=B(t,m(t))$, we obtain that $B_q(t,m(t))$ and $B(t,m(t))$ can be considered as given on $C^0([0,T],\mathcal{T}^n)$.

By the construction, for every $\xi_q(t)$ its quadratic derivative equals $B_q(t,\xi_q(t))$. This means that for every bounded continuous real function f on $C^0([0,T],\mathcal{T}^n)$

that is measurable with respect to $\mathcal{N}_t$, the equality

$$\lim_{\Delta t\to 0}\int_{C^0([0,T],\mathcal{T}^n)}[\frac{(m(t+\Delta t)-m(t))(m(t+\Delta t)-m(t))^*}{\Delta t}$$
$$-B_q(t,m(t))]f(m(\cdot))d\mu_q = 0$$

holds.

Since $B_q(t,m)$ pointwise tends to $B(t,m)$ as $q\to\infty$, $B_q(t,m(t)) = B_q(t,m(\cdot))$ tends to $B(t,m(t)) = B(t,m(\cdot))$ a.s. on $C^0([0,T],\mathcal{T}^n)$ with respect to all measures μ_q and with respect to μ. Specify $\delta > 0$. By Egorov's theorem (see, e.g., [12]) for every i there exists a subset $\tilde{K}^i_\delta \subset C^0([0,T],\mathcal{T}^n)$ such that $(\mu_i)(\tilde{K}^i_\delta) > 1-\delta$ and the sequence $B_q(m(t))$ on $\tilde{K}^i_\delta$ converges to $B(m(t))$ uniformly. Introduce $\tilde{K}_\delta = \bigcup\limits_{i=0}^{\infty}\tilde{K}^i_\delta$. The sequence $B_q(t,m(t))$ on $\tilde{K}_\delta$ for all i converges to $B(t,m(t))$ uniformly and $\mu(\tilde{K}_\delta) > 1-\delta$.

The field $B(t,m(t))$ is continuous on a set of complete measure μ on $C^0([0,T],\mathcal{T}^n)$. Indeed, consider the sequence $\delta_i\to 0$ and the corresponding sequence $\tilde{K}_{\delta_i}$. By construction, $B(m(t))$ is a uniform limit of the sequence of continuous functions on every $\tilde{K}_{\delta_i}$. That is why $B(t,m(t))$ is continuous on every $\tilde{K}_{\delta_i}$, i.e. on any finite union $\bigcup\limits_{i=1}^{n}\tilde{K}_{\delta_i}$. Evidently $\lim\limits_{n\to\infty}\mu(\bigcup\limits_{i=1}^{n}\tilde{K}_{\delta_i}) = 1$.

Taking into account the uniform convergence on $\tilde{K}_\delta$ for all k (see above) we derive from boundedness of $f(m(\cdot))$ that for q large enough

$$\|\int_{\tilde{K}_\delta}[B_q(t,m(t)) - B(t,m(t))]f(m(\cdot))d\mu_q\| < \delta.$$

Since $f(m(\cdot))$ is bounded, there exists a certain number $\Xi > 0$ such that $|f(m(\cdot))| < \Xi$ for all $m(\cdot)$. Recall that all $B_q(t,m)$ and $B(t,m)$ are uniformly bounded, i.e., their norms are not greater than some number $Q > 0$. Then, since

$$\mu_q(C^0([0,T],\mathcal{T}^n)\backslash\tilde{K}_\delta) < \delta$$

for all q large enough,

$$\|\int_{C^0([0,T],\mathcal{T}^n)\backslash\tilde{K}_\delta}[B_q(t,m(t)) - B(t,m(t))]f(m(\cdot))d\mu_q\| < 2\delta Q\Xi$$

for all q large enough. Since δ is an arbitrary positive number,

$$\lim_{q\to\infty}\int_{C^0([0,T],\mathcal{T}^n)}[B_q(t,m(t)) - B(t,m(t))]f(m(\cdot))d\mu_q = 0.$$

The function $B(t,m(t))$ is μ-a.s. continuous and bounded on $C^0([0,T],\mathcal{T}^n)$ (see above). Since in addition the measures μ_q weakly converge to μ, by Lemma from [13, section VI.1]

$$\lim_{k\to\infty}\int_{C^0([0,T],\mathcal{T}^n)}B(t,m(t))f(m(\cdot))d\mu_q = \int_{C^0([0,T],\mathcal{T}^n)}B(t,m(t))f(m(\cdot))d\mu.$$

Evidently

$$\lim_{k\to\infty}\int_{C^0([0,T],\mathcal{T}^n)}[\frac{(m(t+\Delta t)-m(t))(m(t+\Delta t)-m(t))^*}{\Delta t}]f(m(\cdot))d\mu_q$$

$$= \int_{C^0([0,T],\mathcal{T}^n)} [\frac{(m(t+\Delta t)-m(t))(m(t+\Delta t)-m(t))^*}{\Delta t}] f(m(\cdot))d\mu.$$

Thus,

$$\lim_{\Delta t\to 0} \int_{C^0([0,T],\mathcal{T}^n)} [\frac{(m(t+\Delta t)-m(t))(m(t+\Delta t)-m(t))^*}{\Delta t}$$
$$-B(t,m(t))] f(m(\cdot))d\mu = 0.$$

Since $f(m(\cdot))$ is an arbitrary bounded continuous function, measurable with respect to $\mathcal{N}_t$, this means that $D_2\xi(t) = B(t,\xi(t))$. But by construction $B(t,\xi(t)) \in \mathbf{B}(t,\xi(t))$ μ-a.s.

Next step deals with the current velocity of the solution. As well as $B_q(t,m(t))$ and $B(t,m(t))$ (see above), $v_q(t,m(t))$ and $v(t,m(t))$ for any specified t can be considered as given on $C^0([0,T],\mathcal{T}^n)$. By construction $D_S\xi_q(t) + \frac{1}{2}diagB_q(t,m(t)) = v_q(t,\xi_q(t))$ for all k. This means that for every real bounded continuous function f on $C^0([0,T],\mathcal{T}^n)$, measurable with respect to $\mathcal{N}_t$, for all q the equality

$$\lim_{\Delta t\to 0} \int_{C^0([0,T],\mathcal{T}^n)} [\frac{m(t+\Delta t)-m(t-\Delta t)}{2\Delta t} + \frac{1}{2}diagB_q(t,m(t))$$
$$-v_q(m(t))] f(m(\cdot))d\mu_q = 0$$

holds.

Specify an arbitrary $\varepsilon > 0$. Since μ_q weakly converges to μ, there exists $K(\varepsilon)$ such that for $q > K(\varepsilon)$

$$\| \int_{C^0([0,T],\mathcal{T}^n)} [\frac{m(t+\Delta t)-m(t-\Delta t)}{2\Delta t}] f(m(\cdot))d\mu_q$$
$$- \int_{C^0([0,T],\mathcal{T}^n)} [\frac{m(t+\Delta t)-m(t-\Delta t)}{2\Delta t}] f(m(\cdot))d\mu \| < \varepsilon$$

and

$$\| \int_{C^0([0,T],\mathcal{T}^n)} f(m(\cdot))v(m(t))d\mu_q - \int_{C^0([0,T],\mathcal{T}^n)} f(m(\cdot))v(m(t))d\mu \| < \varepsilon.$$

With the same arguments as above, by the use of Egorov's theorem we prove that

$$\lim_{q\to\infty} \int_{C^0([0,T],\mathcal{T}^n)} [v_q(m(t)) - v(m(t))] f(m(\cdot))d\mu_q = 0.$$

and that v is continuous on the set of complete measure. Recall that v is bounded as a selector of bounded set-valued mapping.

Then by Lemma from [13, section VI.1] we obtain

$$\lim_{q\to\infty} \int_{C^0([0,T],\mathcal{T}^n)} v(m(t))f(m(\cdot))d\mu_q = \int_{C^0([0,T],\mathcal{T}^n)} v(m(t))f(m(\cdot))d\mu.$$

Evidently

$$\lim_{q\to\infty} \int_{C^0([0,T],\mathcal{T}^n)} [\frac{(m(t+\Delta t)-m(t-\Delta t))}{2\Delta t} + \frac{1}{2}diagB_q(t,m(t))] f(m(\cdot))d\mu_q$$
$$= \int_{C^0([0,T],\mathcal{T}^n)} [\frac{(m(t+\Delta t)-m(t-\Delta t))}{2\Delta t} + \frac{1}{2}diagB_q(t,m(t))] f(m(\cdot))d\mu.$$

Thus,

$$\lim_{\Delta t\to 0}\int_{C^0([0,T],\mathcal{T}^n)}[\frac{(m(t+\Delta t)-m(t-\Delta t)}{2\Delta t}+\frac{1}{2}diagB_q(t,m(t))$$
$$-v(m(t))]f(m(\cdot))d\mu=0.$$

Since $f(m(\cdot))$ is an arbitrary bounded continuous function, measurable with respect to $\mathcal{N}_t$, this means that $D_S\xi(t)=v(\xi(t))$. But by construction $v(\xi(t))\in \mathbf{v}(\xi(t))$ μ-a.s. By the construction, the measure μ is a weak limit of measures μ_q. Thus the constructed solution is perfect. This completes the proof. □

Remark 3.6. Note that all sequences of ε_q-approximations for all sequences of $\varepsilon_q\to 0$, used in the proof of Theorem 3.5, satisfy the hypothesis of [11, Lemma 3] and so the set of measures $\{\mu_q\}$ (corresponding to all sequences and all q) is weakly compact.

4. Optimal solutions

Let $f(t,m)$ be a continuous bounded real-valued function on $\mathbb{R}\times\mathcal{T}^n$. Introduce the cost criterion in the form

$$J(\xi(\cdot))=E\int_0^T f(t,\xi(t))dt \tag{4.1}$$

We are looking for solutions, for which the value of the criterion is minimal.

Theorem 4.1. *There is a perfect solution of (3.1) that minimizes the value of J.*

Proof. (cf. [6])[1] Since all the measures on $(C^0([0,T],\mathcal{T}^n),\mathcal{C})$, constructed in the proof of Theorem 3.5 for perfect solutions of (3.1), are probabilistic and the function f in (4.1) is bounded, the set of values of J on those solutions is bounded. If that set of values has a minimum, then the corresponding measure μ is the one we are looking for: the coordinate process on the space $(C^0([0,T],\mathcal{T}^n),\mathcal{C}),\mu)$ is an optimal solution.

Suppose that the above-mentioned set of values has no minimum, but then it has a greatest lower bound $\aleph$ that is a limit point in that set. Let μ_i^* be a sequence of measures such that for the corresponding solutions $\xi_i^*(t)$ the values $J(\xi_i^*(t))$ converge to $\aleph$. Every μ_i^* is a weak limit of a sequence of measures μ_{ij} corresponding to some sequence of ε_j-approximations as $j\to\infty$. One can easily see that it is possible to select from the sequence a subsequence (for simplicity we denote it by the same symbol μ_{ij}) such that for the corresponding solutions $\xi_{ij}(t)$ and for all i we obtain the uniform convergence of $J(\xi_{ij}(\cdot))$ to $J(\xi_i^*(\cdot))$ as $j\to\infty$. Then $J(\xi_{ii}(\cdot))\to\aleph$ as $i\to\infty$. Since the set of all measures corresponding to all approximations, is weakly compact (see Remark 3.6), we can select from μ_{ii} a subsequence (denote it by the same symbol μ_{ii}) that weakly converges to a certain measure μ^*. By the construction, for the coordinate process $\xi^*(t)$ on $(C^0([0,T],\mathcal{T}^n),\mathcal{C}),\mu^*)$ we get $J(\xi^*(\cdot))=\aleph$, i.e., the value is minimal. Since μ^* is a limit of μ_{ii}, $\xi^*(t)$ is a perfect solution of (3.1) that we are looking for. □

[1]This proof is in fact coincides with that of [6, Theorem 2] but we include it here for completeness

Recall that there is a standard method of transition from the right-hand side of an equation with feedback control to an inclusion: one has to consider all values of the right-hand side at the given point for all values of control and construct its convex closure. It follows from an obvious analogue of well-known Filippov's lemma that for any solution of the obtained inclusion there exists a control under which this solution is realized as a solution of the equation. In particular, this is valid for optimal solutions of inclusions obtained above.

References

[1] Nelson, E.: Derivation of the Schrödinger equation from Newtonian mechanics, *Phys. Rev.*, **150** (1966) 1079-1085.

[2] Nelson, E.: *Dynamical theory of Brownian motion*, Princeton University Press, Princeton, NJ, 1967.

[3] Nelson, E.: *Quantum fluctuations*, Princeton University Press, Princeton, NJ, 1985.

[4] Azarina S.V. and Gliklikh, Yu.E.: Differential inclusions with mean derivatives, *Dyn. Syst. Appl.*, **16** (2007) 49-71.

[5] Gliklikh, Yu.E.: *Global and Stochastic Analysis with Applications to Mathematical Physics*, Springer-Verlag, London, 2011.

[6] Gliklikh, Yu.E. and Zheltikova O.O.: Stochastic Equations and Inclusions with Mean Derivatives and Some Applications. Optimal Solutions for Inclusions of Geometric Brownian Motion Type, *Methodology and Computing in Applied Probability*, **17**, N 1 (2015) 91-105.

[7] Farinelli, S.: Geometric arbitrage theory and market dynamics, *Journal of geometric mechanics*, **7**, N 4 (2015) 431 − 471.

[8] Parthasarathy, K.R.: *Introduction to Probability and Measure*, Springer-Verlag, New York, NY, 1978.

[9] Gliklikh, Yu.E. and Zheltikova, O.O.: On existence of optimal solutions for stochasic differential inclusions with mean derivatives, *Applicable Analysis*, **93** (2014) 35-45.

[10] Azarina, S.V. and Gliklikh, Yu.E.: On the Solvability of Nonautonomous Stochastic Differential Equations with Current Velocities, *Mathematical Notes* **100**, N 1 (2016) 3 − 10.

[11] Gliklikh, Yu.E. and Makarova, A.V.: On existence of solutions to stochastic differential inclusions with current velocities II, *Journal of Computational and Engineering Mathematics*, **3**, N 1 (2016) 48-60.

[12] Yosida, K.: *Functional Analysis*, Springer-Verlag, Berlin et. al., 1965.

[13] Gihman, I.I. and Skorohod, A.V.: *Theory of Stochastic Processes*, Vol.3, Springer-Verlag, New York, NY, 1979.

Yuri E. Gliklikh & Olga O. Zheltikova: Voronezh State University, Universitetskaya pl. 1, Voronezh, Russia

E-mail address: yeg@math.vsu.ru ksu_ola@mail.ru