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THE STUDY OF THE EFFECTS OF FEAR AND ADDITIONAL FOOD IN A PREDATOR - PREY SYSTEM MATHEMATICALLY

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Abstract: Here we have analyzed a predator - prey model considering the effects of the fear of predation and additional food to the predator species. The fear of predation has more effective impact on prey species growth than direct predation. To maintain the predator growth when growth of prey species is very low then additional food is necessary for predators. We have discussed the local and global stability of the equilibrium points. This study establishes that fear of predation significantly affects the growth rate of the prey species and prey species can be extinct from the system due to fear. We have also showed that the predator species is more stable when additional food is supplied to predators. We have presented some numerical simulations to show the feasibility of the main results.

Keywords: Fear effect, additional food, local stability, global stability, and predation.

1 Introduction

Predator - prey dynamical model has been analyzed for almost a century since Lotka [11] and Volterra [18] established the first predator - prey interaction model on the basis of population dynamics. The continuous time models of predator - prey type interaction with different type of functional response of the predator to its prey have been extensively discussed in the literature [3, 4, 8, 9, 17]. The main characteristics of predator - prey interaction system is predator functional response on prey species, which describes the number of prey consumed per predator per unit time. Functional response has a vital role on the stability and the dynamic behavior of the predator - prey model. Various type of functional response has been used to describe the characteristics of predator - prey relationship. Different types of controlling parameters have been incorporated to analyze the dynamical characteristics of relationship between predator and prey [2, 5, 6, 7, 10].

The aim of this study is to investigate the dynamic behaviors of the predator - prey model considering the fear of predation effects to the prey species and additional food source for predator species. Here we have used Holling type-II functional response as prey consumption rate by the predator species. The additional food can be taken as either non - reproducing prey or some food source. Many researchers have investigated the dynamic characteristics of predator - prey model supplying additional food to predators [14, 15, 16]. From biological point of view, for the case of severe scarcity of prey, predators can switch over other populations (alternative food). Recently, few researchers [12, 13, 20] have investigated the impact of fear effect in a predator - prey system with the help of mathematical modeling. Wang et al [19] first formulated a mathematical model by incorporating fear of the predator on prey species. They have showed that the cost of fear reduces the birth rate of prey. The fear of predation can be more powerful and longer - lasting compared to direct predation. For instance, mule deer spend less time foraging due to the predation risk of mountain lions [1]. The prey species can be extinct from a biological system due to fear of predation. High levels of fear can stabilize the predator - prey dynamics by excluding the existence of periodic solution. Also, the fear implemented in the birth rate of prey makes a significant impact on the growth of the predator population.

Here we have discussed the dynamic characteristics of a predator - prey system considering fear effect of prey and additional food to the predators. We have analyzed the following mathematical system:

$$\frac{dx}{dt} = \frac{xr}{h + ky} - ax - bx^2 - \frac{pxy}{x + c}$$

$$\frac{dy}{dt} = \frac{(ex + m)y}{x + c} - dy^2 \qquad (1)$$

Here, 'x' and 'y' denote the density of prey species and predator species at time t respectively; 'r' is the growth rate of the prey species; 'a' is the date rate of prey population; 'b' is the interspecific competition parameter of the prey species; 'p' is the encounter rate of predators with their prey; 'c' is the half saturation constant for the predator; 'e' is the conversion efficiency of prey biomass into predator biomass; 'k' is the level of fear; 'h' is a positive constant related to fear function; 'm' is the intrinsic growth rate of the predator species and 'd' is the interspecific competition parameter of the predator species. Now if $x = 0$, then we can write $\frac{dy}{dt} = \frac{my}{c} - dy^2$. Therefore, the predator species could be permanent without prey species. Hence we can say that the predator species have alternative food source (additional food). System (1) has to be examined with the following initial conditions: $x(0) > 0$, $y(0) > 0$.

2 Positivity of the model system

The predator - prey model (1) describes the dynamics of a population system and therefore it is important to prove all quantities will be positive for all time. One can easily show that all the solutions (x(t), y(t)) of the system (1) with the initial condition $x(0) > 0$, $y(0) > 0$ remain positive for all $t \geq 0$.

Theorem-1: Every solution of the system (1) with positive initial values $x(0) > 0$, $y(0) > 0$ exists and is unique in the interval $[0,\infty)$.

Proof: Let $Z \equiv (x, y)^T$ and $F(Z) = [F_1(Z), F_2(Z)]^T$ such that

$$F_1(Z) = \frac{xr}{h + ky} - ax - bx^2 - \frac{pxy}{x + c}$$

$$F_2(Z) = \frac{(ex + m)y}{x + c} - dy^2 \qquad (2)$$

Therefore, the system (1) can be written in the form $\dot{Z} = F(Z)$, where $F : C_+ \to R_+^2$ with $Z(0) = Z_0 \in R_+^2$, $F_i \in C^\infty(R_+)$ for i = 1, 2. Thus the vector function F is a locally Lipschitzian and completely continuous function of the variables x, y in the positive quadrant $E = \{(x(t), y(t)); x > 0, y > 0\}$. Therefore we can easily say that any solution (x, y) of the system (1) with positive initial values exists and is unique in the interval [0, c] for all $t \geq 0$, where c is a finite positive real number.

3 Model Analysis

3.1 Equilibrium points of the system with the conditions of existence

There are four equilibrium points of our system as follows:

$E_0 = (0; 0)$

$E_1 = (\frac{r - ah}{bh}; 0)$

$E_2 = (0; \frac{m}{cd})$

$E^* = (x^*, y^*)$.

Trivial equilibrium point E_0 and the prey free equilibrium point E_2 always exist.

The predator free equilibrium point is E_1 and existence condition of this equilibrium point is

$r - ah > 0$ i.e. $R_{11} = \frac{r}{ah} < 1$.

The interior equilibrium point is given by E^* where x^* and y^* satisfies the following set of equations:

$$\frac{r}{h + ky^*} - a - bx^* - \frac{py^*}{x^* + c} = 0$$

$$\frac{(ex^* + m)}{x^* + c} - dy^* = 0 \qquad (3)$$

3.2 Local stability analysis of the system about equilibrium points:

The Jacobian matrix of the system (1) at any arbitrary point (x, y) is given by

$$J(x,y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$$

where, $j_{11} = \frac{r}{h + ky} - a - 2bx - \frac{py}{(x + c)^2}$; $j_{12} = -\frac{xkr}{(h + ky)^2} - \frac{px}{x + c}$; $j_{21} = \frac{(ec - m)y}{(x + c)^2}$ and

$j_{22} = \frac{(ex + m)}{x + c} - 2dy$

Theorem-1: The trivial equilibrium point E_0 is always unstable. If the predators free equilibrium point E_1 exists, then it is unstable. The prey free equilibrium point E_2 is asymptotically stable if
$\frac{rcd}{hcd + km} < a + \frac{pm}{c^2 d}$ and unstable if $\frac{rcd}{hcd + km} > a + \frac{pm}{c^2 d}$.

Proof: The two eigenvalues associated with the Jacobian matrix computed around E_0 are $(\frac{r}{h} - a)$ and $\frac{m}{c}$. Therefore, the equilibrium point E_0 is always unstable since one eigenvalue $\frac{m}{c}$ is positive.

The two eigenvalues associated with the Jacobian matrix computed around E_1 are $\frac{ah - r}{h}$ and $\frac{e(r - ah) + mbh}{(r - ah) + cbh}$. Now, from the existence condition of E_1 we can say that one eigenvalue $\frac{ah - r}{h}$ is negative and other eigenvalue $\frac{e(r - ah) + mbh}{(r - ah) + cbh}$ is positive. Hence, we conclude that if the predators free equilibrium point E_1 exists, then it is unstable.

The two eigenvalues associated with the Jacobian matrix computed around E_2 are $-\frac{m}{c}$ and $\frac{rcd}{hcd + km} - (a + \frac{pm}{c^2 d})$. Here, one eigenvalue is negative. Hence, the prey free equilibrium point E_2 will be asymptotically stable if $\frac{rcd}{hcd + km} < (a + \frac{pm}{c^2 d})$ and unstable if $\frac{rcd}{hcd + km} > (a + \frac{pm}{c^2 d})$.

3.3 Local stability analysis of the interior equilibrium point $E^*(x^*, y^*)$

Theorem-2: The interior equilibrium point $E^*(x^*, y^*)$ of the system (1) is locally asymptotically stable if $R_1^* < 1$ and $R_2^* < 1$ where $(R_1)^* = \frac{py^*}{b(x^* + c)}$ and $(R_2)^* = \frac{ec}{m}$.

Proof: The Variational matrix corresponding to the interior equilibrium point $E^*(x^*, y^*)$ is given by

$$J = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \qquad (5),$$

Where, $e_{11} = -bx^* + \frac{px^*y^*}{(x^* + c)^2}$; $e_{12} = -\frac{rkx^*}{(ky^* + h)^2} - \frac{px^*}{x^* + c}$; $e_{21} = \frac{(ec - m)y^*}{(x^* + c)^2}$ and $e_{22} = -dy^*$.

Now for the local stability of the equilibrium point $E^*(x^*, y^*)$ we first linearize the system (1) using the following transformations:

$$x = x^* + x_1$$

$$y = y^* + y_1 \qquad (6)$$

Where x_1 and y_1 are small perturbation about $E^*(x^*, y^*)$, and then the linear form of the system (1) is given by:

$$\frac{dx_1}{dy} = \left(-bx^* + \frac{px^*y^*}{(x^*+c)^2}\right)x_1 + \left(-\frac{rkx^*}{(ky^*+h)^2} - \frac{px^*}{x^*+c}\right)y_1$$

$$\frac{dy_1}{dy} = \frac{(ec-m)y^*}{(x^*+c)^2}x_1 - dy^*y_1 \qquad (7)$$

We now consider the following positive definite function:

$$V = \frac{(x_1)^2}{2x^*} + \frac{(y_1)^2}{2y^*}$$

Now differentiating V with respect to t, we get

$$\frac{dV}{dt} = \frac{x_1}{x^*}\frac{dx_1}{dt} + \frac{y_1}{y^*}\frac{dy_1}{dt}$$

Now using equation (7) we have,

$$\frac{dV}{dt} = \left(-b + \frac{py^*}{(x^*+c)^2}\right)(x_1)^2 + \left(-\frac{rk}{(ky^*+h)^2} - \frac{p}{x^*+c}\right)x_1y_1 + x_1y_1\left(\frac{(ec-m)}{(x^*+c)^2}\right) - d(y_1)^2$$

So $\frac{dV}{dt}$ will be negative definite if $b > \frac{py^*}{(x^*+c)^2}$ and $ec - m < 0$ i.e. if $(R_1)^* < 1$ and $(R_2)^* < 1$, where $(R_1)^* = \frac{py^*}{b(x^*+c)^2}$ and $(R_2)^* = \frac{ec}{m}$.

Now we conclude that $\frac{dV}{dt}$ is negative definite and the interior equilibrium point $E^*(x^*, y^*)$ of the system (1) is locally asymptotically stable if the conditions of theorem is satisfied.

3.4 Global stability analysis of the interior equilibrium point $E^*(x^*, y^*)$

To discuss the global stability nature of the system (1) around the interior equilibrium point $E^*(x^*, y^*)$, we take a positive definite Lyapunov function as follows:

$$L(x, y) = x + x^* \log\left(\frac{x^*}{x}\right) + y + y^* \log\left(\frac{y^*}{y}\right).$$

Now, computing the time derivative of L(x, y), we get

$$\frac{dL}{dt} = \left(\frac{x-x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y-y^*}{y}\right)\frac{dy}{dt}$$

$$= (x-x^*)\left[\frac{r}{h+ky} - a - bx - \frac{py}{x+c}\right] + (y-y^*)\left\{\frac{ex+m}{x+c} - dy\right].$$

After some algebraic calculations, we obtain

$$\frac{dL}{dt} = -\left[b - \frac{py^*}{(x^*+c)(x+c)}\right](x-x^*)^2 - (x-x^*)(y-y^*)\frac{rk}{(h+ky)(h+ky^*)} - \frac{p(x^*+c)}{(x^*+c)(x+c)}(x-x^*)(y-y^*) + (x-x^*)(y-y^*)\frac{ec-m}{(x+c)(x^*+c)} - d(y-y^*)^2.$$

Therefore, the above expression will be negative definite and the function L(x, y) will be a positive definite Lyapunov function around the interior equilibrium point $E^*(x^*, y^*)$ if

$[b - \frac{py^*}{(x^*+c)(x+c)}] > 0$ and $(ec - m) < 0$. Hence, we conclude that, the system (1) is globally asymptotically stable around the interior equilibrium point $E^*(x^*, y^*)$ if $[b - \frac{py^*}{(x^*+c)(x+c)}] > 0$ and $(ec - m) < 0$.

4 Numerical Discussions

Numerical simulations are important in mathematical modeling. It is observed that the numerical results are in good agreement with our analytical findings. We have studied the system (1) numerically using MATLAB software to get better insight of the proposed model. We have taken the following set of parameter values a = 0.02, b = 0.3, r = 0.5, p = 0.4, m = 1.1, h = 1.2, k = 0.15, d = 0.1, e = 0.7 and c = 1.2 for numerical studies.

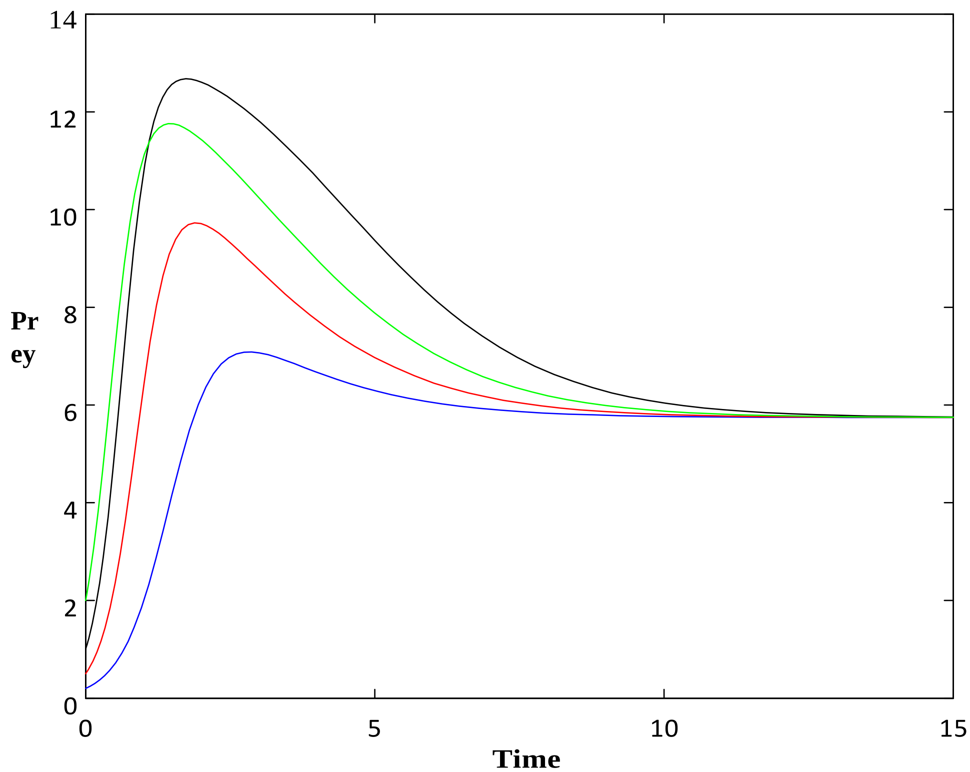

Figure 1: *Dynamic behaviors of the prey species of the system (1) with initial conditions (x(0), y(0)) = (0.2, 2.0),(0.5, 1.0),(1.0, 0.2) and (2.0, 0.5) respectively.*

We have shown the dynamic behaviors of the prey and predator species in Figure - 1 and Figure - 2 with initial conditions (x(0), y(0)) = (0.2, 2.0),(0.5, 1.0),(1.0, 0.2) and (2.0, 0.5) respectively. We have presented the dynamical characteristics of the system (1) with initial conditions (x(0), y(0)) = (0.1, 2.0),(0.6, 2.0),(1.0, 2.0) and (1.6, 2.0) respectively(Figure - 3). From the above diagrams we noticed that the proposed system admits a unique positive equilibrium in every cases and that equilibrium point is globally asymptotically stable.

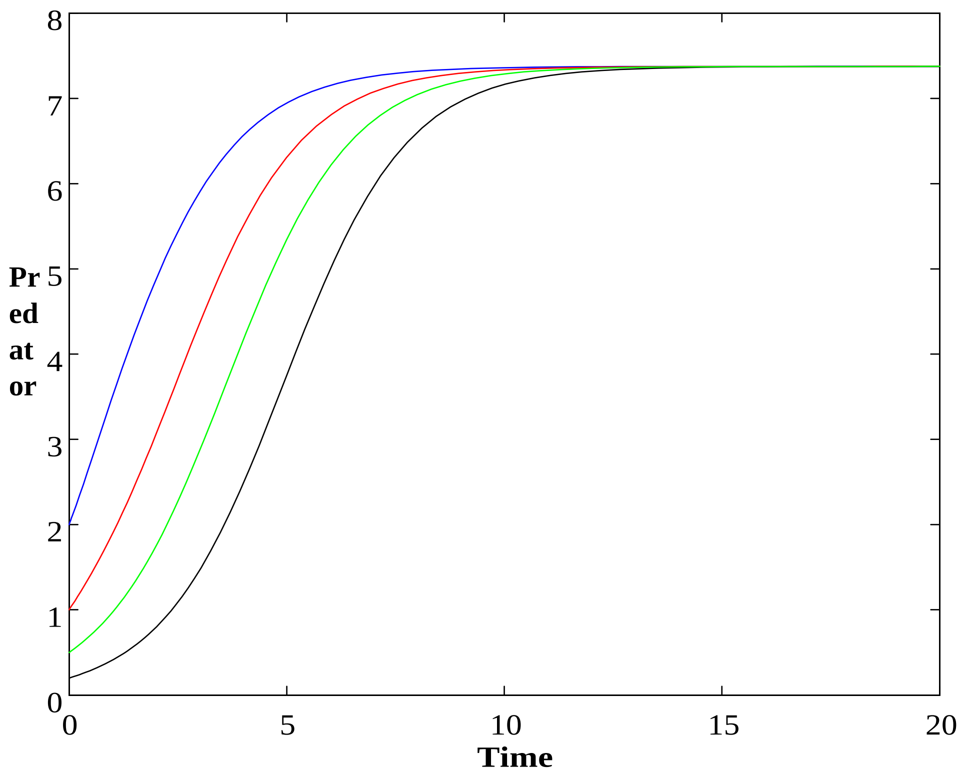

Figure 2: *Dynamic behaviors of the predator species of the system (1) with initial conditions (x(0), y(0)) = (0.2, 2.0),(0.5, 1.0),(1.0, 0.2) and (2.0, 0.5) respectively.*

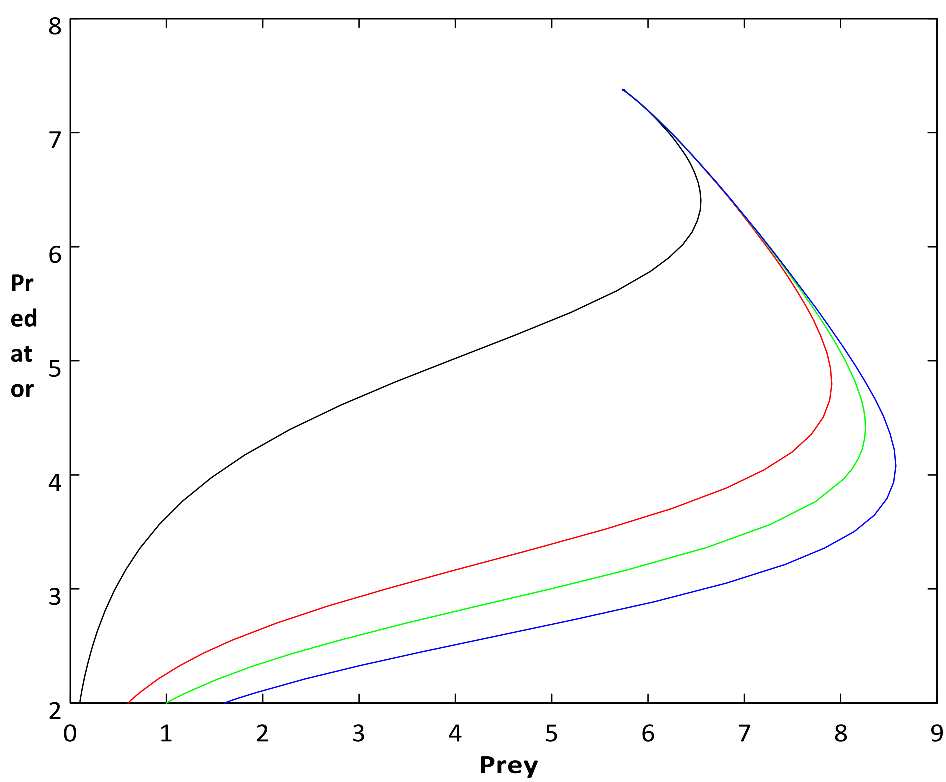

Figure 3: *Dynamic behaviors of the system (1) with initial conditions (x(0), y(0)) = (0.1, 2.0),(0.6, 2.0),(1.0, 2.0) and (1.6, 2.0) respectively.*

To study the effect of fear on prey species we have presented the time series diagram (Figure - 4) of the prey species for different values of the fear parameter (k). We notice that the growth of prey species significantly changed due to fear of predation by the predator species. We also notice that if the fear of predation is increase, then the prey species can be extinct from a biological system. In Figure - 5, we have shown the time series diagram of the predator species for different values of the parameter m. We notice that the predator species exhibit more stable growth for the increasing value of m. Hence we can say that if we supply alternative food to the predator species then the predator species will be more stable to survive.

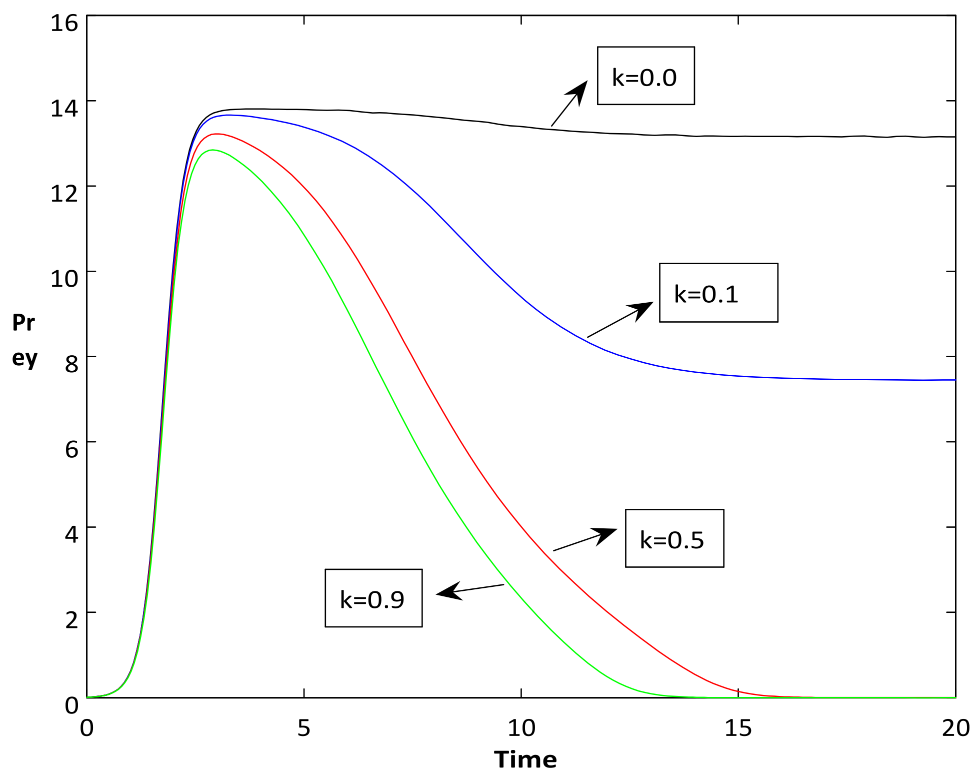

Figure 4: *Dynamic behaviors of the prey species of the system (1) for different values of fear parameter k = 0.0, 0.1, 0.5 and 0.9 respectively.*

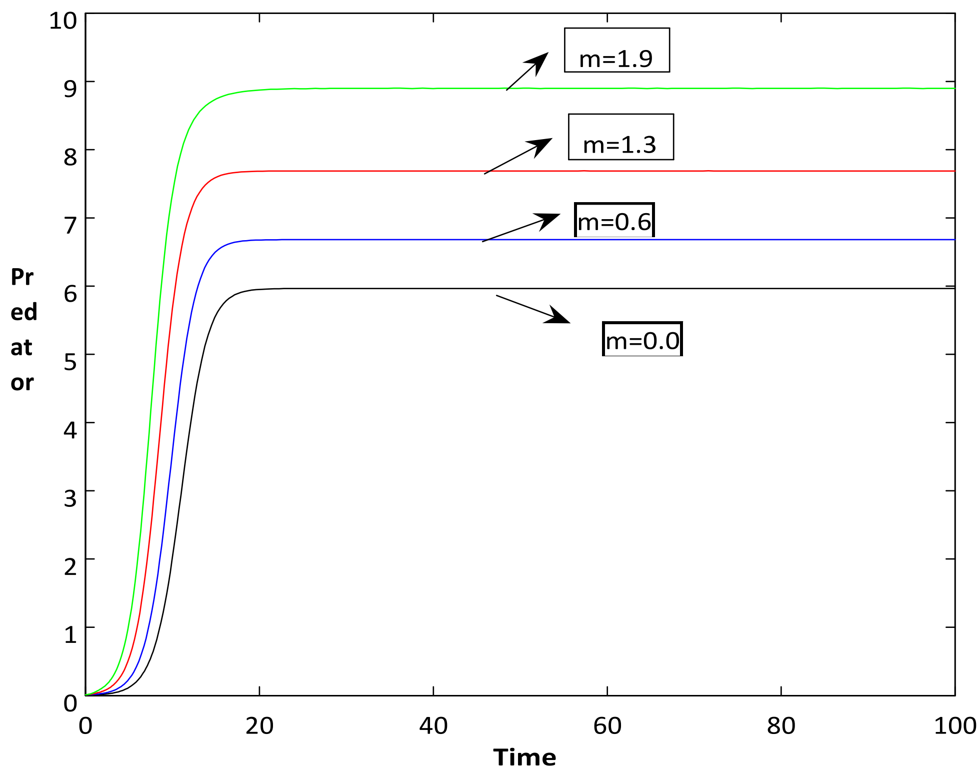

Figure 5: *Dynamic behaviors of the predator species of the system (1) for different values of m = 0.0, 0.6, 1.3 and 1.9 respectively.*

5 Conclusion

In an ecological system, prey - predator interaction is such a biological phenomenon that balances the food web. The sustain ability of predator species depends on their consumption process and the searching strategy for prey. Recently, many researchers study predator-prey model with various biological factors and conditions. Here we study a predator - prey model with fear of predation. We also consider the effect of alternative food to the predator species. We have study the local and global stability of our proposed system around different equilibrium points. From numerical study we notice that the fear of predation has a negative effect on the growth and survival of the prey species, it may be one of the essential factor that leads to the extinction of the prey species. We have also studied the effect of alternative food source to the predator species. Our study indicates that the growth rate of predator will increase with the supply of additional food. Sometimes it has been observed that prey species has a tendency of migration due to fear of predation, and then an alternative food source must exist for sustaining the predator population. Hence, we believe our study can open a new window of study in future.

References

[1] Altendorf, K. B., Laundre, J. W., Gonzales, C. A. L., Brown, J. S.: Assessing effects of predation risk on foraging behavior of mule deer. Journal of Mammalogy 82 (2001), 430-439.

[2] Bhattacharyya, R., Mukhopadhyay, B.: On an eco-epidemiological model with prey harvesting and predator switching: local and global perspectives, Nonlinear Anal.: Real World Applications 11, 3824-3833(2010).

[3] Chattopadhyay J, Arino O.: A predator-prey model with disease in the prey. Nonl. Anal. 36, 747-766(1999).

[4] Das, K. P.: Disease-Induced Chaotic Oscillations and its Possible Control in a Predator-Prey System with Disease in Predator. Differential Equations and Dynamical Systems. 24(2), 215-230(2016).

[5] Das, K. P., Chatterjee, S., Chattopadhyay, J.: Dynamics of NutrientPhytoplankton Interaction in the Presence of Viral Infection and Periodic Nutrient Input. Math. Model. Nat. Phenom. 3(3), 149-169(2008).

[6] Das, K. P., Chatterjee, S., Chattopadhyay, J.: Occurrence of chaos and its possible control in a predator-prey model with density dependent disease-induced mortality on predator population. J. Biol. Syst. 18, 399-435(2010).

[7] Freedman, H. I., Shukla, J. B.: Models for the Effect of Toxicant in SingleSpecies and Predator-Prey Systems. Journal of mathematical biology. 30, 15-30(1991).

[8] Hastings, A., Powell, T.: Chaos in three-species food chain. Ecology. 72 (3), 896-903(1991).

[9] Holling, C. S. Some characteristics of simple types of predation and parasitism. Canadian Entomology. 91, 385-398(1959b).

[10] Holling, C. S.: The functional response of predators to prey density and its role in mimicry and population regulation. Memoirs of the Entomological Society of Canada. 45, 5-60(1965).

[11] Lotka, A. J.: Relation between birth rates and death rates. Science. 26, 21-22(1925).

[12] Pal, S., Pal, N., Samanta, S., Chattopadhyay, J.: Fear effect in prey and hunting cooperation among predators in a Leslie-Gower model. Mathematical Biosciences and Engineering, 16(5), 5146-5179(2019).

[13] Panday, P., Pal, N., Samanta, S., Chattopadhyay, J.: Stability and bifurcation analysis of a three-species food chain model with fear. Int. J. Bifurc. Chaos Appl. Sci. Eng., 28 (2018), 1850009.

[14] Sahoo, B.: A predator-prey model with general Holling interactions in presence of additional food. Int. J. Plant Research, 2 (2012) 47-50.

[15] Srinivasu, P. D. N., Prasad, B. S. R. V., Venkatesulu, M.: Biological control through provision of additional food to predators: a theoretical study. Theor. Popul. Biol., 72 (2007) 111-120.

[16] van Baalen, M., Krivan, V., van Rijn, P. C. J., Sabelis, M. W.: Alternative food, switching predators, and the persistence of predator-prey systems. The American Naturalist, 157, 512-524(2001).

[17] Venturino, E.: The influence of disease on Lotka-Volterra systems, Rky Mt J Math. 24, 381-402(1994).

[18] Volterra, V.: Variazioni e fluttauazioni del numero d'individui in specie animals conviventi, Mem. della Reale Acad. Nazi. dei Lincei, 2, 31- 113(1926). Translation in an appendix to Chapmain's Animal Ecology, New York, 1931.

[19] Wang, X., Zanette, L. Y., Zou, X.: Modelling the fear effect in predator-prey interactions. J. Math. Biol, 73, 1179-1204(2016).

[20] Wang, X., Zou, X.: Modeling the fear effect in predator-prey interactions with adaptive avoidance of predators. Bull. Math. Biol., 79, 1325- 1359(2017).

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